PROBABILITY ON FINITE TRANSITIVE GRAPHS

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The work I present here concerns two classical areas of study in probability: *percolation* and *random walks*. First and foremost these are a source of beautiful mathematics, but they also have numerous and diverse links to various physical problems. In particular, I will describe some of my recent progress on understanding certain *finite* models of these processes, which are both more mathematically delicate than their classically studied infinite analogues, and closer in spirit to the physical problems that motivate them.

In one standard model of percolation, one starts with a graph G and a parameter $p \in [0, 1]$, and then one deletes or retains each edge of G independently at random, with each individual edge retained with probability p, to obtain a random subgraph G_p of G. Of particular classical interest is the probability $\psi_G(p)$ that G_p has an infinite connected component. For every connected graph G there exists a *critical probability* $p_c = p_c(G) \in [0, 1]$ such that $\psi_G(p) = 0$ for $p < p_c$ and $\psi_G(p) = 1$ for $p > p_c$. If p_c is neither 0 nor 1 then we say that there is a *phase transition* at p_c, i.e. a transition from the phase $0 \leq p < p_c$ in which infinite components do not exist, to the phase $p_c < p \leq 1$ in which they do. Phase transitions in general are of considerable importance in physics (for example, the transition of matter between solid, liquid and gas), and percolation turns out to be an excellent model in which to understand them more generally.

Random walks can be used as discrete approximations to Brownian motion, and have links to topics as diverse as electric networks and card shuffling. The *simple random walk* on a graph is a random sequence $X_0, X_1, X_2, \ldots$ of vertices, with X_0 some fixed 'starting vertex', and each X_{n+1} chosen uniformly at random from the neighbours of X_n. A classical question about the simple random walk on an infinite graph is whether it is *transient*. i.e. has positive probability of 'escaping to infinity' without first returning to X_0.

These properties were classically studied in the setting of infinite *transitive* graphs (a graph G is called *transitive* if for every two vertices u, v of G there exists an automorphism φ of G such that $\varphi(u) = v$). In this setting, both the existence of a phase transition for percolation and the transience of the random walk can be characterised completely by a property of the graph called its *degree of growth*, which one can think of as the 'dimension' of the graph. However, keeping in mind the physical motivations and applications of percolation and random walks, it is in some sense more natural to study *finite* graphs. Of course, one has to replace the notions of *escaping to infinity* and *infinite component* with suitable analogues that make sense in a finite setting; for the sake of brevity I will omit the details of this, but there are a number of precise conjectures in this direction. In particular, Benjamini (2001) conjectured an optimal sufficient condition on a finite transitive graph for percolation to exhibit a phase transition, and Benjamini & Kozma (2002) conjectured an optimal sufficient condition for the random walk to be 'transient' (in a suitable sense).

Despite some small progress towards these conjectures by various authors, until recently they appeared out of reach. Indeed, there are various reasons why these finite questions are inherently harder to analyse than their infinite analogues. One is that the notion of 'dimension' is somewhat subtle in a finite transitive graph: given some large $n \in \mathbb{N}$, and letting m vary between 1 and n, at what point does an $m \times n$ torus change from being '1-dimensional' to '2-dimensional'? Nonetheless, in recent work with Romain Tessera I have completely resolved and even strengthened the Benjamini–Kozma conjecture on random walks, and with Tom Hutchcroft I have proved Benjamini's percolation conjecture under all but the very weakest versions of the hypothesis. See Igor Pak's blog https://tinyurl.com/2p8kzen7 for more background on why the percolation conjecture in particular was hard.

One of the main reasons we have been able to make such sudden progress on these conjectures is a certain structure theory of transitive graphs developed recently by Tessera and me. This theory has a long history, starting with theorems of Gromov and Trofimov in the 1980s showing that 'low-dimensional' transitive graphs are rich in algebraic structure. About ten years ago, Breuillard, Green and Tao used *approximate groups* to give a more quantitative version of Gromov's theorem that made it more useful for finite graphs. Tessera and I have, amongst other things, refined this result even further, and proved a similarly quantitative version of Trofimov's theorem, again using approximate groups. In proving the results I have described, my coauthors and I combined this theory with various other probabilistic, geometric, algebraic and combinatorial techniques. For example, in the work on percolation we introduced a new (and independently interesting) technique in the seemingly distant subject of *additive combinatorics*.

I believe I will be able to present these topics to a lay audience very effectively; for example, see https://youtu.be/fljnOAggS7Y for a short example of me explaining random walks and their links to group theory and to electric networks.