HOMOGENEOUS LORENTZ MANIFOLDS
WITH SIMPLE ISOMETRY GROUP

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ABSTRACT. Let $H$ be a closed, noncompact subgroup of a simple Lie group $G$, such that $G/H$ admits an invariant Lorentz metric. We show that if $G = \text{SO}(2, n)$, with $n \geq 3$, then the identity component $H^*$ of $H$ is conjugate to $\text{SO}(1, n)^*$. Also, if $G = \text{SO}(1, n)$, with $n \geq 3$, then $H^*$ is conjugate to $\text{SO}(1, n-1)^*$.

1. INTRODUCTION

1.1. Definition. • A Minkowski form on a real vector space $V$ is a nondegenerate quadratic form that is isometric to the form $-x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ on $\mathbb{R}^{n+1}$, where $\dim V = n + 1 \geq 2$.

• A Lorentz metric on a smooth manifold $M$ is a choice of Minkowski metric on the tangent space $T_pM$, for each $p \in M$, such that the form varies smoothly as $p$ varies.

A. Zeghib [Ze1] classified the compact homogeneous spaces that admit an invariant Lorentz metric. In this note, we remove the assumption of compactness, but add the restriction that the transitive group $G$ is almost simple. Our starting point is a special case of a theorem of N. Kowalsky.

1.2. Theorem (N. Kowalsky, cf. [Ko3, Thm. 5.1]). Let $G/H$ be a nontrivial homogeneous space of a connected, almost simple Lie group $G$ with finite center. If there is a $G$-invariant Lorentz metric on $G/H$, then either

1) there is also a $G$-invariant Riemannian metric on $G/H$; or
2) $G$ is locally isomorphic to either $\text{SO}(1, n)$ or $\text{SO}(2, n)$, for some $n$.

As explained in the following elementary proposition, it is easy to characterize the homogeneous spaces that arise in Conclusion (1) of Theorem 1.2, although it is probably not reasonable to expect a complete classification.

1.3. Notation. We use $\mathfrak{g}$ to denote the Lie algebra of a Lie group $G$, and $\mathfrak{h} \subset \mathfrak{g}$ to denote the Lie algebra of a Lie subgroup $H$ of $G$.

1.4. Proposition (cf. [Ko3, Thm. 1.1]). Let $G/H$ be a homogeneous space of a Lie group $G$, such that $\mathfrak{g}$ is simple and $\dim G/H \geq 2$. The following are equivalent.

1) The homogeneous space $G/H$ admits both a $G$-invariant Riemannian metric and a $G$-invariant Lorentz metric.
2) The closure of $\text{Ad}_G H$ is compact, and leaves invariant a one-dimensional subspace of $\mathfrak{g}$ that is not contained in $\mathfrak{h}$.

The two main results of this note examine the cases that arise in Conclusion (2) of Theorem 1.2. It is well known [Ko2, Egs. 2 and 3] that \(\text{SO}(1, n)^0/\text{SO}(1, n - 1)^0\) and \(\text{SO}(2, n)^0/\text{SO}(1, n)^0\) have invariant Lorentz metrics. Also, for any discrete subgroup \(\Gamma\) of \(\text{SO}(1, 2)\), the Killing form provides an invariant Lorentz metric on \(\text{SO}(1, 2)^0/\Gamma\). We show that these are essentially the only examples.

Note that \(\text{SO}(1, 1)\) and \(\text{SO}(2, 2)\) fail to be almost simple. Thus, in 1.2(2), we may assume

- \(G\) is locally isomorphic to \(\text{SO}(1, n)\), and \(n \geq 2\); or
- \(G\) is locally isomorphic to \(\text{SO}(2, n)\), and \(n \geq 3\).

2.3'. **Proposition.** Let \(G\) be a Lie group that is locally isomorphic to \(\text{SO}(1, n)\), with \(n \geq 2\). If \(H\) is a closed subgroup of \(G\), such that

- the closure of \(\text{Ad}_G H\) is not compact, and
- there is a \(G\)-invariant Lorentz metric on \(G/H\),

then either

1) after any identification of \(g\) with \(\text{so}(1, n)\), the subalgebra \(h\) is conjugate to a standard copy of \(\text{so}(1, n - 1)\) in \(\text{so}(1, n)\), or

2) \(n = 2\) and \(H\) is discrete.

3.5'. **Theorem.** Let \(G\) be a Lie group that is locally isomorphic to \(\text{SO}(2, n)\), with \(n \geq 3\). If \(H\) is a closed subgroup of \(G\), such that

- the closure of \(\text{Ad}_G H\) is not compact, and
- there is a \(G\)-invariant Lorentz metric on \(G/H\),

then, after any identification of \(g\) with \(\text{so}(2, n)\), the subalgebra \(h\) is conjugate to a standard copy of \(\text{so}(1, n)\) in \(\text{so}(2, n)\).

N. Kowalsky announced a much more general result than Theorem 3.5' in [Ko2, Thm. 4], but it seems that she did not publish a proof before her premature death. She announced a version of Proposition 2.3' (with much more general hypotheses and a somewhat weaker conclusion) in [Ko2, Thm. 3], and a proof appears in her Ph.D. thesis [Ko1, Cor. 6.2].

1.5. **Remark.** It is easy to see that there is a \(G\)-invariant Lorentz metric on \(G/H\) if and only if there is an \((\text{Ad}_G H)\)-invariant Minkowski form on \(g/h\). Thus, although Proposition 2.3' and Theorem 3.5' are geometric in nature, they can be restated in more algebraic terms. It is in such a form that they are proved in §2 and §3.

Proposition 2.3' and Theorem 3.5' are used in work of S. Adams [Ad3] on nontame actions on Lorentz manifolds. See [Zi, Ko3, AS, Ze2, Ad1, Ad2] for some other research concerning actions of Lie groups on Lorentz manifolds.

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2. **Homogeneous spaces of \(\text{SO}(1, n)\)**

The following lemma is elementary.
2.1. Lemma. Let \( \pi \) be the standard representation of \( g = so(1, k) \) on \( \mathbb{R}^{k+1} \), and let \( g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \) be an Iwasawa decomposition of \( g \).

1) The representation \( \pi \) has only one positive weight (with respect to \( a \)), and the corresponding weight space is 1-dimensional.

2) There are subspaces \( V \) and \( W \) of \( \mathbb{R}^{k+1} \), such that
   (a) \( \dim(\mathbb{R}^{k+1}/V) = 1 \);
   (b) \( \dim W = 1 \);
   (c) \( \pi(a)V \subset W \);
   (d) for all nonzero \( u \in \mathfrak{n} \), we have \( \pi(u)^2 \mathbb{R}^{k+1} = W \); and
   (e) for all nonzero \( u \in \mathfrak{n} \) and \( v \in \mathbb{R}^{k+1} \), we have \( \pi(u)^2 v = 0 \) if and only if \( v \in V \).

2.2. Corollary. Let \( \mathfrak{h} \) be a subalgebra of a real Lie algebra \( g \), let \( Q \) be a Minkowski form on \( g/\mathfrak{h} \), and define \( \pi : N_G(\mathfrak{h}) \rightarrow GL(\mathfrak{g}/\mathfrak{h}) \) by \( \pi(g)(v + \mathfrak{h}) = (Ad_g v) + \mathfrak{h} \).

1) Suppose \( T \) is a connected Lie subgroup of \( G \) that normalizes \( H \), such that \( \pi(T) \subset SO(Q) \) and \( Ad_T T \) is diagonalizable over \( \mathbb{R} \). Then, for any ordering of the \( T \)-weights on \( g \), the subalgebra \( \mathfrak{h} \) contains codimension-one subspaces of both \( g^+ \) and \( g^- \), where \( g^+ \) is the sum of all the positive weight spaces of \( T \), and \( g^- \) is the sum of all the negative weight spaces of \( T \).

2) If \( U \) is a connected Lie subgroup of \( G \) that normalizes \( H \), such that \( \pi(U) \subset SO(Q) \) and \( Ad_U U \) is unipotent, then there are subspaces \( V/\mathfrak{h} \) and \( W/\mathfrak{h} \) of \( g/\mathfrak{h} \), such that
   (a) \( \dim(\mathfrak{g}/V) = 1 \);
   (b) \( \dim(\mathfrak{W}/\mathfrak{h}) = 1 \);
   (c) \( [V, u] \subset W \);
   (d) for each \( u \in u \), either \( W = \mathfrak{h} + (\text{ad}_g u)^2 \mathfrak{g} \), or \( [\mathfrak{g}, u] \subset \mathfrak{h} \); and
   (e) for all \( u \in u \), we have \( (\text{ad}_g u)^2 V \subset \mathfrak{h} \).

2.3. Proposition. Let \( H \) be a Lie subgroup of \( G = SO(1, n) \), with \( n \geq 2 \), such that
   - the closure of \( H \) is not compact; and
   - there is an \((Ad_G H)\)-invariant Minkowski form on \( g/\mathfrak{h} \).

Then either
1) \( H^o \) is conjugate to a standard copy of \( SO(1, n-1) \) in \( SO(1, n) \), or
2) \( n = 2 \) and \( H^o \) is trivial.

Proof. Let \( \overline{H} \) be the Zariski closure of \( H \), and note that the Minkowski form is also invariant under \( Ad_G \overline{H} \). Replacing \( H \) by a finite-index subgroup, we may assume \( \overline{H} \) is Zariski connected.

Let \( G = KAN \) be an Iwasawa decomposition of \( G \).

Case 1. Assume \( n \geq 3 \) and \( A \subset \overline{H} \). From Corollary 2.2(1), we see that \( \mathfrak{h} \) contains codimension-one subspaces of both \( \mathfrak{n} \) and \( \mathfrak{n}^- \). (Note that this implies \( H^o \) is nontrivial.) This implies that \( \overline{H} \) is reductive. (Because \( (H \cap N)^o \) unip \( \overline{H} \) is a unipotent subgroup that intersects \( N \) nontrivially (and \( R\text{-rank} G = 1 \), it must be contained in \( N \), so unip \( \overline{H} \subset N \). Similarly, unip \( \overline{H} \subset N^- \). Therefore unip \( \overline{H} \subset N \cap N^- = e \).) Then, since \( \overline{H} \) contains a codimension-one subgroup of \( N \), and since \( A \subset \overline{H} \), it follows that \( \overline{H} \) is conjugate to either \( SO(1, n-1) \) or \( SO(1, n) \). Because \( H^o \) is a nontrivial, connected, normal subgroup of \( \overline{H} \), we conclude that \( H^o \) is conjugate to either \( SO(1, n-1)^o \) or \( SO(1, n)^o \). Because \( g/\mathfrak{h} \neq 0 \) (else
dim g/h = 0 < 2, which contradicts the fact that there is a Minkowski form on g/h), we see that \( H^o \) is conjugate to SO(1, n - 1).

**Case 2.** Assume \( n \geq 3 \) and \( H \) does not contain any nontrivial hyperbolic elements. The Levi subgroup of \( H \) must be compact, and the radical of \( H \) must be unipotent, so choose a compact \( M \) and a nontrivial unipotent subgroup \( U \) such that \( H = M \ltimes U \). Replacing \( H \) by a conjugate, we may assume, without loss of generality, that \( U \subset N \).

Let us show, for every nonzero \( u \in u \), that \([g, u] \not\subset h \). From the Morosov Lemma [Ja, Thm. 17(1), p. 100], we know there exists \( v \in g \), such that \([v, u] \) is hyperbolic (and nonzero). If \([v, u] \subset h \), this contradicts the fact that \( H \) does not contain nontrivial hyperbolic elements.

Let \( V/h \) and \( W/h \) be subspaces of \( g/h \) as in Corollary 2.2. Because \((ad_u)^2 g = n \) for every nonzero \( u \in n \), we have \( W = n + h \) (see 2.2(2d)), so \( dim n/(h \cap n) = 1 \) (see 2.2(2b)) and

\[
[u, V] \subset W = n + h \subset n + h = n + m
\]

(see 2.2(2c)).

Assume, for the moment, that \( n \geq 4 \). Then

\[
dim u + dim(V \cap n^-) \geq \dim(h \cap n) + \dim(V \cap n^-) \geq (\dim n - 1) + (\dim n^- - 1) = (n - 2) + (n - 2) \geq n > \dim n.
\]

This implies that there exist \( u \in u \) and \( v \in V \cap n^- \), such that \( (u, v) \cong sl(2, \mathbb{R}) \), with \([u, v]\) hyperbolic (and nonzero). This contradicts the fact that \( m + n \) has no nontrivial hyperbolic elements.

We may now assume that \( n = 3 \). For any nonzero \( u \in n \), we have

\[
dim[u, V] \geq \dim[u, g] - 1 = \dim n + 1 > \dim n,
\]

so \([u, V] \not\subset n \). Then, from (2.4), we conclude that \( m \neq 0 \), so \( m \) acts irreducibly on \( n \). This contradicts the fact that \( h \cap n \) is a codimension-one subspace of \( n \) that is normalized by \( m \).

**Case 3.** Assume \( n = 2 \). We may assume \( H^o \) is nontrivial (otherwise Conclusion (2) holds). We must have \( dim g/h \geq 2 \), so we conclude that \( dim H^o = 1 \) and \( dim g/h = 2 \). Because \( SO(1, 1) \) consists of hyperbolic elements, this implies that \( Ad_G h \) acts diagonalizably on \( g/h \), for every \( h \in H \). Therefore \( H^o \) is conjugate to \( A \), and, hence, to \( SO(1, 1)^o \).

3. HOMOGENEOUS SPACES OF SO(2, n)

**3.1. Theorem** (Borel-Tits [BT2, Prop. 3.1]). Let \( H \) be an \( F \)-subgroup of a reductive algebraic group \( G \) over a field \( F \) of characteristic zero. Then there is a parabolic \( F \)-subgroup \( P \) of \( G \), such that \( \text{unip} H \subset \text{unip} P \) and \( H \subset N_G(\text{unip} H) \subset P \).

**3.2. Notation.** Let \( k = \lfloor n/2 \rfloor \). Identifying \( \mathbb{C}^{k+1} \) with \( \mathbb{R}^{2k+2} \) yields an embedding of \( SU(1, k) \) in \( SO(2, 2k) \). Then the inclusion \( \mathbb{R}^{2k+2} \to \mathbb{R}^{2k+3} \) yields an embedding of \( SU(1, k) \) in \( SO(2, 2k + 1) \). Thus, we may identify \( SU(1, \lfloor n/2 \rfloor) \) with a subgroup of \( SO(2, n) \).

We use the following well-known result to shorten one case of the proof of Theorem 3.5.

**3.3. Lemma** ([OW, Lem. 6.8]). If \( L \) is a connected, almost-simple subgroup of \( SO(2, n) \), such that \( \mathbb{R} \)-rank \( L = 1 \) and \( \text{dim} L > 3 \), then \( L \) is conjugate under \( O(2, n) \) to a subgroup of either \( SO(1, n) \) or \( SU(1, \lfloor n/2 \rfloor) \).
3.4. Corollary. Let $L$ be a connected, reductive subgroup of $G = \text{SO}(2,n)$, such that $\mathbb{R}$-rank $L = 1$. Then $\dim U \leq n - 1$, for every connected, unipotent subgroup $U$ of $L$.

Furthermore, if $\dim U = n - 1$, then either

1) $L$ is conjugate to $\text{SO}(1,n)^*$; or
2) $n$ is even, and $L$ is conjugate under $\text{O}(2,n)$ to $\text{SU}(1,n/2)$.

3.5. Theorem. Let $H$ be a Lie subgroup of $G = \text{SO}(2,n)$, with $n \geq 3$, such that

- the closure of $H$ is not compact, and
- there is an $(\text{Ad}_G H)$-invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$.

Then $H^0$ is conjugate to a standard copy of $\text{SO}(1,n)^0$ in $\text{SO}(2,n)$.

Proof. Let $\overline{H}$ be the Zariski closure of $H$, and note that the Minkowski form is also invariant under $\text{Ad}_G \overline{H}$. Replacing $H$ by a finite-index subgroup, we may assume $\overline{H}$ is Zariski connected.

Let $G = KAN$ be an Iwasawa decomposition of $G$. For each real root $\phi$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{a}$), let $\mathfrak{g}_\phi$ be the corresponding root space, and let proj$_{\phi}: \mathfrak{g} \to \mathfrak{g}_\phi$ and proj$_{\phi - \phi}: \mathfrak{g} \to \mathfrak{g}_\phi + \mathfrak{g}_{-\phi}$ be the natural projections. Fix a choice of simple real roots $\alpha$ and $\beta$ of $\mathfrak{g}$, such that $\dim \mathfrak{g}_\alpha = 1$ and $\dim \mathfrak{g}_\beta = n - 2$ (so the positive real roots are $\alpha$, $\beta$, $\alpha + \beta$, and $\alpha + 2\beta$). Replacing $N$ by a conjugate under the Weyl group, we may assume $n = \mathfrak{g}_\alpha + \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. From the classification of parabolic subgroups [BT1, Prop. 5.14, p. 99], we know that the only proper parabolic subalgebras of $\mathfrak{g}$ that contain $\mathfrak{n}_\mathfrak{g}(n)$ are

$$n_\mathfrak{g}(n), \mathfrak{p}_\alpha = n_\mathfrak{g}(n) + \mathfrak{g}_{-\alpha}, \text{ and } \mathfrak{p}_\beta = n_\mathfrak{g}(n) + \mathfrak{g}_\beta.$$

Case 1. Assume $\overline{H}$ contains nontrivial hyperbolic elements. Let $t = \overline{H} \cap \mathfrak{a}$. Replacing $H$ by a conjugate, we may assume $t \neq 0$.

Subcase 1.1. Assume $t \in \{\ker(\alpha + \beta), \ker \beta\}$.

Subsubcase 1.1.1. Assume $\overline{H}$ is reductive. We may assume $t = \ker(\alpha + \beta)$ (if necessary, replace $H$ with its conjugate under the Weyl reflection corresponding to the root $\alpha$). Then, from Corollary 2.2(1), we see that $\mathfrak{h}$ contains a codimension-one subspace of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$. (Note that this implies $H^0$ is nontrivial.)

Let $n' = \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$, so $n'$ is the Lie algebra of a maximal unipotent subgroup of $G$. (In fact, $n'$ is the image of $n$ under the Weyl reflection corresponding to the root $\alpha$.) From the preceding paragraph, we have

$$\dim(\overline{H} \cap n') \geq \dim(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}) = n - 1.$$

Therefore, Corollary 3.4 implies that $\overline{H}$ is conjugate (under $\text{O}(2,n)$) to either $\text{SO}(1,n)$ or $\text{SU}(1,n/2)$. It is easy to see that $\overline{H}$ is not conjugate to $\text{SU}(1,n/2)$. (See [OW, proof of Thm. 1.5] for an explicit description of $\text{SU}(1,n/2) \cap n$. If $n$ is even, then $n > 3$, so $\text{SU}(1,n/2)$ does not contain a codimension-one subspace of any $(n - 2)$-dimensional root space, but $\mathfrak{h}$ does contain a codimension-one subspace of $\mathfrak{g}_\beta$. Therefore, we conclude that $\overline{H}$ is conjugate to $\text{SO}(1,n)$. Then, because $H^0$ is a nontrivial, connected, normal subgroup of $\overline{H}$, we conclude that $H^0 = (\overline{H})^0$ is conjugate to $\text{SO}(1,n)^0$.

Subcase 1.1.2. Assume $\overline{H}$ is not reductive. Let $P$ be a maximal parabolic subgroup of $G$ that contains $\overline{H}$ (see Theorem 3.1). By replacing $P$ and $H$ with conjugate subgroups,
we may assume that $P$ contains the minimal parabolic subgroup $N_G(N)$. Therefore, the classification of parabolic subalgebras (3.6) implies that $P$ is either $P_\alpha$ or $P_\beta$.

**Subsubcase 1.1.2.1.** Assume $\mathfrak{t} = \text{ker}(\alpha + \beta)$. From Corollary 2.2(1), we see that $\mathfrak{h}$ (and hence also $\mathfrak{p}$) contains codimension-one subspaces of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_\alpha$. Because $\mathfrak{p}_\alpha$ does not contain such a subspace of $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_\alpha$, we conclude that $P = P_\beta$. Furthermore, because the intersection of $\mathfrak{p}_\beta$ with each of these subspaces does have codimension one, we conclude that $\mathfrak{h}$ has precisely the same intersection; therefore $(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta) + (\mathfrak{g}_\beta + \mathfrak{g}_\alpha) \subseteq \mathfrak{h}$. Hence $\mathfrak{h} \supseteq [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We now have

$$(\text{ad}_{\mathfrak{g}} \mathfrak{g}_{\alpha+\beta})^2 \mathfrak{g} = \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} \equiv 0 \quad (\text{mod } \mathfrak{h}),$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}_{\alpha+\beta}] \supseteq [\mathfrak{g}_{-\alpha-\beta}, \mathfrak{g}_{\alpha+\beta}] \supseteq \ker \beta.$$  

This contradicts the fact that $\mathfrak{h} \cap \mathfrak{a} = \mathfrak{t} = \ker(\alpha + \beta)$.

**Subsubcase 1.1.2.2.** Assume $\mathfrak{t} = \ker \beta$. From Corollary 2.2(1), we see that $\mathfrak{h}$ (and hence also $\mathfrak{p}$) contains a codimension-one subspace of $\mathfrak{g}_{\alpha} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. Because neither $\mathfrak{p}_\alpha$ nor $\mathfrak{p}_\beta$ contains such a subspace, this is a contradiction.

**Subcase 1.2.** Assume $\mathfrak{t} \in \{\ker \alpha, \ker(\alpha + 2\beta)\}$. We may assume $\mathfrak{t} = \ker \alpha$ (if necessary, replace $H$ with its conjugate under the Weyl reflection corresponding to the root $\beta$). From Corollary 2.2(1), we see that $\mathfrak{h}$ contains a codimension-one subspace of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. Because any codimension-one subalgebra of a nilpotent Lie algebra must contain the commutator subalgebra, we conclude that $\mathfrak{h}$ contains $\mathfrak{g}_{\alpha+\beta}$. Then we have

$$(\text{ad}_{\mathfrak{g}} \mathfrak{g}_{\alpha+\beta})^2 \mathfrak{g} = \mathfrak{g}_{\alpha+2\beta} \equiv 0 \quad (\text{mod } \mathfrak{h}),$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supseteq [\mathfrak{g}, \mathfrak{g}_{\alpha+\beta}] \supseteq \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}.$$  

Similarly, we also have $\mathfrak{h} \supseteq \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. It is now easy to show that $\mathfrak{h} \supseteq \mathfrak{g}_\phi$ for every real root $\phi$, so $\mathfrak{h} = \mathfrak{g}$. This contradicts the fact that $\mathfrak{g}/\mathfrak{h} \neq 0$.

**Subcase 1.3.** Assume $\mathfrak{t}$ contains a regular element of $\mathfrak{a}$. Replacing $H$ by a conjugate under the Weyl group, we may assume that $\mathfrak{n}$ is the sum of the positive root spaces, with respect to $\mathfrak{t}$. Then, from Corollary 2.2(1), we see that $\mathfrak{h}$ contains codimension-one subspaces of both $\mathfrak{n}$ and $\mathfrak{n}^{-}$. Therefore, $\mathfrak{h}$ contains codimension-one subspaces of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$ and $\mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$, so the argument of Subcase 1.2 applies.

**Case 2.** Assume $\mathfrak{h}$ does not contain nontrivial hyperbolic elements. The Levi subgroup of $H$ must be compact, and the radical of $\overline{H}$ must be unipotent, so choose a compact $M$ and a nontrivial unipotent subgroup $U$ such that $\overline{H} = M \ltimes U$. Choose subspaces $V/\mathfrak{h}$ and $W/\mathfrak{h}$ of $\mathfrak{g}/\mathfrak{h}$ as in Corollary 2.2(2).

Let $P$ be a proper parabolic subgroup of $G$, such that $U \subset \text{unip } P$ and $H \subset P$ (see Theorem 3.1). Replacing $H$ and $P$ by conjugates, we may assume, without loss of generality, that $P$ contains the minimal parabolic subgroup $N_G(N)$ (so unip $P \subset N$). From the classification of parabolic subalgebras (3.6), we know that there are only three possibilities for $P$. We consider each of these possibilities separately.

First, though, let us show that:

\[(3.7)\quad \text{for every nonzero } u \in \mathfrak{u}, \text{ we have } [\mathfrak{g}, u] \not\subseteq \mathfrak{h}.\]
From the Morosov Lemma [Ja, Thm. 17(1), p. 100], we know there exists \( v \in \mathfrak{g} \), such that \([v, u]\) is hyperbolic (and nonzero). If \([v, u] \in \mathfrak{h}\), this contradicts the fact that \(\mathfrak{h}\) does not contain nontrivial hyperbolic elements.

**Subcase 2.1.** Assume \( P = N_{\mathfrak{g}}(N) \) is a minimal parabolic subgroup of \( G \).

**Subsubcase 2.1.1.** Assume \( \text{proj}_{\beta} u \neq 0 \). Choose \( u \in \mathfrak{u} \), such that \( \text{proj}_{\beta} u \neq 0 \), and let \( Z = (\text{ad}_u)^2\mathfrak{g}_{-\alpha - 2\beta} \). (So \( \dim Z = 1 \), \( \text{proj}_{\alpha} Z \neq 0 \), and \( \text{proj}_{\alpha - \beta} Z = 0 \).) From Corollary 2.2(2d), we know that \( Z \subset W \). Then, because \( \text{proj}_{\alpha} \mathfrak{h} \subset \text{proj}_{\alpha} \mathfrak{p} = 0 \), we conclude, from Corollary 2.2(2b), that \( W = \mathfrak{h} + Z \).

Because \( W = \mathfrak{h} + Z \subset \mathfrak{p} + Z \), we have \( \text{proj}_{\alpha - \beta} W = 0 \). Therefore, because \( \text{proj}_{\beta} u \neq 0 \), we conclude, from Corollary 2.2(2c), that \( \text{proj}_{\alpha - 2\beta} V = 0 \), so Corollary 2.2(2a) implies that \( V = \ker(\text{proj}_{\alpha - 2\beta}) \). In particular, we have \( \mathfrak{g}_{-\beta} \subset V \), so Corollary 2.2(2c) implies \( \mathfrak{g}_{-\beta, \mathfrak{u}} \subset W \). Therefore, we have

\[
\begin{align*}
[\mathfrak{g}_{-\beta}, \text{proj}_\beta u] & \subset [\mathfrak{g}_{-\beta}, u + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta})] = [\mathfrak{g}_{-\beta}, u] + [\mathfrak{g}_{-\beta}, \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}]
\subset W + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}) = \mathfrak{h} + Z + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}) \subset \mathfrak{m} + \mathfrak{n} + Z.
\end{align*}
\]

Because \( \text{proj}_{\alpha}[\mathfrak{g}_{-\beta}, \text{proj}_\beta u] = 0 \), we conclude that \( \mathfrak{g}_{-\beta, \mathfrak{u}} \subset \mathfrak{m} + \mathfrak{n} \). This contradicts the fact that \( \mathfrak{m} + \mathfrak{n} \) does not contain nontrivial hyperbolic elements.

**Subsubcase 2.1.2.** Assume \( \text{proj}_\beta u = 0 \). Replacing \( H \) by a conjugate under \( N \), we may assume \( \mathfrak{m} \subset \mathfrak{h}_0 \), so \( \text{proj}_\beta \mathfrak{h} = 0 \).

We have \( u \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta} \), so \( (\text{ad}_u)^2\mathfrak{g} \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta} \) for every \( u \in \mathfrak{u} \). Thus, Corollary 2.2(2d) implies \( W \subset (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}) + \mathfrak{h} \).

We have

\[
\text{proj}_{\beta \mathfrak{g}_{-\beta - \beta}} W \subset \text{proj}_{\beta \mathfrak{g}_{-\beta}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}) + \text{proj}_{\beta \mathfrak{g}_{-\beta}} \mathfrak{h} = 0,
\]

so Corollary 2.2(2c) implies that \( \text{proj}_{\beta \mathfrak{g}_{-\beta}} ((\text{ad}_u) V) = 0 \).

**Subsubsubcase 2.1.2.1.** Assume \( \text{proj}_\alpha u \neq 0 \), for some \( u \in \mathfrak{u} \). From the conclusion of the preceding paragraph, we know that \( \text{proj}_{\beta} ((\text{ad}_u) V) = 0 \). Because \( \text{proj}_{\beta} u = 0 \) and \( \text{proj}_\alpha \neq 0 \), this implies \( \text{proj}_{\alpha - \beta} V = 0 \), so \( V = \ker(\text{proj}_{\alpha - \beta}) \) (see 2.2(2a)). In particular, \( \mathfrak{g}_{-\alpha} \subset V \), so Corollary 2.2(2c) implies

\[
\begin{align*}
[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] & \subset [u + (\mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}), \mathfrak{g}_{-\alpha}] \subset [u, V] + [\mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta}, \mathfrak{g}_{-\alpha}]
\subset W + \mathfrak{g}_\beta \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}.
\end{align*}
\]

This contradicts the fact that \( \mathfrak{m} + \mathfrak{n} \) does not contain nontrivial hyperbolic elements.

**Subsubsubcase 2.1.2.2.** Assume \( \text{proj}_{\alpha + \beta} u \neq 0 \), for some \( u \in \mathfrak{u} \). From Subsubsubcase 2.1.2.1, we may assume \( \text{proj}_\alpha u = 0 \). Because \( 0 = \text{proj}_{\beta \mathfrak{g}_{-\beta}} ((\text{ad}_u) V) \) has codimension \( \leq 1 \) in \( \text{proj}_{\beta \mathfrak{g}_{-\beta}} ((\text{ad}_u) \mathfrak{g}) \) (see 2.2(2a)), which contains the 2-dimensional subspace \( \text{proj}_{\beta \mathfrak{g}_{-\beta}} ([u, \mathfrak{g}_{-\alpha - 2\beta} + \mathfrak{g}_{-\alpha}]) \), we have a contradiction.

**Subsubsubcase 2.1.2.3.** Assume \( u = \mathfrak{g}_{\alpha + 2\beta} \). (This argument is similar to Subsubsubcase 2.1.2.1.) Because \( \text{proj}_{\beta} ((\text{ad}_u) V) = 0 \), we know that \( \text{proj}_{\alpha - \beta} V = 0 \), so \( V = \ker(\text{proj}_{\alpha - \beta}) \) (see 2.2(2a)). In particular, \( \mathfrak{g}_{-\alpha - 2\beta} \subset V \), so Corollary 2.2(2c) implies

\[
[\mathfrak{g}_{\alpha + 2\beta}, \mathfrak{g}_{-\alpha - 2\beta}] \subset [u, V] \subset W \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}.
\]

This contradicts the fact that \( \mathfrak{m} + \mathfrak{n} \) does not contain nontrivial hyperbolic elements.
Subcase 2.2. Assume \( P = P_\alpha \). We may assume there exists \( x \in \mathfrak{h} \), such that \( \text{proj}_{-\alpha} x \neq 0 \) (otherwise, \( H \subset N_G(N) \), so Subcase 2.1 applies). Note that, because \( U \subset \text{unip} P \), we have \( \text{proj}_\alpha u = 0 \).

Subsubcase 2.2.1. Assume \( \text{proj}_{\alpha + \beta} u \neq 0 \). Choose \( u \in u \), such that \( \text{proj}_{\alpha + \beta} u \neq 0 \). Then \([x, u] \in [\mathfrak{h}, u] \subset u\), and \([x, u], u\) is a nonzero element of \( \mathfrak{g}_{\alpha + 2\beta} \), so we see that \( \mathfrak{g}_{\alpha + 2\beta} \subset [u, u] \). Because every unipotent subgroup of \( SO(1, k) \) is abelian, we conclude that \( \text{ad}_g \mathfrak{g}_{\alpha + 2\beta} \) acts trivially on \( \mathfrak{g}/\mathfrak{h} \), which means \( \mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha + 2\beta}] \). This contradicts (3.7).

Subsubcase 2.2.2. Assume \( \text{proj}_{\alpha + \beta} u = 0 \). We may assume, furthermore, that \( \text{proj}_\beta \mathfrak{h} \neq 0 \) (otherwise, by replacing \( H \) with its conjugate under the Weyl reflection corresponding to the root \( \alpha \), we could revert to Subcase 2.1). Then, because \([\mathfrak{h}, u] \subset u\), we must have \( \text{proj}_\beta u = 0 \). Thus, \( u = \mathfrak{g}_{\alpha + 2\beta} \). From Corollary 2.2(2d), we have

\[
W = [\mathfrak{g}, \mathfrak{g}_{\alpha + 2\beta}, \mathfrak{g}_{\alpha + 2\beta}] + \mathfrak{h} = \mathfrak{g}_{\alpha + 2\beta} + \mathfrak{h} \subset u + \mathfrak{h} = \mathfrak{h},
\]

so

\[
W \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta}) \subset \overline{\mathfrak{h}} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta}) - (\overline{\mathfrak{h}} \cap u) \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta}) = u \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta}) = \mathfrak{g}_{\alpha + 2\beta} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta}) = 0.
\]

On the other hand, from Corollary 2.2(2c), we know that \( W \) contains a codimension-one subspace of \([\mathfrak{g}, \mathfrak{g}_{\alpha + 2\beta}]\), so \( W \) contains a codimension-one subspace of \( \mathfrak{g}_\beta + \mathfrak{g}_{\alpha + \beta} \). This is a contradiction.

Subcase 2.3. Assume \( P = P_\beta \). Note that, because \( U \subset \text{unip} P \), we have \( \text{proj}_\beta u = 0 \).

From Corollary 2.2(2d), we have

\[
W = \mathfrak{h} + (\text{ad}_g u)^2 \mathfrak{g} \subset \mathfrak{h} + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha + \beta} + \mathfrak{g}_{\alpha + 2\beta})
\]

\[
= \mathfrak{h} + \text{unip} \mathfrak{p}_\beta \subset (m + u) + \text{unip} \mathfrak{p}_\beta = m + \text{unip} \mathfrak{p}_\beta.
\]

Subsubcase 2.3.1. Assume there is some nonzero \( u \in u \), such that \( \text{proj}_\alpha u = 0 \). Replacing \( H \) by a conjugate (under \( G_{-\beta} \)), we may assume \( \text{proj}_{\alpha + \beta} u \neq 0 \).

Let \( V' = V \cap (\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha - \beta}) \). Because \( V' \) contains a codimension-one subspace of \( \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha - \beta} \) (see Corollary 2.2(2a)), one of the following two subsubcases must apply.

Subsubcase 2.3.1.1. Assume there exists \( v \in V' \), such that \( \text{proj}_{-\alpha - \beta} v = 0 \). From Corollary 2.2(2c), we have \([u, v] \in W\). Then, because \([u, v] \) is a nonzero element of \( \mathfrak{g}_\beta \), we conclude that

\[
0 \neq W \cap \mathfrak{g}_\beta \subset (m + \text{unip} \mathfrak{p}_\beta) \cap \mathfrak{g}_\beta = 0.
\]

This contradicts the fact that \( M \), being compact, has no nontrivial unipotent elements.

Subsubcase 2.3.1.2. Assume \( \text{proj}_{-\alpha - \beta} V' = \mathfrak{g}_{-\alpha - \beta} \). For \( v \in V' \), we have \( \text{proj}_0[u, v] = [\text{proj}_{\alpha + \beta} u, \text{proj}_{-\alpha - \beta} v] \). Thus, there is some \( v \in V' \), such that \( \text{proj}_0[u, v] \) is hyperbolic (and nonzero). On the other hand, from Corollary 2.2(2c), we have \([u, v] \in W = m + \text{unip} \mathfrak{p}_\beta \). This contradicts the fact that \( m \subset \overline{\mathfrak{h}} \) does not contain nonzero hyperbolic elements.

Subsubcase 2.3.2. Assume \( \text{proj}_\alpha u \neq 0 \), for every nonzero \( u \in u \). Fix some nonzero \( u \in u \). Because \( \dim u = 1 \), we must have \( \dim u = 1 \) (so \( u = \text{Rt} u \)). Replacing \( H \) by a conjugate (under \( G_\beta \)), we may assume \( \text{proj}_{\alpha + \beta} u = 0 \). Also, we may assume \( \text{proj}_{\alpha + 2\beta} u \neq 0 \) (otherwise, we could revert to Subcase 2.3.1 by replacing \( H \) with its conjugate under the Weyl reflection corresponding to the root \( \beta \)).
Let $t = \{u, g_{-\alpha} + g_{-\alpha-2\beta}\}$. Because $\langle g_{\alpha}, g_{-\alpha} \rangle$ and $\langle g_{\alpha+2\beta}, g_{-\alpha-2\beta} \rangle$ centralize each other, we see that $t = \{g_{\alpha}, g_{-\alpha}\} + \{g_{\alpha+2\beta}, g_{-\alpha-2\beta}\}$ is a two-dimensional subspace of $g$ consisting entirely of hyperbolic elements. Because $V$ contains a codimension-one subspace of $g_{-\alpha} + g_{-\alpha-2\beta}$ (see Corollary 2.2(2a)), and $[u, V] \subset W$ (see Corollary 2.2(2c)), we see that $W$ contains a codimension-one subspace of $t$, so $W$ contains nontrivial hyperbolic elements. This contradicts the fact that $W \subset m + \text{unip } p_{\beta}$ does not contain nontrivial hyperbolic elements. □

REFERENCES


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