THE LAURENT PHENOMENON

SERGEY FOMIN AND ANDREI ZELEVINSKY

ABSTRACT. A composition of birational maps given by Laurent polynomials need not be given by Laurent polynomials; however, sometimes—quite unexpectedly—it does. We suggest a unified treatment of this phenomenon, which covers a large class of applications. In particular, we settle in the affirmative a conjecture of D. Gale and R. Robinson on integrality of generalized Somos sequences, and prove the Laurent property for several multidimensional recurrences, confirming conjectures by J. Propp, N. Elkies, and M. Kleber.

CONTENTS

1. Introduction 1
2. The Caterpillar Lemma 4
3. One-dimensional recurrences 7
4. Two- and three-dimensional recurrences 10
5. Homogeneous exchange patterns 19
References 21

1. INTRODUCTION

In this paper, we suggest a unified explanation for a number of instances in which certain recursively defined rational functions prove, unexpectedly, to be Laurent polynomials. We begin by presenting several instances of this Laurent phenomenon established in the paper.

Example 1.1. (The cube recurrence) Consider a 3-dimensional array

\[(y_{i,j,k} : (i,j,k) \in \mathcal{H})\]

whose elements satisfy the recurrence

\[y_{i,j,k} = \frac{\alpha y_{i-1,j,k} y_{i,j-1,k-1} + \beta y_{i,j-1,k} y_{i-1,j,k-1} + \gamma y_{i,j,k-1} y_{i-1,j-1,k}}{y_{i-1,j,k-1}}.\]  

Here \(\mathcal{H}\) can be any non-empty subset of \(\mathbb{Z}^3\) satisfying the following conditions:

(1.2) if \((i,j,k) \in \mathcal{H}\), then \((i',j',k') \in \mathcal{H}\) whenever \(i \leq i', j \leq j', k \leq k'\);

(1.3) for any \((i',j',k') \in \mathcal{H}\), the set \(\{(i,j,k) \in \mathcal{H} : i < i', j < j', k < k'\}\) is finite.
Theorem 1.2. Let $H_{\text{init}} = \{(a, b, c) \in \mathcal{H} : (a - 1, b - 1, c - 1) \not\in \mathcal{H}\}$. For every $(i, j, k) \in \mathcal{H}$, the entry $y_{i,j,k}$ is a Laurent polynomial with coefficients in $\mathbb{Z}[\alpha, \beta, \gamma]$ in the initial entries $y_{a,b,c}$, for $(a, b, c) \in H_{\text{init}}$.

The cube recurrence (with $\alpha = \beta = \gamma = 1$) was introduced by James Propp [10], who was also the one to conjecture Laurentness in the case when $\mathcal{H} \subset \mathbb{Z}^3$ is given by the condition $i + j + k \geq 0$; in this case $H_{\text{init}}$ consists of all $(a, b, c) \in \mathcal{H}$ such that $a + b + c \in \{0, 1, 2\}$. Another natural choice of $\mathcal{H}$ was suggested by Michael Kleber: $\mathcal{H} = \mathbb{Z}_{\geq 0}^3$, in which case $H_{\text{init}} = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : abc = 0\}$.

Example 1.3. (The Gale-Robinson sequence) Let $p$, $q$, and $r$ be distinct positive integers, let $n = p + q + r$, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_{k+n} = \frac{\alpha y_{k+p} y_{k+n-p} + \beta y_{k+q} y_{k+n-q} + \gamma y_{k+r} y_{k+n-r}}{y_k}.$$  

(1.4)

David Gale and Raphael Robinson conjectured (see [7] and [8, E15]) that every term of such a sequence is an integer provided $y_0 = \cdots = y_{n-1} = 1$ and $\alpha, \beta, \gamma$ are positive integers. Using Theorem 1.2, we prove the following stronger statement.

Theorem 1.4. As a function of the initial terms $y_0, \ldots, y_{n-1}$, every term of the Gale-Robinson sequence is a Laurent polynomial with coefficients in $\mathbb{Z}[\alpha, \beta, \gamma]$.

We note that the special case $\alpha = \beta = \gamma = 1, p = 1, q = 2, r = 3, n = 6$ (resp., $r = 4, n = 7$) of the recurrence (1.4) is the Somos-6 (resp., Somos-7) recurrence [7].

Example 1.5. (Octahedron recurrence) Consider the 3-dimensional recurrence

$$y_{i,j,k} = \frac{\alpha y_{i+1,j,k-1} y_{i-1,j,k-1} + \beta y_{i,j+1,k-1} y_{i,j-1,k-1}}{y_{i,j,k-2}}$$

(1.5)

for an array $(y_{ijk})_{(i,j,k)\in \mathcal{H}}$ whose indexing set $\mathcal{H}$ is contained in the lattice

$$L = \{(i, j, k) \in \mathbb{Z}^3 : i + j + k \equiv 0 \mod 2\}$$

(1.6)

and satisfies the following analogues of conditions (1.2)–(1.3):

(1.7) if $(i, j, k) \in \mathcal{H}$, then $(i', j', k') \in \mathcal{H}$ whenever $|i' - i| + |j' - j| \leq k' - k$;

(1.8) for any $(i', j', k') \in \mathcal{H}$, the set $\{(i, j, k) \in \mathcal{H} : |i' - i| + |j' - j| \leq k' - k\}$ is finite.

Theorem 1.6. Let $H_{\text{init}} = \{(a, b, c) \in \mathcal{H} : (a, b, c-2) \not\in \mathcal{H}\}$. For every $(i, j, k) \in \mathcal{H}$, the entry $y_{i,j,k}$ is a Laurent polynomial with coefficients in $\mathbb{Z}[\alpha, \beta]$ in the initial entries $y_{a,b,c}$, for $(a, b, c) \in H_{\text{init}}$.

The octahedron recurrence on the half-lattice

$$\mathcal{H} = \{(i, j, k) \in L : k \geq 0\}$$

(1.9)

was studied by W. H. Mills, D. P. Robbins, and H. Rumsey in their pioneering work [9] on the Alternating Sign Matrix Conjecture (cf. [1] and [10, Section 10] for further references); in particular, they proved the special case of Theorem 1.6 for this choice of $\mathcal{H}$.

Example 1.7. (Two-term version of the Gale-Robinson sequence) Let $p$, $q$, and $n$ be positive integers such that $p < q \leq n/2$, and let the sequence $y_0, y_1, \ldots$ satisfy the recurrence

$$y_{k+n} = \frac{\alpha y_{k+p} y_{k+n-p} + \beta y_{k+q} y_{k+n-q}}{y_k}. $$

(1.10)
Using Theorem 1.6, one can prove that this sequence also exhibits the Laurent phenomenon.

**Theorem 1.8.** As a function of the initial terms \( y_0, \ldots, y_{n-1} \), every term \( y_m \) is a Laurent polynomial with coefficients in \( \mathbb{Z}[\alpha, \beta] \).

We note that in the special case \( \alpha = \beta = 1, p = 1, q = 2, n = 5 \) (resp., \( n = 4 \)), (1.10) becomes the Somos-5 (resp., Somos-4) recurrence [7].

The last example of the Laurent phenomenon presented in this section is of a somewhat different kind; it is inspired by [2].

**Example 1.9.** Let \( n \geq 3 \) be an integer, and consider a quadratic form

\[
P(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 + \sum_{i < j} \alpha_{ij} x_i x_j.
\]

Define the rational transformations \( F_1, \ldots, F_n \) by

\[
F_i : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \frac{P_{x_i=0}}{x_i}, x_{i+1}, \ldots, x_n).
\]

**Theorem 1.10.** For any sequence of indices \( i_1, \ldots, i_m \), the composition map \( G = F_{i_1} \circ \cdots \circ F_{i_m} \) is given by

\[
G : x = (x_1, \ldots, x_n) \mapsto (G_1(x), \ldots, G_n(x)),
\]

where \( G_1, \ldots, G_n \) are Laurent polynomials with coefficients in \( \mathbb{Z}[\alpha_{ij} : i < j] \).

This paper is an outgrowth of [6], where we initiated the study of a new class of commutative algebras, called cluster algebras, and established the Laurent phenomenon in that context. Here we prove the theorems stated above, along with a number of related results, using an approach inspired by [6]. The first step is to reformulate the problem in terms of generalized exchange patterns (cf. [6, Definition 2.1]), which consist of clusters and exchanges among them. The clusters are distinguished finite sets of variables, each of the same cardinality \( n \). An exchange operation on a cluster \( x \) replaces a variable \( x \in x \) by a new variable \( x' = \frac{P}{x} \), where \( P \) is a polynomial in the \( n-1 \) variables \( x - \{ x \} \). Each of the above theorems can be restated as saying that any member of the cluster obtained from an initial cluster \( x_0 \) by a particular sequence of exchanges is a Laurent polynomial in the variables from \( x_0 \). Theorem 1.10 is explicitly stated in this way; in the rest of examples above, the rephrasing is less straightforward.

Our main technical tool is “The Caterpillar Lemma” (Theorem 2.1), which establishes the Laurent phenomenon for a particular class of exchange patterns (see Figure 1). This is a modification of the namesake statement [6, Theorem 3.2], and its proof closely follows the argument in [6]. (We note that none of the two statements is a formal consequence of another.)

In most applications, including Theorems 1.2 and 1.6 above, the “caterpillar” patterns to which Theorem 2.1 applies, are not manifestly present within the original setup. Thus, we first complete it by creating additional clusters and exchanges, and then apply the Caterpillar Lemma.

The paper is organized as follows. The Caterpillar Lemma is proved in Section 2. Subsequent sections contain its applications. In particular, Theorems 1.2, 1.4, 1.6, and 1.8 are proved in Section 4, while Theorem 1.10 is proved in Section 5. Other instances of the Laurent phenomenon treated in this paper include generalizations of each of the following: Somos-4 sequences (Example 3.3), Elkies's
"knight recurrence" (Example 4.1), frieze patterns (Example 4.3) and number walls (Example 4.4).

We conjecture that in all instances of the Laurent phenomenon established in this paper, the Laurent polynomials in question have nonnegative integer coefficients. In other contexts, similar nonnegativity conjectures were made earlier in [4, 5, 6].

Acknowledgments. We thank Jim Propp for introducing us to a number of beautiful examples of the Laurent phenomenon, and for very helpful comments on the first draft of the paper. In particular, it was he who showed us how to deduce Theorem 1.8 from Theorem 1.6.

This paper was completed during our stay at the Isaac Newton Institute for Mathematical Sciences (Cambridge, UK), whose support and hospitality are gratefully acknowledged.

2. The Caterpillar Lemma

Let us fix an integer \( n \geq 2 \), and let \( T \) be a tree whose edges are labeled by the elements of the set \( \{1, 2, \ldots, n\} \), so that the edges emanating from each vertex receive different labels. By a common abuse of notation, we will sometimes denote by \( T \) the set of the graph’s vertices. We will write \( t \xrightarrow{k} t' \) if vertices \( t, t' \in T \) are joined by an edge labeled by \( k \).

From now on, let \( \mathbb{A} \) be a unique factorization domain (the ring of integers \( \mathbb{Z} \) or a suitable polynomial ring would suffice for most applications). Assume that a nonzero polynomial \( P \in \mathbb{A}[x_1, \ldots, x_n] \), not depending on \( x_k \), is associated with every edge \( t \xrightarrow{k} t' \) in \( T \). We will write \( t \xrightarrow{p} t' \) or \( t \xrightarrow{k/p} t' \), and call \( P \) the exchange polynomial associated with the given edge. The entire collection of these polynomials is called a generalized exchange pattern on \( T \). (In [6], we introduced a much narrower notion of an exchange pattern; hence the terminology.)

We fix a root vertex \( t_0 \in T \), and introduce the initial cluster \( x(t_0) \) of \( n \) independent variables \( x_1(t_0), \ldots, x_n(t_0) \). To each vertex \( t \in T \), we then associate a cluster \( x(t) \) consisting of \( n \) elements \( x_1(t), \ldots, x_n(t) \) of the field of rational functions \( \mathbb{A}(x_1(t_0), \ldots, x_n(t_0)) \). The elements \( x_i(t) \) are uniquely determined by the following exchange relations, for every edge \( t \xrightarrow{k/p} t' \):

\[
\begin{align*}
\text{(2.1)} & \quad x_i(t) = x_i(t') \quad \text{for any } i \neq k; \\
\text{(2.2)} & \quad x_k(t) x_k(t') = P(x(t)).
\end{align*}
\]

(One can recursively compute the \( x_i(t) \)'s, moving away from the root. Since the exchange polynomial \( P \) does not depend on \( x_k \), the exchange relation (2.2) does not change if we apply it in the opposite direction.)

We next introduce a special class of “caterpillar” patterns, and state conditions on their exchange polynomials that will imply Laurentness.

For \( m \geq 1 \), let \( T_{n,m} \) be the tree of the form shown in Figure 1. The tree \( T_{n,m} \) has \( m \) vertices of degree \( n \) in its “spine” and \( m(n-2) + 2 \) vertices of degree 1. We label every edge of the tree by an element of \( [n] \), so that the \( n \) edges emanating from each vertex on the spine receive different labels. We let the root \( t_0 \) be a vertex in \( T_{n,m} \) that does not belong to the spine but is connected to one of its ends. This gives rise to the orientation of the spine, with all the arrows pointing away from \( t_0 \) (see Figure 1). We assign a nonzero exchange polynomial
$P \in A[x_1, \ldots, x_n]$ to every edge $t \rightarrow t'$ of $T_{n,m}$, thus obtaining an exchange pattern.

For a rational function $F = F(x,y,\ldots)$, we will denote by $F|_{x \rightarrow g(x,y,\ldots)}$ the result of substituting $g(x,y,\ldots)$ for $x$ into $F$. To illustrate, if $F(x,y) = xy$, then $F|_{x \rightarrow \frac{y}{x}} = \frac{y^2}{x}$.

**Theorem 2.1.** (The Caterpillar Lemma) Assume that a generalized exchange pattern on $T_{n,m}$ satisfies the following conditions:

(2.3) For any edge $\bullet \xrightarrow{\frac{k}{p}} \bullet$, the polynomial $P$ does not depend on $x_k$, and is not divisible by any $x_i$, $i \in [n]$.

(2.4) For any two edges $\bullet \xrightarrow{\frac{i}{Q}} \bullet \xrightarrow{\frac{j}{Q}} \bullet$, the polynomials $P$ and $Q_0 = Q|_{x_i = 0}$ are coprime elements of $A[x_1, \ldots, x_n]$.

(2.5) For any three edges $\bullet \xrightarrow{\frac{i}{Q}} \bullet \xrightarrow{\frac{j}{Q}} \bullet \xrightarrow{\frac{i}{R}} \bullet$ labeled $i,j,i$, we have

$$L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow Q_0^i},$$

where $b$ is a nonnegative integer, $Q_0 = Q|_{x_i = 0}$, and $L$ is a Laurent monomial whose coefficient lies in $A$ and is coprime with $P$.

Then each element $x_i(t)$, for $i \in [n]$, $t \in T_{n,m}$, is a Laurent polynomial in $x_1(t_0), \ldots, x_n(t_0)$, with coefficients in $A$.

(Note the orientation of edges in (2.4)-(2.5).)

**Proof.** Our argument is essentially the same as in [6, Theorem 3.2]. For $t \in T_{n,m}$, let

$$L(t) = A[x_1(t)^{\pm 1}, \ldots, x_n(t)^{\pm 1}]$$
de note the Laurent polynomial ring in the cluster $x(t)$ with coefficients in $A$. We view each $L(t)$ as a subring of the ambient field of rational functions $A(x(t_0))$.

In this notation, our goal is to show that every cluster $x(t)$ is contained in $L(t_0)$. We abbreviate $L_0 = L(t_0)$. Note that $L_0$ is a unique factorization domain, so any two elements $x,y \in L_0$ have a well-defined greatest common divisor $gcd(x,y)$, which is an element of $L_0$ defined up to a multiple from the group $L_0^\times$ of invertible elements in $L_0$; the group $L_0^\times$ consists of Laurent monomials in $x_1(t_0), \ldots, x_n(t_0)$ whose coefficient belongs to $A^\times$, the group of invertible elements of $A$.

To prove that all $x(t)$ are contained in $L_0$, we proceed by induction on $m$, the size of the spine. The claim is trivial for $m = 1$, so let us assume that $m > 2$, and furthermore assume that our statement is true for all "caterpillars" with smaller
spine. It is thus enough to prove that $x(t_{\text{head}}) \subset L_0$, where $t_{\text{head}}$ is one of the vertices most distant from $t_0$ (see Figure 1).

We assume that the path from $t_0$ to $t_{\text{head}}$ starts with the following two edges: $t_0 \xrightarrow{\frac{i}{P}} t_1 \xrightarrow{\frac{j}{Q}} t_2$. Let $t_3 \in T_{n,m}$ be the vertex such that $t_2 \xrightarrow{\frac{i}{R}} t_3$. The following lemma plays a crucial role in our proof.

**Lemma 2.2.** The clusters $x(t_1)$, $x(t_2)$, and $x(t_3)$ are contained in $L_0$. Furthermore, $\gcd(x_i(t_3), x_i(t_1)) = \gcd(x_j(t_2), x_i(t_1)) = 1$.

**Proof.** The only element in the clusters $x(t_1)$, $x(t_2)$, and $x(t_3)$ whose inclusion in $L_0$ is not immediate from (2.1)-(2.2) is $x_i(t_3)$. To simplify the notation, let us denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, and $v = x_i(t_3)$, so that these variables appear in the clusters at $t_0, \ldots, t_3$, as shown below:

$$
\begin{array}{c}
\bullet\quad y, z \quad \bullet \quad x, u \quad \bullet \quad v, u \\
\xrightarrow{\frac{i}{P}} \quad \xrightarrow{\frac{j}{Q}} \quad \xrightarrow{\frac{i}{R}} \\
t_0 \quad t_1 \quad t_2 \quad t_3
\end{array}
$$

Note that the variables $x_k$, for $k \not\in \{i, j\}$, do not change as we move among the four clusters under consideration. The lemma is then restated as saying that

\begin{align*}
(2.6) & \quad v \in L_0; \\
(2.7) & \quad \gcd(z, u) = 1; \\
(2.8) & \quad \gcd(z, v) = 1. 
\end{align*}

Another notational convention will be based on the fact that each of the polynomials $P, Q, R$ has a distinguished variable on which it depends, namely $x_j$ for $P$ and $R$, and $x_i$ for $Q$. (In view of (2.3), $P$ and $R$ do not depend on $x_i$, while $Q$ does not depend on $x_j$.) With this in mind, we will routinely write $P$, $Q$, and $R$ as polynomials in one (distinguished) variable. For example, we rewrite the formula in (2.5) as

\begin{equation}
R\left(\frac{Q(0)}{y}\right) = L(y)Q(0)P(y),
\end{equation}

where we denote $L(y) = L|_{x_j=y}$. In the same spirit, the notation $Q', R'$, etc., will refer to the partial derivatives with respect to the distinguished variable.

We will prove the statements (2.6), (2.7), and (2.8) one by one, in this order. We have:

$$
z = \frac{P(y)}{x};
$$

$$
u = \frac{Q(x)}{y} = \frac{Q}{y} \left(\frac{P(y)}{x}\right);$$

$$
u = \frac{R(u)}{z} = \frac{R}{z} \left(\frac{Q(x)}{y}\right) = \frac{R}{z} \left(\frac{Q(x)}{y}\right) - \frac{R}{z} \left(\frac{Q(0)}{y}\right) + \frac{R}{z} \left(\frac{Q(0)}{y}\right).$$

Since

$$
\frac{R}{z} \left(\frac{Q(0)}{y}\right) - \frac{R}{z} \left(\frac{Q(0)}{y}\right) \in L_0
$$
and

\[ \frac{R \left( \frac{Q(0)}{y} \right)}{z} = \frac{L(y)Q(0)^bP(y)}{z} = L(y)Q(0)^b x \in \mathcal{L}_0, \]

(2.6) follows.

We next prove (2.7). We have

\[ u = \frac{Q(x)}{y} = \frac{Q(0)}{y} \mod z. \]

Since \( x \) and \( y \) are invertible in \( \mathcal{L}_0 \), we conclude that \( \gcd(z, u) = \gcd(P(y), Q(0)) = 1 \) (using (2.4)).

It remains to prove (2.8). Let

\[ f(z) = R \left( \frac{Q(z)}{y} \right). \]

Then

\[ v = \frac{f(z) - f(0)}{z} = L(y)Q(0)^bx. \]

Working \( \mod z \), we obtain:

\[ \frac{f(z) - f(0)}{z} \equiv f'(0) = R' \left( \frac{Q(0)}{y} \right) \cdot \frac{Q'(0)}{y}. \]

Hence

\[ v \equiv R' \left( \frac{Q(0)}{y} \right) \cdot \frac{Q'(0)}{y} + L(y)Q(0)^bx \mod z. \]

Note that the right-hand side is a polynomial of degree 1 in \( z \) whose coefficients are Laurent polynomials in the rest of the variables of the cluster \( x(t_0) \). Thus (2.8) follows from \( \gcd(L(y)Q(0)^b, P(y)) = 1 \), which is a consequence of (2.4)–(2.5). \( \Box \)

We can now complete the proof of Theorem 2.1. We need to show that any variable \( X = x_k(t_{\text{head}}) \) belongs to \( \mathcal{L}_0 \). Since both \( t_1 \) and \( t_3 \) are closer to \( t_{\text{head}} \) than \( t_0 \), we can use the inductive assumption to conclude that \( X \) belongs to both \( \mathcal{L}(t_1) \) and \( \mathcal{L}(t_3) \). Since \( X \in \mathcal{L}(t_1) \), it follows from (2.1) that \( X \) can be written as \( X = f/x(t_1)^a \) for some \( f \in \mathcal{L}_0 \) and \( a \in \mathbb{Z}_{\geq 0} \). On the other hand, since \( X \in \mathcal{L}(t_3) \), it follows from (2.1) and from the inclusion \( x(t_3) \in \mathcal{L}_0 \) provided by Lemma 2.2 that \( X \) has the form \( X = g/x(t_3)^b \cdot x(t_3)^c \) for some \( g \in \mathcal{L}_0 \) and some \( b, c \in \mathbb{Z}_{\geq 0} \). The inclusion \( X \in \mathcal{L}_0 \) now follows from the fact that, by the last statement in Lemma 2.2, the denominators in the two obtained expressions for \( X \) are coprime in \( \mathcal{L}_0 \). \( \Box \)

3. ONE-DIMENSIONAL RECURRENCES

In this section, we apply Theorem 2.1 to study the Laurent phenomenon for sequences \( y_0, y_1, \ldots \) given by recursions of the form

\[ y_{m+n} y_m = F(y_{m+1}, \ldots, y_{m+n-1}), \tag{3.1} \]

where \( F \in \mathbb{N}[x_1, \ldots, x_{n-1}] \).

For an integer \( m \), let \( (m) \) denote the unique element of \( [n] = \{1, \ldots, n\} \) satisfying \( m \equiv (m) \mod n \). We define the polynomials \( F_1, \ldots, F_n \in \mathbb{N}[x_1, \ldots, x_n] \) by

\[ F_m = F(x_{(m+1)}, x_{(m+2)}, \ldots, x_{(m-1)}); \tag{3.2} \]
thus $F_m$ does not depend on the variable $z_m$. We introduce the infinite "cyclic exchange pattern"

$$t_0 \quad \overset{(0)}{F(0)} \quad t_1 \quad \overset{(1)}{F(1)} \quad t_2 \quad \overset{(2)}{F(2)} \quad t_3 \quad \overset{(3)}{F(3)} \quad t_4 \quad \cdots,$$

and let the cluster at each point $t_m$ consist of the variables $y_m, \ldots, y_{m+n-1}$, labeled within the cluster according to the rule $y_s = x_{(s)}(t_m)$. Then equations (3.1) become the exchange relations associated with this pattern.

To illustrate, let $n = 4$. Then the clusters will look like this:

$$t_0 \quad \overset{4}{t_1} \quad \overset{1}{t_2} \quad \overset{2}{t_3} \quad \overset{3}{t_4} \quad \cdots$$

In order to include this situation into the setup of Section 2 (cf. Figure 1), we create an infinite "caterpillar tree" whose "spine" is formed by the vertices $t_m, m > 0$. We thus attach the missing $n-2$ "legs" with labels in $[n] - \{m-1, m\}$, to each vertex $t_m$.

Our next goal is to state conditions on the polynomial $F$ which make it possible to assign exchange polynomials satisfying (2.3)-(2.5) to the newly constructed legs.

The first requirement (cf. (2.3)) is:

$$\text{(3.4) The polynomial } F \text{ is not divisible by any } x_i, i \in [n-1].$$

For $m \in [n-1]$, we set

$$\text{(3.5) } Q_m = F_m |_{x_n \to 0} = F(x_{m+1}, \ldots, x_{n-1}, 0, x_1, \ldots, x_{m-1}).$$

Our second requirement is

$$\text{(3.6) Each } Q_m \text{ is an irreducible element of } \mathbb{A}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}].$$

To state our most substantial requirement, we recursively define a sequence of polynomials $G_{n-1}, \ldots, G_1, G_0 \in \mathbb{A}[x_1, \ldots, x_{n-1}]$; more precisely, each $G_m$ will be defined up to a multiple in $\mathbb{A}^\times$. (Later, $G_1, \ldots, G_{n-2}$ will become the exchange polynomials assigned to the "legs" of the caterpillar labeled by $n = (0)$; see Figure 2.)

We set $G_{n-1} = F$, and obtain each $G_{m-1}$ from $G_m$, as follows. Let

$$\tilde{G}_{m-1} = G_m |_{x_n \to x_m}.$$
Let $L$ be a Laurent monomial in $x_1, \ldots, x_{n-1}$, with coefficient in $A$, such that

\begin{equation}
G_{m-1} = \frac{\tilde{G}_{m-1}}{L},
\end{equation}

is a polynomial in $A[x_1, \ldots, x_{n-1}]$ not divisible by any $x_i$ or by any non-invertible scalar in $A$. Such an $L$ is unique up to a multiple in $A^\times$. Finally, we set

\begin{equation}
G_{m-1} = \frac{\tilde{G}_{m-1}}{Q^b_m},
\end{equation}

where $Q^b_m$ is the maximal power of $Q_m$ that divides $\tilde{G}_{m-1}$. With all this notation, our final requirement is:

\begin{equation}
G_0 = F.
\end{equation}

**Theorem 3.1.** Let $F$ be a polynomial in the variables $x_1, \ldots, x_{n-1}$ with coefficients in a unique factorization domain $A$ satisfying conditions (3.4), (3.6), and (3.10). Then every term of the sequence $(y_i)$ defined by the recurrence

\begin{equation}
y_{m+n} = \frac{F(y_{m+1}, \ldots, y_{m+n-1})}{y_m}
\end{equation}

is a Laurent polynomial in the initial $n$ terms, with coefficients in $A$.

**Proof.** To prove the Laurentness of some $y_N$, we will apply Theorem 2.1 to the caterpillar tree constructed as follows. We set $t_{\text{head}} = t_{N-n+1}$, this corresponds to the first cluster containing $y_N$. As a path from $t_0$ to $t_{\text{head}}$, we take a finite segment of (3.3):

\begin{equation}
t_0 \rightarrow (0) \rightarrow F_{(0)} \rightarrow (1) \rightarrow F_{(1)} \rightarrow (2) \rightarrow F_{(2)} \rightarrow \cdots \rightarrow \frac{N-1}{F_{(N-1)}} \rightarrow \frac{N}{F_{(N)}} \rightarrow t_{N-n+1}.
\end{equation}

We then define the exchange polynomial $G_{j,k-1}$ associated with the leg labeled $j$ attached to a vertex $t_k$ on the spine (see Figure 3) by

\begin{equation}
G_{j,k-1} = G_{k-j-1}(x_{j+1}, \ldots, x_n, x_1, \ldots, x_{j-1}),
\end{equation}

where in the right-hand side, we use the polynomials $G_1, \ldots, G_{n-2}$ constructed in (3.7)–(3.9) above.

![Figure 3](image)

It remains to verify that this exchange pattern satisfies (2.3), (2.4), and (2.5). Condition (2.3) for the edges appearing in (3.11) is immediate from (3.4), while for the rest of the edges, it follows from the definition of $\tilde{G}_{m-1}$ in (3.8).

Turning to (2.4), we first note that we may assume $i = (0) = n$ (otherwise apply a cyclic shift of indices). Under this assumption, we can identify the polynomials $P$ and $Q_n$ in (2.4) with the polynomials $G_{m-1}$ and $Q_m$ in (3.9), for some value of $m$. (The special case of $P$ attached to one of the edges in (3.11) corresponds to $m = 1$, ...
and its validity requires \((3.10)\). Then the condition \(\gcd(G_{m-1}, Q_m) = 1\) follows from \((3.6)\) and the choice of the exponent \(b\) in \((3.9)\).

Finally, \((2.5)\) is ensured by the construction \((3.7)-(3.9)\), which was designed expressly for this purpose. As before, the special case of \(P\) attached to one of the edges in \((3.11)\) holds due to \((3.10)\).

In the rest of this section, we give a few applications of Theorem 3.1. In all of them, conditions \((3.4)\) and \((3.6)\) are immediate, so we concentrate on the verification of \((3.10)\).

**Example 3.2.** Let \(a\) and \(b\) be positive integers, and let the sequence \(y_0, y_1, \ldots\) satisfy the recurrence

\[
y_k = \frac{y_{k-2}^a y_{k-1}^b + 1}{y_{k-3}}.
\]

We claim that every term of the sequence is a Laurent polynomial over \(\mathbb{Z}\) in \(y_0, y_1,\) and \(y_2\). To prove this, we set \(n = 3\) and construct the polynomials \(G_2, G_1,\) and \(G_0\) using \((3.7)-(3.9)\). Initializing \(G_2 = F(x_0, x_2) = x_0^2 x_2^2 + 1,\) we obtain:

\[
Q_2 = F(0, x_1) = 1, \quad \tilde{G}_1 = F_{|x_1 \leftarrow x_2} = x_1^a x_2^b + 1, \quad G_1 = \tilde{G}_1 = x_1^a + x_2^b,
\]

\[
Q_1 = F(x_2, 0) = 1, \quad \tilde{G}_0 = G_1_{|x_1 \leftarrow x_2} = x_1^{-a} + x_2^b, \quad G_0 = \tilde{G}_0 = 1 + x_1^a x_2^2 = F,
\]

as desired.

**Example 3.3.** (Generalized Somos-4 sequence) Let \(a, b,\) and \(c\) be positive integers, and let the sequence \(y_0, y_1, \ldots\) satisfy the recurrence

\[
y_k = \frac{y_{k-3}^a y_{k-1}^b + y_{k-2}^b}{y_{k-4}}.
\]

(The Somos-4 sequence \([7]\), introduced by Michael Somos, is the special case \(a = c = 1, b = 2\).) Again, each \(y_i\) is a Laurent polynomial in the initial terms \(y_0, y_1,\) \(y_2,\) and \(y_3\). To prove this, we set \(n = 4\) and compute \(G_3, \ldots, G_0\) using \((3.7)-(3.9)\) and beginning with \(G_3 = F = x_1^2 x_2^2 + x_2^4.

\[
Q_3 = F(0, x_1, x_2) = x_1^2, \quad G_3_{|x_3 \leftarrow x_2} = x_1^{a+bc} x_3^{-c} + x_2^b, \quad G_2 = x_1^{a+bc} + x_2^b x_3^c,
\]

\[
Q_2 = F(x_3, 0, x_1) = x_2^2 x_3^2, \quad G_2_{|x_3 \leftarrow x_2} = x_2^{a+bc} x_3^{-b-c} x_1^{a+bc}, \quad G_1 = x_1^a x_2^b + x_3^{a+b-c},
\]

\[
Q_1 = F(x_2, x_3, 0) = x_3^2, \quad G_1_{|x_3 \leftarrow x_2} = x_1^{-a} x_2^b x_3^a + x_3^{a+b-c}, \quad G_0 = x_1^4 + x_2^2 x_3^2 = F,
\]

and the claim follows.

**Remark 3.4.** The Laurent phenomena in Theorems \(1.4\) and \(1.8\) can also be proved by applying Theorem 3.1: in the former (resp., latter) case, the polynomial \(F\) is given by \(F = \alpha x_0 x_{n-1} + \beta x_1 x_{n-2} + \gamma x_2 x_{n-3}\) (resp., \(F = \alpha x_0 x_{n-1} + \beta x_1 x_{n-2}\)). The proofs are straightforward but rather long. Shorter proofs, based on J. Propp’s idea of viewing one-dimensional recursions as “projections” of multi-dimensional ones, are given in Section 4 below.

### 4. TWO- AND THREE-DIMENSIONAL RECURRENCES

In this section, we use the strategy of Section 3 to establish the Laurent phenomenon for several recurrences involving two- and three-dimensional arrays. Our first example generalizes a construction (and the corresponding Laurentness conjecture) suggested by Noam Elkies and communicated by James Propp. Even though the
Laurent phenomenon in this example can be deduced from Theorem 1.6, we choose to give a self-contained treatment, for the sake of exposition.

**Example 4.1.** (*The knight recurrence*) Consider a two-dimensional array \((y_{ij})_{i,j \geq 0}\) whose entries satisfy the recurrence

\[
y_{i,j}y_{i-2,j-1} = \alpha y_{i,j-1}y_{i-2,j} + \beta y_{i-1,j}y_{i-1,j-1}.
\]

We will prove that every \(y_{ij}\) is a Laurent polynomial in the initial entries

\[
Y_{\text{init}} = \{y_{ab} : a < 2 \text{ or } b < 1\},
\]

with coefficients in the ring \(A = \mathbb{Z}[\alpha, \beta]\).

We will refer to \(Y_{\text{init}}\) as the initial cluster, even though it is an infinite set. Notice, however, that each individual \(y_{ij}\) only depends on finitely many variables \(\{y_{ab} \in Y_{\text{init}} : a \leq i, b \leq j\}\).

Similarly to Section 3, we will use the exchange relations (4.1) to create a sequence of clusters satisfying the Caterpillar Lemma (Theorem 2.1).

This is done in the following way. Let us denote by \(\mathcal{H} = \mathbb{Z}_{\geq 0}^2\) the underlying set of indices; for \(h = (i, j) \in \mathcal{H}\), we will write \(y_h = y_{ij}\). The variables of the initial cluster have labels in the set

\[
H_{\text{init}} = \{(i, j) \in \mathcal{H} : i < 2 \text{ or } j < 1\}.
\]

In Figure 4, the elements of \(H_{\text{init}}\) are marked by *'s.

![Figure 4. The initial cluster and the equivalence classes \((h)\)](image)

We introduce the product partial order on \(\mathcal{H}:

\[
(i_1, j_1) \leq (i_2, j_2) \iff (i_1 \leq i_2) \text{ and } (j_1 \leq j_2).
\]

For an element \(h = (i, j) \in \mathcal{H} - H_{\text{init}},\) let us denote \(h^- = (i - 2, j - 1);\) in this notation, the exchange relation (4.1) expresses the product \(y_h \cdot y_{h^-}\) as a polynomial in the variables \(y_{h'},\) for \(h^- < h' < h\).

We write \(h^- \sim h,\) and extend this to an equivalence relation \(\sim\) on \(\mathcal{H} .\) The equivalence class of \(h\) is denoted by \(\langle h \rangle.\) These classes are shown as slanted lines in Figure 4. All our exchange polynomials will belong to the ring \(A[x_a : a \in \mathcal{H}/\sim].\)

Note that \(H_{\text{init}}\) has exactly one representative from each equivalence class. We will now construct a sequence of subsets \(H_0 = H_{\text{init}}, H_1, H_2, \ldots,\) each having this
property, using the following recursive rule. Let us fix a particular linear extension of the partial order (4.2), say,

\((i_1, j_1) \leq (i_2, j_2) \iff (i_1 + j_1 < i_2 + j_2) \text{ or } (i_1 + j_1 = i_2 + j_2 \text{ and } i_1 \leq i_2)\).

Restricting this linear ordering to the complement \(\mathcal{H} - H_{\text{init}}\) of the initial cluster, we obtain a numbering of the elements of this complement by positive integers:

\[ h_0 = (2, 1), \ h_1 = (2, 2), \ h_2 = (3, 1), \ h_3 = (2, 3), \ h_4 = (3, 2), \ h_5 = (4, 1), \ h_6 = (2, 4), \ h_7 = (3, 2), \ h_8 = (4, 2), \]

and so on. Having constructed \(H_m\), we let \(H_{m+1} = H_m \cup \{h_m\} - \{h_m\}\). To illustrate, the set \(H_0\) is shown in Figure 5.

\[\begin{array}{|c|}
\hline
0 & 0 \\
\hline
\end{array}\]

**Figure 5.** Indexing set \(H_0\)

We next create the infinite exchange pattern

\[
(4.3) \quad t_0 \xrightarrow{P(h_0)} t_1 \xrightarrow{P(h_1)} t_2 \xrightarrow{P(h_2)} t_3 \xrightarrow{P(h_3)} t_4 \xrightarrow{\cdots}.
\]

(cf. (3.3)) The cluster at each point \(t_m\) is given by \(x(t_m) = \{y_h : h \in H_m\}\); as before, each cluster variable \(y_h\) corresponds to the variable \(x(h)\). The exchange polynomial \(P(h)\) for an edge \(\bullet \xrightarrow{(h)} \bullet\) with \(h = (i, j)\) is given by

\[
(4.4) \quad P(h) = \alpha x_{(i, j-1)} x_{(i-2, j)} + \beta x_{(i-1, j)} x_{(i-1, j-1)}.
\]

Then equations (4.1) become the exchange relations associated with this pattern.

To establish the Laurent phenomenon, we will complete the caterpillar pattern by attaching "legs" to each vertex \(t_m\) and assigning exchange polynomials to these legs so that the appropriate analogues of conditions (3.4), (3.6) and (3.10) are satisfied. In order to do so, we now work over the polynomial ring \(\mathbb{A}[x_a : a \in \mathcal{H}/\sim]\) in infinitely many indeterminates, the number of legs attached to every vertex \(t_m\) will also be infinite (one for every label a different from \(h_{m-1}\) and \(h_m\)). This will not matter much for our argument though: to prove the Laurentness for any \(y_{h_m}\), we will simply restrict our attention to the finite part of the infinite caterpillar tree lying between \(t_0\) and \(t_{\text{head}} = t_{m+1}\), and to the legs labeled by \(h_k\) for \(0 \leq k \leq m\).

The role of conditions (3.4) and (3.6) is now played by the observation that each exchange polynomial \(P(h)\) is not divisible by any variable \(x_a\), and furthermore every specialization \(P(h)|_{x_a=0}\) is an irreducible element of the Laurent polynomial ring.
To formulate the analogue of (3.10), let us fix an equivalence class $a \in \mathcal{H}/\sim$ and concentrate on defining the exchange polynomials for the legs labeled by $a$ and attached to the vertices squeezed between two consecutive occurrences of the label $a$ on the spine:

\[
(4.5) \quad \bullet \xrightarrow{a} \frac{a_1}{P_a} \bullet \xrightarrow{a_2} \cdots \xrightarrow{a_{N-3}} \frac{a_{N-1}}{P_a} \bullet.
\]

We note that the labels $a_1, \ldots, a_{N-1} \in \mathcal{H}/\sim$ appearing on the spine between these two occurrences of $a$ are distinct. For $m = N - 2, N - 3, \ldots, 1$, we denote by $G_m$ the exchange polynomial to be associated with the $a$-labeled leg attached between the edges labeled $a_m$ and $a_{m+1}$ (cf. (4.5)).

The polynomials $G_m$ are defined with the help of a recursive procedure analogous to (3.7)–(3.9). We initialize $G_{N-1} = P_a$, and obtain each $G_{m-1}$ from $G_m$, as follows. The step (3.7) is replaced by

\[
G_{m-1} = G_m \bigg|_{x_{a_m} = \frac{a_m}{x_{a_m}}},
\]

with

\[
Q_m = P_{a_m} \bigg|_{x_a = 0}.
\]

We then compute $G_{m-1}$ and $G_m$ exactly as in (3.8)–(3.9). By the argument given in the proof of Theorem 3.1, the equality $G_0 = P_a$ would imply the desired Laurentness (cf. (3.10)).

To simplify computations, we denote the equivalence classes "surrounding" $a$, as shown below:

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

\[
\cdots \quad q \quad p \quad f \quad c \quad a \quad \cdots
\]

\[
\cdots \quad f \quad c \quad a \quad e \quad b \quad \cdots
\]

\[
\cdots \quad a \quad e \quad b \quad g \quad d \quad \cdots
\]

\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]

In other words, if $a = (i, j)$, then $b = ((i, j - 1))$, $c = ((i - 1, j))$, etc. With this notation, we can redraw the pattern (4.5) as follows:

\[
(4.8) \quad \bullet \xrightarrow{a} \bullet \xrightarrow{b} \cdots \xrightarrow{f} \bullet \xrightarrow{e} \bullet \xrightarrow{d} \cdots \xrightarrow{c} \bullet \xrightarrow{b} \cdots \bullet \xrightarrow{a} \bullet,
\]

for appropriate values of $k$ and $\ell$.

We will call a value of $m$ essential if $G_{m-1} \neq G_m$. We are going to see that the essential values of $m$ are those for which $a_m \in \{b, c, e, f\}$ in the notation of (4.8), these values are $\ell + 1$, $\ell$, $k + 1$, and $k$.

We initialize $G_{N-1} = P_a = \alpha x_b x_f + \beta x_c x_e$. The values of $m$ in the interval $\ell < m < N$ are not essential since the variable $x_{a_m}$ does not enter $P_a$, which is furthermore not divisible by $Q_m$ (because the latter involves variables absent in $P_a$).
The first essential value is $m = \ell + 1$, with $a_m = b$:

$$Q_{\ell+1} = P_b|_{x_{\ell+1} = 0} = (a x_a x_{d} + \beta x_{c} x_{g})|_{x_{\ell+1} = 0} = \beta x_{c} x_{g},$$

$$\tilde{G}_{\ell} = P_a|_{x_{\ell+1} = 0} = \alpha x_{a} x_{f} + \beta x_{c} x_{e},$$

$$G_\ell = a x_{f} x_{f} + \alpha x_{b} x_{b}.$$

Step $m = \ell$ (here $a_m = c$):

$$Q_{\ell} = P_c|_{x_{\ell} = 0} = (a x_{c} x_{p} + \beta x_{a} x_{f})|_{x_{\ell} = 0} = \alpha x_{a} x_{p},$$

$$\tilde{G}_{\ell-1} = G_{\ell}|_{x_{\ell} = 0} = \alpha x_{a} x_{f} + x_{b} x_{c} x_{e},$$

$$G_{\ell-1} = x_{c} x_{g} x_{f} + x_{b} x_{c} x_{p}.$$

Notice that $G_{\ell-1}$ does not involve $x_d$, so the value $m = k + 2$ is not essential, as are the rest of the values in the interval $k + 1 < m < \ell$.

Step $m = k + 1$, with $a_m = a$:

$$Q_{k+1} = P_c|_{x_{k+1} = 0} = (a x_{c} x_{g} + \beta x_{a} x_{b})|_{x_{k+1} = 0} = \alpha x_{c} x_{g},$$

$$\tilde{G}_{k} = x_{c} x_{g} x_{f} + x_{b} x_{p} x_{c},$$

$$G_{k} = x_{f} x_{c} + \alpha x_{b} x_{p}.$$

Step $m = k$, with $a_m = f$:

$$Q_{k} = P_f|_{x_{k} = 0} = (a x_{f} x_{p} + \beta x_{c} x_{p})|_{x_{k} = 0} = \beta x_{c} x_{p},$$

$$\tilde{G}_{k-1} = \beta x_{c} x_{e} + \alpha x_{b} x_{p},$$

$$G_{k-1} = \beta x_{c} x_{e} + \alpha x_{b} x_{f}.$$

The values of $m$ in the interval $0 < m < k$ are not essential since none of the corresponding variables $x_{a_m}$ appears in $G_{k-1}$; in particular, $m = 1$ is not essential, since $G_{k-1}$ does not involve $x_{g}$. Hence

$$G_{0} = G_{k-1} = \beta x_{c} x_{e} + \alpha x_{b} x_{f} = P_a,$$

as desired. The Laurentness is proved.

**Remark 4.2.** The Laurent phenomenon for the recurrence (4.1) actually holds in greater generality. Specifically, one can replace $\mathcal{H}$ by any subset of $\mathbb{Z}^2$ which satisfies the following analogues of conditions (1.2)–(1.3) and (1.7)–(1.8):

(4.9) if $h \in \mathcal{H}$, then $h' \in \mathcal{H}$ whenever $h \leq h'$;

(4.10) for any $h' \in \mathcal{H}$, the set $\{ h \in \mathcal{H} : h \leq h' \}$ is finite.

Then take $H_{\text{init}} = \{ h \in \mathcal{H} : h^- \notin \mathcal{H} \}$.

The proof of Laurentness only needs one adjustment, concerning the choice of a linear extension $\preceq$. Specifically, while proving that $y_h$ is given by a Laurent polynomial, take a finite set $\mathcal{H}(h) \subset \mathcal{H}$ containing $h$ and satisfying the conditions

(4.11) if $h' \in \mathcal{H}(h)$, then $h'' \in \mathcal{H}(h)$ whenever $h'' \leq h'$ and $h'' \notin \mathcal{H}$;

(4.12) for any $h' \in \mathcal{H}$ such that $h' \leq h$, there exists $h'' \in \mathcal{H}(h)$ such that $h'' \geq h$ and $h'' \sim h$.

(The existence of $\mathcal{H}(h)$ follows from (4.9)–(4.10).) Then define $\preceq$ exactly as before on the set $\mathcal{H}(h)$; set $h' \preceq h''$ for any $h' \in \mathcal{H}(h)$ and $h'' \in \mathcal{H} - \mathcal{H}(h)$, and define $\preceq$ on
the complement $\mathcal{H} - \mathcal{H}^{(h)}$ by an arbitrary linear extension of $\preceq$. These conditions ensure that the sets $H_m$ needed in the proof of Laurentness of the given $y_h$ are well defined, and that the rest of the proof proceeds smoothly.

Armed with the techniques developed above in this section, we will now prove the main theorems stated in the introduction.

**Proof of Theorem 1.2.** Our argument is parallel to that in Example 4.1, so we skip the steps which are identical in both proofs. For simplicity of exposition, we present the proof in the special case $\mathcal{H} = \mathbb{Z}_{\geq 0}^2$; the case of general $\mathcal{H}$ requires the same adjustments as those described in Remark 4.2.

We define the product partial order $\preceq$ and a compatible linear order $\succeq$ on $\mathcal{H}$ by

$$(i_1, j_1, k_1) \preceq (i_2, j_2, k_2) \quad \text{iff} \quad (i_1 \leq i_2) \text{ and } (j_1 \leq j_2) \text{ and } (k_1 \leq k_2),$$

$$(i_1, j_1, k_1) \succeq (i_2, j_2, k_2) \quad \text{iff} \quad (i_1 + j_1 + k_1 < i_2 + j_2 + k_2)$$

or $(i_1 + j_1 + k_1 = i_2 + j_2 + k_2$ and $i_1 + j_1 < i_2 + j_2)$

or $(i_1 + j_1 = i_2 + j_2$ and $k_1 = k_2$ and $i_1 \leq i_2)$.

For $h = (i, j, k)$, we set $h^- = (i - 1, j - 1, k - 1)$; thus, the exchange relation (1.1) expresses the product $y_h \cdot y_{h^-}$ as a polynomial in the variables $y_{h'}$, for $h^- < h'$.

All the steps in Example 4.1 leading to the creation of the infinite exchange pattern (4.3) are repeated verbatim. Instead of (4.4), the exchange polynomials $P_{(h)}$ along the spine are now given by

$P_{((i,j,k))} = \alpha x_{(i-1,j,k-1)} x_{(i,j-1,k-1)} + \beta x_{(i,j,k-1)} x_{(i-1,j,k-1)} + \gamma x_{(i,j,k-1)} x_{(i-1,j-1,k-1)}$.

The role of (4.7) is now played by Figure 6, which shows the "vicinity" of an equivalence class $a$. This figure displays the orthogonal projection of $\mathcal{H}$ along the vector $(1,1,1)$. Thus the vertices represent equivalence classes in $\mathcal{H}/\sim$. For example, if $a = ((i,j,k))$, then

$$b = ((i,j,k-1)), \quad c = ((i,j-1,k)), \quad d = ((i-1,j,k)),$$

$$e = ((i,j-1,k-1)), \quad f = ((i-1,j,k-1)), \quad g = ((i-1,j-1,k))$$.

With this notation, we have:

$$P_a = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g.$$  

With the polynomials $G_1, G_2, \ldots$ defined as in (4.5), the essential values of $m$ are those for which $a_m \in \{b, c, d, e, f, g\}$. (The verification that the rest of the variables are not essential is left to the reader.) We denote these values by $m_1, \ldots, m_5$, respectively.

The computation of the polynomials $G_m$ begins by initializing

$$G_{N-1} = P_a = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g.$$  

Step $m = m_1, a_m = b$:

$$Q_{m_1} = P_b|_{x_d = 0} = \alpha x_f x_q + \beta x_e x_p;$$

$$G_{m_1 - 1} = G_{m_1}|_{x_f = \frac{x_d}{\alpha}} = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g x_p + \frac{x_i x_d x_e + \beta x_c x_f + \gamma x_b x_g x_p}{x_b};$$

$$G_{m_1 - 1} = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g x_p + \beta x_c x_f x_q + \beta x_b x_g x_p.$$
FIGURE 6. The cube recurrence

Step $m = m_2$, $a_m = c$:

$$Q_{m_2} = P_{i | x_s \leftarrow 0} = \alpha x_g x_r + \gamma x_c x_s;$$

$$\tilde{G}_{m_2-1} = \alpha x_g x_d x_e + \alpha x_g x_r x_f + \alpha x_c x_g x_q + \beta x_c x_g x_p;$$

$$G_{m_2-1} = \alpha x_g x_c x_d x_e + \alpha x_g x_f x_g x_s + \alpha x_g x_f x_q x_e + \alpha x_g x_f x_g x_p.$$

Step $m = m_3$, $a_m = d$:

$$Q_{m_3} = P_{d | x_s \leftarrow 0} = \beta x_g x_u + \gamma x_f x_u;$$

$$\tilde{G}_{m_3-1} = \alpha x_b x_d x_e + \frac{\beta x_b x_c x_d x_e}{x_d} + \alpha x_c x_d x_g x_q + \beta x_c x_g x_p;$$

$$G_{m_3-1} = \alpha x_b x_c x_d x_g x_q + \alpha x_b x_e x_f x_s + \alpha x_b x_d x_c x_f x_s + \alpha x_b x_d x_g x_p + \alpha x_b x_d x_f x_g x_e.$$
Step $m = m_0$, $a_m = 0$:

$$Q_{m_0} = P_g|_{x_s = 0} = \alpha x_c x_u + \beta x_d x_s;$$

$$\tilde{G}_{m_0 - 1} = \frac{Q_{m_0}}{x_s} (\alpha x_d x_e + \beta x_c x_f) + \gamma x_b Q_{m_0};$$

$$G_{m_0 - 1} = \alpha x_d x_e + \beta x_c x_f + \gamma x_b x_g = P_a,$$

completing the proof. \qed

We will now deduce the Gale-Robinson conjecture from Theorem 1.2.

**Proof of Theorem 1.4.** To prove the Laurentness of a given element $y_N$ of the Gale-Robinson sequence $(y_m)$, we define the array $(z_{ijk})_{(i,j,k)\in \mathcal{H}}$ by setting $z_{ijk} = y_{N+pi+qj+rk}$, with the indexing set

$$\mathcal{H} = \mathcal{H}(N) = \{(i,j,k) \in \mathbb{Z}^3 : N + pi + qj + rk \geq 0\}.$$

Then (1.4) implies that the $z_{ijk}$ satisfy the cube recurrence (1.1). Note that $\mathcal{H}$ satisfies the conditions (1.2)-(1.3). Thus Theorem 1.2 applies to $(z_{ijk})$, with $H_{init} = \{(a,b,c) \in \mathbb{Z}^3 : 0 \leq N + pa + qb + rc < n\}$. It remains to note that $y_N = x_{000}$, while for any $(a,b,c) \in H_{init}$, we have $z_{abc} = y_m$ with $0 \leq m < n$. \qed

**Proof of Theorem 1.6.** This theorem is proved by the same argument as Theorem 1.2. We treat the Mills-Robbins-Rumsey special case (1.9) (cf. also (1.6)); similarly to Theorem 1.2, the case of general $\mathcal{H}$ requires the standard adjustment described in Remark 4.2. We use the partial order on the lattice $L$ defined by

$$(i,j,k) \leq (i',j',k') : |i' - i| + |j' - j| \leq k' - k.$$

For $h = (i,j,k) \in L$, we set $h^- = (i,j,k-2)$, and define the equivalence relation $\sim$ accordingly. Figure 7 shows equivalence classes "surrounding" a given class $a$ (cf. Figure 6).

![Figure 7](image)

The initialization polynomial $G_{N-1} = P_a$ is given by $P_a = \alpha x_c x_d + \beta x_b x_e$. The table below displays $a_m$, $Q_m$, $\tilde{G}_{m-1}$, and $G_{m-1}$ for all essential values of $m$.

<table>
<thead>
<tr>
<th>$a_m$</th>
<th>$Q_m$</th>
<th>$\tilde{G}_{m-1}$</th>
<th>$G_{m-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b$</td>
<td>$\alpha x_p x_q$</td>
<td>$\alpha x_c x_d + \alpha \beta x_x x_e x_s$</td>
<td>$x_b x_c x_d + \beta x_c x_p x_q$</td>
</tr>
<tr>
<td>$c$</td>
<td>$\beta x_2 x_r$</td>
<td>$\beta x_x x_e x_b x_d + \beta x_c x_p x_q$</td>
<td>$x_b x_d x_r + x_c x_e x_p$</td>
</tr>
<tr>
<td>$d$</td>
<td>$\beta x_p x_s$</td>
<td>$\beta x_x x_d x_b x_r + x_c x_e x_p$</td>
<td>$\beta x_b x_r x_s + x_c x_d x_e$</td>
</tr>
<tr>
<td>$e$</td>
<td>$\alpha x_r x_s$</td>
<td>$\beta x_b x_r x_s + \alpha x_x x_e x_d x_e$</td>
<td>$\beta x_b x_e + \alpha x_c x_d$</td>
</tr>
</tbody>
</table>
We see that \( G_0 = G_{e-1} = P_a \), completing the proof. \( \square \)

**Proof of Theorem 1.8.** The proof mimics the above proof of Theorem 1.4. To prove the Laurentness of an element \( y_N \) of the sequence \( (y_m) \), we define the array \( (z_{ijk})_{(i,j,k) \in \mathcal{H}} \) by setting \( z_{ijk} = y_{N+\ell(i,j,k)} \), where \( \ell(i,j,k) = n \frac{i+j+k}{2} - pi - qj \). The indexing set \( \mathcal{H} \) is now given by

\[
\mathcal{H} = \mathcal{H}(N) = \{(i,j,k) \in \mathbb{Z}^3 : N + \ell(i,j,k) \geq 0\}.
\]

Then (1.10) implies that the \( z_{ijk} \) satisfy the octahedron recurrence (1.5). It is easy to check that \( \mathcal{H} \) satisfies the conditions (1.7)--(1.8). Thus Theorem 1.6 applies to \( (z_{ijk}) \), with \( H_{\text{init}} = \{(a,b,c) \in L : 0 \leq N + \ell(a,b,c) < n\} \), and the theorem follows. \( \square \)

We conclude this section by a couple of examples in which the Laurent phenomenon is established by the same technique as above. In each case, we provide:

- a picture of the equivalence classes “surrounding” a given class \( a \), which plays the role of (4.7) in Example 4.1;
- the initialization polynomial \( G_{N-1} = P_a \);
- a table showing \( a_m, Q_m, G_{m-1}, \) and \( G_{m-1} \) for all essential values of \( m \).

**Example 4.3.** *(Frieze patterns)* The generalized frieze pattern recurrence (cf., e.g., [3, 11]) is

\[
y_{ij}y_{i-1,j-1} = \epsilon y_{i,j-1}y_{i-1,j} + \beta,
\]

where \( \epsilon \in \{1, -1\} \). To prove Laurentness (over \( \mathbb{Z}[\beta] \)), refer to Figure 8. Then \( P_a = \epsilon x_b x_c + \beta \), and the essential steps are:

<table>
<thead>
<tr>
<th></th>
<th>( a_m )</th>
<th>( Q_m )</th>
<th>( \tilde{G}_{m-1} )</th>
<th>( G_{m-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>( \beta )</td>
<td>( \frac{\epsilon \beta x_b}{x_b} + \beta )</td>
<td>( \epsilon x_c + x_b )</td>
<td>( \beta + \epsilon^{-1} x_b x_c )</td>
</tr>
</tbody>
</table>

![Figure 8](image)

**Example 4.4.** *(Number walls)* Consider the 2-dimensional recurrence

\[
y_{ij}y_{i,j-2} = y_{i-1,j-1}y_{i+1,j-1} + y_{i,j-1},
\]

where \( \epsilon \in \{1, -1\} \). To prove Laurentness (over \( \mathbb{Z}[\beta] \)), refer to Figure 9. Then \( P_a = \epsilon x_b x_c + \beta \), and the essential steps are:

<table>
<thead>
<tr>
<th></th>
<th>( a_m )</th>
<th>( Q_m )</th>
<th>( \tilde{G}_{m-1} )</th>
<th>( G_{m-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>( \beta )</td>
<td>( \frac{\epsilon \beta x_b}{x_b} + \beta )</td>
<td>( \epsilon x_c + x_b )</td>
<td>( \beta + \epsilon^{-1} x_b x_c )</td>
</tr>
</tbody>
</table>

![Figure 9](image)
where \( p, q, \) and \( r \) are nonnegative integers. To prove Laurentness, refer to Figure 9. Then \( P_a = x_d^p x_f^r + x_e^q \), and the essential steps are:

\[
\begin{array}{cccc}
\quad & a_m & Q_m & \tilde{G}_{m-1} & G_{m-1} \\
\hline
b & x_f^q & x_d^p \left( \frac{x_e^r}{x_b} \right)^r + x_e^q & x_d^p x_f^r & x_d^p x_e^q + x_e^q x_a^r \\
c & x_g^p x_f^r & x_d^p x_e^q + \left( \frac{x_e^r}{x_a} \right)^q x_b^r & x_d^p x_g^q & x_g^p x_e^q + x_e^q x_a^r \\
d & x_g^p \left( \frac{x_e^r}{x_d} \right)^p x_e^q + x_g^p x_e^q & x_g^p x_e^q & x_g^p & x_e^q x_a^r
\end{array}
\]

\[j \quad \begin{array}{cccc}
\hline
\quad & a & b & c & d \\
\hline
\quad & g & a & b & \quad \quad i
\end{array}\]

\textbf{Figure 9}

\textbf{Remark 4.5.} As pointed out by J. Propp, the Laurent phenomenon for certain special cases of Examples 4.3 and 4.4 can be obtained by specialization of Example 1.5.

\section{5. Homogeneous Exchange Patterns}

In this section, we deduce Theorem 1.10 and a number of similar results from the following corollary of Theorem 2.1.

\textbf{Corollary 5.1.} Let \( \Lambda \) be a unique factorization domain. Assume that a collection of nonzero polynomials \( P_1, \ldots, P_n \in A[x_1, \ldots, x_n] \) satisfies the following conditions:

\begin{enumerate}
\item[(5.1)] Each \( P_k \) does not depend on \( x_k \), and is not divisible by any \( x_i, \ i \in [n] \).
\item[(5.2)] For any \( i \neq j \), the polynomials \( P_i \stackrel{\text{def}}{=} (P_j)|_{x_i=0} \) and \( P_i \) are coprime.
\item[(5.3)] For any \( i \neq j \), we have
\[
L \cdot P_i^b \cdot P_i = P_i|_{x_i=\frac{P_j}{x_j}},
\]
where \( b \) is a nonnegative integer, and \( L \) is a Laurent monomial whose coefficient lies in \( \Lambda \) and is coprime with \( P_i \).
\end{enumerate}

Let us define the rational transformations \( F_i, \ i \in [n], \) by

\[
F_i: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, P_i, x_{i+1}, \ldots, x_n).
\]

Then any composition of the form \( F_{i_1} \circ \cdots \circ F_{i_m} \) is given by Laurent polynomials with coefficients in \( \Lambda \).

\textbf{Proof.} Let \( T_n \) denote a regular tree of degree \( n \) whose edges are labeled by elements of \( [n] \) so that all edges incident to a given vertex have different labels. Assigning \( P_i \) as an exchange polynomial for every edge of \( T_n \) labeled by \( i \), we
obtain a “homogeneous” exchange pattern on $\mathbb{T}_n$ satisfying conditions (2.3)–(2.5) in Theorem 2.1. This implies the desired Laurentness.

**Example 5.2.** Let $n \geq 3$ be an integer, and let $P$ be a quadratic form given by

$$P(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} \alpha_{ij} x_i x_j.$$  

Theorem 1.10 is a special case of Corollary 5.1 for $P_i = P|_{x_i=0}$ and $A = \mathbb{Z}[\alpha_{ij} : i < j]$. Conditions (5.1)–(5.2) are clear. To verify (5.3), note that

$$P_i = P_{ji} + x_i^2 + x_i \left( \sum_k \alpha_{kj} x_k + \sum_\ell \alpha_{ij} x_\ell \right),$$

where $k$ (resp. $\ell$) runs over all indices such that $k \neq i$ and $k < j$ (resp. $\ell \neq i$ and $\ell > j$). It follows that

$$P_i|_{x_j = \frac{r_{ji}}{x_j^2}} = P_{ji} + \frac{P_{ji}}{x_j^2} \left( \sum_k \alpha_{kj} x_k + \sum_\ell \alpha_{ij} x_\ell \right) = P_i^2 P_k,$$

verifying (5.3).

In the remainder of this section, we list a few more applications of Corollary 5.1. In each case, the verification of its conditions is straightforward.

**Example 5.3.** Let $P$ and $Q$ be monic palindromic polynomials in one variable:

$$P(x) = (1 + x^d) + \alpha_1 (x + x^{d-1}) + \alpha_2 (x^2 + x^{d-2}) + \ldots;$$
$$Q(x) = (1 + x^e) + \beta_1 (x + x^{e-1}) + \beta_2 (x^2 + x^{e-2}) + \ldots.$$

Then every member of the sequence $y_0, y_1, \ldots$ defined by the recurrence

$$y_k = \begin{cases} 
\frac{\mu^2 P(y_{k-1}/\lambda)}{y_{k-2}} & \text{if } k \text{ is odd;} \\
\frac{\lambda^2 Q(y_{k-1}/\mu)}{y_{k-2}} & \text{if } k \text{ is even}.
\end{cases}$$

is a Laurent polynomial in $y_0$ and $y_1$ with coefficients in $A = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}, \alpha_i, \beta_i]$. This follows from Corollary 5.1 with $n = 2$, $P_1 = \mu^2 P(x_2/\lambda)$, and $P_2 = \lambda^2 Q(x_1/\mu)$.

**Example 5.4.** Consider the sequence $y_0, y_1, \ldots$ defined by the recurrence

$$y_k = \frac{y_{k-1}^2 + cy_{k-1} + d}{y_{k-2}}.$$ 

Every term of this sequence is a Laurent polynomial in $y_0$ and $y_1$ with coefficients in $\mathbb{Z}[c, d]$.

**Example 5.5.** Define the rational transformations $F_1, F_2, F_3$ by

$$F_1 : (x_1, x_2, x_3) \mapsto (\frac{x_2 + x_3^2 + x_2^2 x_3}{x_1}, x_2, x_3),$$

$$F_2 : (x_1, x_2, x_3) \mapsto (x_1, \frac{x_1 + x_3}{x_2}, x_3),$$

$$F_3 : (x_1, x_2, x_3) \mapsto (x_1, \frac{x_2}{x_3}, \frac{x_2 + x_1^2 + x_2^2 x_1}{x_3}).$$

Then any composition $F_{i_1} \circ F_{i_2} \circ \cdots$ is given by $(x_1, x_2, x_3) \mapsto (G_1, G_2, G_3)$, where $G_1, G_2, G_3$ are Laurent polynomials in $x_1, x_2, x_3$ over $\mathbb{Z}$. 

REFERENCES


Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: fomin@umich.edu

Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: andrei@neu.edu