Convergence in capacity

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1. Introduction

The purpose of this paper is to study convergence of Monge-Ampère measures associated to sequences of plurisubharmonic functions defined on a hyperconvex subdomain $\Omega$ of $\mathbb{C}^n$.

The concept of convergence in capacity was introduced in [X], where it was proved that for any uniformly bounded sequence $\varphi_j$ of plurisubharmonic functions that converges in $\mathbb{C}^n$-capacity we have that $(dd^c\varphi_j)^n$ converges weak* to $(dd^c\varphi)^n$, $j \to +\infty$.

We generalize this result:

**Theorem 1.1.** Assume $\mathcal{F} \ni u_0 \leq u_j \in \mathcal{F}$ and that $u_j$ converges to $u$ in $\mathbb{C}^n$-capacity. Then $(dd^c u_j)^n$ converges weak* to $(dd^c u)^n$, $j \to +\infty$.

We first recall some definitions. See [C1] and [C2] for details.

The class $\mathcal{F}$ consists of all plurisubharmonic functions $\varphi$ on $\Omega$ such that there is a sequence $\varphi_j \in \mathcal{E}_0, \varphi_j \searrow \varphi, j \to +\infty$ and $\sup \int_{\Omega} (dd^c \varphi_j)^n < +\infty$, where $\mathcal{E}_0$ is the class of bounded plurisubharmonic functions $\psi$ such that $\lim_{z \to \xi} \psi(z) = 0$, $\forall \xi \in \partial \Omega$ and $\int_{\Omega} (dd^c \psi)^n < +\infty$.

The following definition was introduced in [X]: A sequence $\varphi_j \in \mathcal{F}$ converges to $\varphi$ in $\mathbb{C}^n$-capacity if

$$\text{cap}\{\{z \in k : |\varphi - \varphi_j| > \delta\}\} \to 0, \quad j \to +\infty \quad \forall \ k \subset \subset \Omega, \quad \forall \ \delta > 0.$$ 

For $\omega \subset \subset \Omega$, $\text{cap}(\omega) = \int_{\Omega} (dd^c h_\omega)^n$ where $h_\omega^*$ is the smallest upper semicontinuous majorant of $h_\omega(z) = \sup\{\varphi(z) ; \varphi \in \mathcal{E}_0 ; \varphi|_\partial \omega \leq -1\}$.

Finally, we write $u_j \rightharpoonup u$, $j \to +\infty$ if $u_j$ converges weak* to $u$, $j \to +\infty$. 

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2. Proofs

Lemma 2.1. Suppose $\mu$ is a positive measure on $\Omega$ which vanishes on all pluripolar sets and $\mu(\bar{\Omega}) < +\infty$. If $u_0, u_j \in \mathcal{F}$, $u_0 \leq u_j \sim u$, $j \to +\infty$ and if $\int u_0 d\mu > -\infty$, then $\lim_{j \to +\infty} \int u_j d\mu = \int u d\mu$.

Proof. Denote by $dV$ the Lebesgue measure and choose $\bar{u}_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$, $\bar{u}_j \geq u_j$ such that $\int_{\Omega} (\bar{u}_j - u_j)(d\mu + dV) < \frac{1}{j}$. Then $u \sim u_j$, $\lim_{j \to +\infty} \int u_j d\mu - \int \bar{u}_j d\mu = 0$ so it is enough to prove that

$$\lim_{j \to +\infty} \int \bar{u}_j d\mu = \int u d\mu.$$  

Thus we can assume $u_j \in \mathcal{E}_0 \cap C(\bar{\Omega})$.

By Theorem 6.3 in [C1] there is a $\psi \in \mathcal{E}_0$, $f \in L^1((dd^c\psi)^n)$ with

$$\mu = f(dd^c\psi)^n$$

so by lemma 5:2 in [C1], for every $p < +\infty$,

$$\lim_{j \to +\infty} \int u_j d\mu_p = \int u d\mu_p$$

there $\mu_p = \min(f, p)(dd^c\psi)^n$.

Now

$$\lim_{j \to +\infty} \int u_j d\mu = \lim_{j \to +\infty} \int u_j d\mu_p +$$

$$+ \lim_{j \to +\infty} \int u_j (f - \min(f, p))(dd^c\psi)^n \geq$$

$$\geq \int u d\mu_p + \int u_0 (f - \min(f, p))d\mu$$

$$\to \int u d\mu, \ p \to +\infty$$

by monotone convergence. On the other hand, by Fatous lemma,

$$\lim_{j \to +\infty} \int u_j d\mu \leq \int u d\mu$$

which gives the desired conclusion. \qed

Proof. Proof of the theorem.

We prove that

$$\lim_{j \to +\infty} \int h(dd^c u_j)^n = \int h(dd^c u)^n, \ \forall h \in \mathcal{E}_0,$$

which is enough by Lemma 3:1 in [C1].

Suppose $\omega_1, \omega_2, \ldots, \omega_{n-1} \in \mathcal{F}$, $h \in \mathcal{E}_0$. It follows from the assumption that $u_j \sim u$, $j \to +\infty$ so $\lim_{j \to +\infty} \int \omega_1 dd^c u_j \wedge dd^c \omega_2 \wedge \cdots \wedge dd^c h = \int \omega_1 dd^c u \wedge \cdots \wedge dd^c h$ by the lemma.
Suppose now that
\[ \lim_j \int \omega_1(dd^c u_j)^p \wedge dd^c \omega_{p+1} \wedge \cdots \wedge dd^c h = \int \omega_1(dd^c u)^p \wedge \cdots \wedge dd^c h. \]
for \(1 \leq q \leq p \leq n - 2\). We claim
\[ \lim_j \int \omega_1(dd^c u_j)^{p+1} \wedge dd^c \omega_{p+2} \wedge \cdots \wedge dd^c \omega_{n-1} \wedge dd^c h = \int \omega_1(dd^c u)^{p+1} \wedge dd^c \omega_{p+2} \wedge \cdots \wedge dd^c \omega_{n-1} \wedge dd^c h. \]
Given \( \varepsilon > 0 \) choose \( k \subset \subset \Omega \) such that \( \{ z \in \Omega; h < -\varepsilon \} \subset k \) and then a subsequence \( u_{j_k} \) such that
\[ \int_{\Omega} \left( \sum_{l=1}^{\infty} h^*_z \{ h \leq u_{j_k} - \varepsilon \} \right)^n < 1 \]
and denote by
\[ h_N = \max \left( \sum_{l=n}^{\infty} h^*_z \{ h \leq u_{j_k} - \varepsilon \}, -1 \right). \]
Then \( h_N \to 0, N \to +\infty \) outside a pluripolar set and
\[ \int \omega_1(dd^c u_{j_k})^{p+1} \wedge \cdots \wedge dd^c h - \int \omega_1(dd^c u_{j_k})^p dd^c u \wedge \cdots \wedge dd^c h = \]
\[ = \int (u_{j_k} - u)(dd^c u_{j_k})^p \wedge dd^c \omega_1 \wedge \cdots \wedge dd^c h = \]
\[ = \int -h_N(u_{j_k} - u)(dd^c u_{j_k})^p \wedge dd^c \omega_1 \wedge \cdots \wedge dd^c h + \]
\[ + \int (1 + h_N)(u_{j_k} - u)(dd^c u_{j_k})^p \wedge dd^c \omega_1 \wedge \cdots \wedge dd^c (h - h_\varepsilon) + \]
\[ + \int (1 + h_N)(u_{j_k} - u)(dd^c u_{j_k})^p \wedge dd^c \omega_1 \wedge \cdots \wedge dd^c h = I_{j_k} + II_{j_k} + III_{j_k}. \]
where \( h_\varepsilon = \max(h, -\varepsilon) \).

Thus
\[ |I_{j_k}| \leq 2 \int (-h_N)(-u_0)(dd^c u_{j_k})^p \wedge \cdots \wedge dd^c h \leq \]
\[ \leq \int [-u_0 + \max(u_0, -R)](dd^c u_{j_k})^p \wedge \cdots \wedge dd^c h + \]
\[ + R \int -h_N(dd^c u_{j_k})^p \wedge \cdots \wedge dd^c h \]
\[ \to \int [-u_0 + \max(u_0, -R)](dd^c u)^p \wedge \cdots \wedge dd^c h + \]
\[ + R \int (-h_N)(dd^c u)^p \wedge \cdots \wedge dd^c h, j_k \to +\infty. \]
Now, the first term on the right hand side is small if \( R \) is large, the second is small if \( N \) is even larger.

\[
|III_{j_k}| \leq \int -u_0(\partial \bar{\partial} u_{j_k})^p \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} h_\varepsilon \leq \\
\leq \int -u_0(\partial \bar{\partial} u_0)^p \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} h_\varepsilon \leq \\
\leq \int -h_\varepsilon(\partial \bar{\partial} u_0)^{p+1} \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} \omega_{n-1} \leq \\
\varepsilon \int (\partial \bar{\partial} u_0)^{p+1} \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} \omega_{n-1}
\]

so it remains to estimate

\[
|II_{j_k}| \leq 2\varepsilon \left( \int (\partial \bar{\partial} u_{j_k})^p \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} (h + h_\varepsilon) \right) \leq \\
\leq 4\varepsilon \int (\partial \bar{\partial} u_0)^p \wedge \partial \bar{\partial} \omega_1 \wedge \cdots \wedge \partial \bar{\partial} h.
\]

which proves the claim.

Using the claim and repeating the chain of inequalities above, we conclude that

\[
\int h(\partial \bar{\partial} u_{j_k})^n - \int h(\partial \bar{\partial} u)^n = \\
\int (u_{j_k} - u)(\partial \bar{\partial} u_{j_k})^{n-1} \wedge h + \\
\int u(\partial \bar{\partial} u_{j_k})^{n-1} \wedge h - \int u(\partial \bar{\partial} u)^{n-1} \wedge h \to 0, j_k \to +\infty.
\]

This proves the theorem, since we have proved that every subsequence of \( u_j \) contains a subsequence \( u_{j_k} \) such that \((\partial \bar{\partial} u_{j_k})^n\) converges weak* to \((\partial \bar{\partial} u)^n\), \( t \to +\infty \).

\[\square\]

References