MIXED TORIC RESIDUES AND CALABI-YAU COMPLETE INTERSECTIONS

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ABSTRACT. Using Cayley trick, we define the notions of mixed toric residues and mixed Hessians associated with $r$ Laurent polynomials $f_1, \ldots, f_r$. We conjecture that the values of mixed toric residues on the mixed Hessians are determined by mixed volumes of the Newton polytopes of $f_1, \ldots, f_r$. Using mixed toric residues, we generalize our Toric Residue Mirror Conjecture to the case of Calabi-Yau complete intersections in Gorenstein toric Fano varieties obtained from nef-partitions of reflexive polytopes.

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1. Introduction

This paper is the continuation of our previous work [BM1] where we proposed a toric mirror symmetry test using toric residues. The idea of this test has appeared in the paper of Morrison-Plesser [MP] who observed that the coefficients of some power series expansions of unnormalized Yukawa couplings for mirrors of Calabi-Yau hypersurfaces in toric varieties $\mathbb{P}$ can be interpreted as generating functions for intersection numbers of divisors on some sequences of toric varieties $\mathbb{P}_\beta$ parametrized by lattice points $\beta$ in the Mori cone $K_{\text{eff}}(\mathbb{P})$ of $\mathbb{P}$. Due to results of Mavlyutov [Mav], it is known that the unnormalized Yukawa couplings can be computed using toric residues introduced by Cox [Cox]. In our paper [BM1], we formulated a general mathematical conjecture, so called Toric Residue Mirror Conjecture, which describes some power series expansions of the toric residues in terms of intersection numbers of divisors on a sequence of simplicial toric varieties $\mathbb{P}_\beta$ (we call them Morrison-Plesser moduli spaces). This conjecture includes all examples of mirror symmetry for Calabi-Yau hypersurfaces in Gorenstein toric varieties associated with reflexive polytopes. Since the toric mirror symmetry construction exists also for Calabi-Yau complete intersection in Gorenstein toric Fano varieties [Bo, BB1], it is natural to try to extend our conjecture to this more general situation.

The case of Calabi-Yau complete intersections of $r$ hypersurfaces

$$f_1(t) = \cdots = f_r(t) = 0, \quad r > 1,$$

defined by Laurent polynomials $f_1(t), \ldots, f_r(t) \in \mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]$ in $d$-dimensional toric varieties $\mathbb{P}$ was not considered by Morrison and Plesser in [MP]. We remark that in this case one does not get a connection to the "quantum cohomology ring" [Bat] as in the hypersurface case. This difference is explained by the consideration of a nonreflexive $(d+r−1)$-dimensional polytope $\Delta$, so called Cayley polytope, and its secondary polytope $\text{Sec}(\Delta)$. The Cayley polytope $\Delta$ appears from the Cayley trick which introduces $r$ additional $r$ variables $t_{d+1}, \ldots, t_{d+r}$ and a new polynomial $F(t) := \sum_{j=1}^r t_{d+j}f_j(t)$. We consider the usual toric residue $\text{Res}_F$ associated with $F$ and define the $k$-mixed toric residue $\text{Res}_F^k$ corresponding to a positive integral solution $k = (k_1, \ldots, k_r)$ of the equation $k_1 + \cdots + k_r = d+r$ as a $k$-th homogeneous component of $\text{Res}_F$. We expect that the $k$-mixed toric residues are similar to the usual toric residues. In particular, we introduce the notion of $k$-mixed Hessian $H_F^k$ of Laurent polynomials $f_1, \ldots, f_r$ and conjecture that the value of $\text{Res}_F^k$ on $H_F^k$ is exactly the mixed volume

$$V(\underbrace{\Delta_1, \ldots, \Delta_1}_{k_1-1}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{k_r-1}),$$

where $\Delta_1, \ldots, \Delta_r$ are Newton polytopes of $f_1, \ldots, f_r$. 
Our generalization of the Toric Residue Mirror Conjecture for Calabi-Yau complete intersections uses the notions of the nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ of $d$-dimensional reflexive polytope $\Delta$ [Bo]. In this situation, one obtains a dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ and two more reflexive polytopes:

$$\nabla^* = \text{conv}\{\Delta_1, \ldots, \Delta_r\}, \quad \Delta^* = \text{conv}\{\nabla_1, \ldots, \nabla_r\}.$$ 

It is important that special coherent triangulations of $\nabla^*$ define coherent triangulations of the Cayley polytope $\tilde{\Delta} := \Delta_1 \ast \cdots \ast \Delta_r$. Therefore the choice of such a triangulation $T$ of $\nabla^*$ determines a vertex $v_T$ of the secondary polytope $\text{Sec}(\tilde{\Delta})$ and a partial projective simplicial crepant desingularization $\mathbb{P} := \mathbb{P}_{\Sigma(T)}$ of the Gorenstein toric variety $\mathbb{P}_{\nabla^*}$. So one obtains a sequence of simplicial toric varieties $\mathbb{P}_\beta$ associated with the lattice points $\beta$ in the Mori cone $K_{\text{eff}}(\mathbb{P})$ of $\mathbb{P}$. We conjecture that the generating function of intersection numbers

$$I_P(a) = \sum_{\beta \in K_{\text{eff}}(\mathbb{P})} I(P, \beta) a^\beta$$

coincides with the power series expansion of the $k$-mixed toric residue

$$R_P(a) = \text{Res}_P^K(P(a, t))$$

at the vertex $v_T \in \text{Sec}(\tilde{\Delta})$. The precise formulation of this conjecture is given in Section 4.

In Sections 5, 6 we check our conjecture for nef-partitions corresponding to Calabi-Yau complete intersections in weighted projective spaces $\mathbb{P}(w_1, \ldots, w_n)$ and in product of projective spaces $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_p}$. The final section is devoted to applications of the Toric Residue Mirror Conjecture to the computation of Yukawa couplings for Calabi-Yau complete intersections.

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2. TORIC RESIDUES

In this section we remind necessary well-known facts about toric residues (see [Cox, CDS, BM1]).

Let $\widetilde{M}$ and $\widetilde{N} = \text{Hom}(\widetilde{M}, \mathbb{Z})$ be two free abelian groups of rank $\tilde{d}$ dual to each other. We denote by

$$\langle \ast, \ast \rangle : \widetilde{M} \times \widetilde{N} \to \mathbb{Z}$$

the natural bilinear pairing, and by $\widetilde{M}_\mathbb{R}$ (resp. $\widetilde{N}_\mathbb{R}$) the real scalar extension of $\widetilde{M}$ (resp. $\widetilde{N}$).
Definition 2.1 ([BB2]). A $\tilde{d}$-dimensional rational polyhedral cone $C$ ($\tilde{d} > 0$) in $\tilde{M}_\mathbb{R}$ is called Gorenstein if it is strongly convex (i.e., $C + (-C) = \{0\}$), there exists an element $n_C \in \tilde{N}$ such that $\langle x, n_C \rangle > 0$ for any nonzero $x \in C$, and all vertices of the $(\tilde{d} - 1)$-dimensional convex polytope

$$\Delta(C) = \{ x \in C : \langle x, n_C \rangle = 1 \}$$

belong to $\tilde{M}$. The polytope $\Delta(C)$ is called the supporting polytope of $C$. For any $m \in C \cap \tilde{M}$, we define the degree of $m$ as

$$\deg m = \langle m, n_C \rangle.$$

Definition 2.2. Let $\tilde{\Delta} = \Delta(C)$ be the supporting polytope for a Gorenstein cone $C \subset \tilde{M}_\mathbb{R}$. We denote by $S_{\tilde{\Delta}}$ the semigroup $\mathbb{C}$-algebra of the monoid of lattice points $C \cap \tilde{M}$. In order to transform the additive semigroup operation in $C \cap \tilde{M}$ into a multiplicative form in $S_{\tilde{\Delta}}$, we write $t^m$ for the element in $S_{\tilde{\Delta}}$ corresponding to $m \in C$. One can consider $S_{\tilde{\Delta}}$ as a graded $\mathbb{C}$-algebra:

$$S_{\tilde{\Delta}} = \bigoplus_{l=0}^{\infty} S_{\tilde{\Delta}}^{(l)},$$

where the $l$-th homogeneous component $S_{\tilde{\Delta}}^{(l)}$ has a $\mathbb{C}$-basis consisting of all $t^m$ such that $m \in C \cap \tilde{M}$ and $\deg m = l$. We define also the homogeneous ideal

$$I_{\tilde{\Delta}} = \bigoplus_{l=0}^{\infty} I_{\tilde{\Delta}}^{(l)}$$

in $S_{\tilde{\Delta}}$ whose $\mathbb{C}$-basis consists of all $t^m$ such that $m$ is a lattice point in the interior of $C$.

Definition 2.3. An element

$$g := \sum_{m \in \tilde{\Delta} \cap \tilde{M}} a_m t^m \in S_{\tilde{\Delta}}^{(1)}, \quad a_m \in \mathbb{C}$$

is called $\tilde{\Delta}$-regular if for some $\mathbb{Z}$-basis $n_1, \ldots, n_{\tilde{d}}$ of $\tilde{N}$ the elements

$$g_i := \sum_{m \in \tilde{\Delta} \cap \tilde{M}} a_m \langle m, n_i \rangle t^m, \quad i = 1, \ldots, \tilde{d}$$

form a regular sequence in $S_{\tilde{\Delta}}$. We define the matrix $G := (g_{ij})_{1 \leq i, j \leq \tilde{d}}$, where

$$g_{ij} := \sum_{m \in \tilde{\Delta} \cap \tilde{M}} a_m \langle m, n_i \rangle \langle m, n_j \rangle t^m, \quad i, j = 1, \ldots, \tilde{d}.$$

The element

$$H_g := \det G$$
is called Hessian of $g$.

Remark 2.4. a) The definition of $\bar{\Delta}$-regularity does not depend on the choice of $\mathbb{Z}$-basis $n_1, \ldots, n_d$ of $N$. In many applications the lattice vector $n_C$ will be included in $\{n_1, \ldots, n_d\}$.

b) If $\bar{\Delta} \cap \bar{M} = \{m_1, \ldots, m_\mu\}$, then by [CDS, Proposition 1.2], one has

$$H_g = \sum_{1 \leq i_1 < \cdots < i_\mu \leq \mu} (\det(m_{i_1}, \ldots, m_{i_\mu}))^2 t^{m_{i_1} + \cdots + m_{i_\mu}}.$$ 

In particular, $H_g$ is independent on the choice of the $\mathbb{Z}$-basis $n_1, \ldots, n_d$ and $H_g \in I^{(d)}_{\bar{\Delta}}$.

c) The graded $\mathbb{C}$-algebra $S_{\bar{\Delta}}$ is Cohen-Macaulay and $I_{\bar{\Delta}}$ is its dualizing module. If $g$ is $\bar{\Delta}$-regular in $S_{\bar{\Delta}}$, then

$$S_g := S_{\bar{\Delta}} / \langle g_1, \ldots, g_d \rangle S_{\bar{\Delta}},$$

is a graded finite-dimensional ring and

$$I_g := I_{\bar{\Delta}} / \langle g_1, \ldots, g_d \rangle I_{\bar{\Delta}}$$

is a graded $S_g$-module together with a non-degenerate pairing

$$S_g^{(l)} \times I_g^{(d-l)} \to I_g^{(d)} \simeq \mathbb{C}, \quad l = 0, \ldots, d - 1.$$ induced by the $S_g$-module structure.

Definition 2.5. By toric residue corresponding to a $\bar{\Delta}$-regular element $g \in S_{\bar{\Delta}}^{(1)}$ we mean the $\mathbb{C}$-linear mapping

$$\text{Res}_g : I^{(d)}_{\bar{\Delta}} \to \mathbb{C}$$

which is uniquely determined by two conditions:

(i) $\text{Res}_g(h) = 0$ for any $h \in \langle g_1, \ldots, g_d \rangle I_{\bar{\Delta}}$;

(ii) $\text{Res}_g(H_g) = \text{Vol}(\bar{\Delta})$, where $\text{Vol}(\bar{\Delta})$ denotes the volume of the $(d - 1)$-dimensional polytope $\bar{\Delta}$ multiplied by $(d - 1)!$.

Let $\mathbb{P}_{\bar{\Delta}} := \text{Proj} S_{\bar{\Delta}}$ be $(d - 1)$-dimensional toric variety associated with the polytope $\bar{\Delta}$ and $\mathcal{O}_{\mathbb{P}_{\bar{\Delta}}}(1)$ the corresponding ample sheaf on $\mathbb{P}_{\bar{\Delta}}$. Then one has the canonical isomorphisms of graded rings

$$S_{\bar{\Delta}} \cong \bigoplus_{l \geq 0} H^0(\mathbb{P}_{\bar{\Delta}}, \mathcal{O}_{\mathbb{P}_{\bar{\Delta}}}(l))$$

and graded modules

$$I_{\bar{\Delta}} \cong \bigoplus_{l \geq 0} H^0(\mathbb{P}_{\bar{\Delta}}, \omega_{\mathbb{P}_{\bar{\Delta}}}(l)).$$
where $\omega_{P_{\tilde{\Delta}}}$ is the dualizing sheaf on $P_{\tilde{\Delta}}$. In particular, we obtain a canonical isomorphism

$$I^{(\tilde{d})}_{\tilde{\Delta}} \cong H^0(P_{\tilde{\Delta}}, \omega_{P_{\tilde{\Delta}}}(\tilde{d})).$$

The following statement is a simple reformulation of Theorem 2.9(i) in [BM1]:

**Proposition 2.6.** Let $n_1, \ldots, n_{\tilde{d}}$ be a $\mathbb{Z}$-basis of $\tilde{N}$ such that $n_1 = n_C$. Denote by $m_1, \ldots, m_{\tilde{d}}$ the dual $\mathbb{Z}$-basis of $\tilde{M}$. For any elements $h \in I^{(\tilde{d})}_{\tilde{\Delta}}$ and $g \in S_{\tilde{\Delta}}^{(1)}$, we define a rational differential $(\tilde{d} - 1)$-form on $P_{\tilde{\Delta}}$:

$$\Omega(h, g) := \frac{h}{g_1 \cdots g_{\tilde{d}}} \frac{dt^{m_2}}{t^{m_2}} \wedge \cdots \wedge \frac{dt^{m_{\tilde{d}}}}{t^{m_{\tilde{d}}}}.$$

If $g$ is $\tilde{\Delta}$-regular, then

$$\text{Res}_g(h) = \sum_{\xi \in V_g} \text{res}_\xi(\Omega(h, g)),$$

where $V_g = \{ \xi \in P_{\tilde{\Delta}} : g_2(\xi) = \cdots = g_{\tilde{d}}(\xi) = 0 \}$ is the set of common zeros of $g_2, \ldots, g_{\tilde{d}}$ and $\text{res}_\xi(\Omega(h, g))$ is the local Grothendieck residue of the form $\Omega(h, g)$ at the point $\xi \in V_g$.

In particular, if all the common roots of $g_2, \ldots, g_{\tilde{d}}$ are simple and contained in the open dense $(\tilde{d} - 1)$-dimensional torus $T \subset P_{\tilde{\Delta}}$, then

$$\text{Res}_g(h) = \sum_{\xi \in V_g} \frac{p(\xi)}{g_1(\xi) H_g^1(\xi)},$$

where $H_g^1$ is the determinant of the matrix $G^1 := (g_{i,j})_{2 \leq i, j \leq \tilde{d}}$.

**Definition 2.7 ([BB1]).** A Gorenstein cone $C$ is called **reflexive** if the dual cone

$$\check{C} = \{ y \in \tilde{N}_R : \langle x, y \rangle \geq 0 \quad \forall x \in C \}$$

is also Gorenstein, i.e., there exists $m_\check{C} \in \check{M}$ such that $\langle m_\check{C}, y \rangle > 0$ for all $y \in \check{C} \setminus \{0\}$, and all vertices of the supporting polytope

$$\Delta(\check{C}) = \{ y \in \check{C} : \langle m_\check{C}, y \rangle = 1 \}$$

belong to $\tilde{N}$. We will call the integer $r = \langle m_\check{C}, n_C \rangle$ the **index of $C$** (or $\check{C}$). A $(\tilde{d} - 1)$-dimensional lattice polytope $\Delta$ is called **reflexive** if it is a supporting polytope of some $\tilde{d}$-dimensional reflexive Gorenstein cone $C$ of index 1. Moreover, the supporting polytope $\tilde{\Delta}^*$ of the dual cone $\check{C}$ is also reflexive polytope which is called **dual** (or **polar**) to $\check{\Delta}$.
If $C$ is a reflexive Gorenstein cone of index $r$, then $I_\Delta$ is a principal ideal generated by the element $t^{m \cdot \phi}$ of degree $r$. So one obtains the canonical isomorphism $I^{(f)}_\Delta \cong S^{(d-r)}_\Delta$. In particular, there exists the toric residue mapping
\[ \text{Res}_g : S^{(d-r)}_\Delta \to \mathbb{C} \]
which is uniquely determined by the conditions:
(i) $\text{Res}_g(h) = 0$ for any $h \in (g_1, \ldots, g_d)_S_\Delta$;
(ii) $\text{Res}_g(H'_g) = \text{Vol}(\Delta)$, where $H_g = t^{m \cdot \phi} H'_g$.

3. Cayley trick and mixed toric residues

Let $M$ be a free abelian group of rank $d$, $M_\mathbb{R} := M \otimes \mathbb{R}$, and $\Delta \subset M_\mathbb{R}$ a convex $d$-dimensional polytope with vertices in $M$. We assume that there exist $r$ convex polytopes $\Delta_1, \ldots, \Delta_r$ with vertices in $M$ such that $\Delta$ can be written as the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ (here we do not require that all polytopes $\Delta_1, \ldots, \Delta_r$ have maximal dimension $d$).

**Definition 3.1.** We set $\widetilde{M} := M \oplus \mathbb{Z}_r^r$, $\widetilde{d} := d + r$ and define the $\widetilde{d}$-dimensional Gorenstein cone $C = C(\Delta_1, \ldots, \Delta_r)$ in $\widetilde{M}_\mathbb{R} := M_\mathbb{R} \oplus \mathbb{R}^r$ as follows
\[ C := \{(\lambda_1 x_1 + \cdots + \lambda_r x_r, \lambda_1, \ldots, \lambda_r) \in \widetilde{M}_\mathbb{R} : \lambda_i \geq 0, x_i \in \Delta_i, i = 1, \ldots, r\}. \]
The $(d + r - 1)$-dimensional polytope $\Delta_1 \ast \cdots \ast \Delta_r$ defined as the intersection of the cone $C$ with the affine hyperplanes $\sum_{i=1}^r \lambda_i = 1$
\[ \Delta_1 \ast \cdots \ast \Delta_r := \{(\lambda_1 x_1 + \cdots + \lambda_r x_r, \lambda_1, \ldots, \lambda_r) : \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1, x_i \in \Delta_i\}, \]
will be called Cayley polytope associated with the Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$. It is clear that all vertices of $\Delta_1 \ast \cdots \ast \Delta_r$ are contained in $\widetilde{M}$ and
\[ \Delta_1 \ast \cdots \ast \Delta_r = \text{conv}((\Delta_1 \times \{b_1\}) \cup \cdots \cup (\Delta_r \times \{b_r\})), \]
where $\{b_1, \ldots, b_r\}$ is the standard basis of $\mathbb{Z}^r$. For fixed polytopes $\Delta_1, \ldots, \Delta_r$ we denote $\Delta_1 \ast \cdots \ast \Delta_r$ simply by $\Delta$.

**Definition 3.2.** Define $S_\Delta := \mathbb{C}[C \cap \widetilde{M}]$ to be the semigroup algebra of the monoid $C \cap \widetilde{M}$ over complex numbers. The algebra $S_\Delta$ has a natural $\mathbb{Z}_{\geq 0}^r$-grading defined by the last $r$ coordinates of lattice points in $\widetilde{M}$. By choosing an isomorphism $M \cong \mathbb{Z}^d$, we can identify $S_\Delta$ with a $\mathbb{Z}_{\geq 0}^r$-graded monomial subalgebra in
\[ \mathbb{C}[t_{d+1}^{\pm 1}, \ldots, t_{d+r}^{\pm 1}], \]
where the \( \mathbb{Z}_{\geq 0} \)-grading is considered with respect to the last \( r \) variables \( t_{d+1}, \ldots, t_{d+r} \).

We denote by \( I_{\Delta} \) the \( \mathbb{Z}_{\geq 0} \)-graded monomial ideal in \( S_{\Delta} \) generated by all lattice points in the interior of \( \bar{C} \). For any \( k = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 0}^r \), we denote by \( S^{(k)}_{\Delta} \) (resp. \( I^{(k)}_{\Delta} \)) the \( k \)-homogeneous component of \( S_{\Delta} \) (resp. \( I_{\Delta} \)). We will use also the total \( \mathbb{Z}_{\geq 0} \)-grading on \( S_{\Delta} \) and \( I_{\Delta} \). For any nonnegative integer \( l \), we denote the corresponding \( l \)-homogeneous components of \( S_{\Delta} \) and \( I_{\Delta} \) by \( S^{(l)}_{\Delta} \) and \( I^{(l)}_{\Delta} \) respectively.

So one has:

\[
S^{(l)}_{\Delta} = \bigoplus_{|k|=l} S^{(k)}_{\Delta}, \quad I^{(l)}_{\Delta} = \bigoplus_{|k|=l} I^{(k)}_{\Delta},
\]

where \( |k| = k_1 + \cdots + k_r \).

Let \( f_1(t), \ldots, f_r(t) \) be Laurent polynomials in \( \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \) such that \( \Delta_i \) is the Newton polytope of \( f_i \) (\( 1 \leq i \leq r \)). We set

\[
F(t) := t_{d+1}f_1(t) + \cdots + t_{d+r}f_r(t).
\]

It is easy to see that \( \bar{\Delta} = \Delta_1 \ast \cdots \ast \Delta_r \) is the Newton polytope of \( F \). Moreover, using the decomposition

\[
S^{(l)}_{\Delta} = \bigoplus_{|k|=l} S^{(k)}_{\Delta} = \bigoplus_{i=1}^r S^{(k)}_{\Delta},
\]

we see that every Laurent polynomial \( G \) in \( \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, t_{d+1}, \ldots, t_{d+r}] \) with the Newton polytope \( \bar{\Delta} \) can be obtained from the sequence of arbitrary Laurent polynomials \( g_1, \ldots, g_r \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \) by the formula \( G = t_{d+1}g_1 + \cdots + t_{d+r}g_r \), where \( \Delta_i \) is the Newton polytope of \( g_i \) (\( 1 \leq i \leq r \)). The above correspondence \( \{f_1, \ldots, f_r\} \mapsto F \) is usually called Cayley trick. We call \( F = t_{d+1}f_1 + \cdots + t_{d+r}f_r \) the Cayley polynomial associated with \( f_1, \ldots, f_r \).

**Definition 3.3.** Let \( \Delta_1, \ldots, \Delta_r \subset M_{\mathbb{R}} \) be convex polytopes with vertices in \( M \) such \( \Delta = \Delta_1 + \cdots + \Delta_r \), has dimension \( d \). We say that \( r \) Laurent polynomials

\[
f_i(t) = \sum_{m \in \Delta_i \cap M} a^{(i)}_m t^m, \quad i = 1, \ldots, r
\]

form a \( \bar{\Delta} \)-regular sequence if the corresponding Cayley polynomial \( F \) is \( \bar{\Delta} \)-regular, i.e., the polynomials

\[
F_i := t_i \partial / \partial t_i F, \quad i = 1, \ldots, d + r
\]

form a regular sequence in \( S_{\Delta} \).

**Definition 3.4.** Let \( f_1(t), \ldots, f_r(t) \) be Laurent polynomials with Newton polytopes \( \Delta_1, \ldots, \Delta_r \) as above, \( F = t_{d+1}f_1 + \cdots + t_{d+r}f_r \) the corresponding Cayley polynomial, and

\[
H_F := \det \left( t_i \frac{\partial F_j}{\partial t_j} \right)_{1 \leq i, j \leq d + r} = \det \left( \left( t_i \frac{\partial}{\partial t_i} \right) \left( t_j \frac{\partial}{\partial t_j} \right) F \right)_{1 \leq i, j \leq d + r} \in I^{(d+r)}_{\Delta} \subset S^{(d+r)}_{\Delta}
\]
the Hessian of $F$. For any $k = (k_1, \ldots, k_r)$ with $|k| = d + r$ we define $H^k_F \in I^k_\Delta$ to be the $k$-homogeneous component of $H_F$. The polynomial $H^k_F$ will be called $k$-mixed Hessian of $f_1, \ldots, f_r$.

**Remark 3.5.** Since the last $r$ rows of the matrix
\[
\begin{pmatrix}
(t_i \frac{\partial}{\partial t_i}) & (t_j \frac{\partial}{\partial t_j}) & F \\
\end{pmatrix}_{i,j \leq d+r}
\]
are divisible respectively by $t_{d+1}, \ldots, t_{d+r}$, the Hessian $H_F$ is divisible by the monomial $t_{d+1} \cdots t_{d+r}$. Therefore $H^k_F = 0$ if one of the coordinates $k_i$ of $k - (k_1, \ldots, k_r)$ is zero. In particular, one has
\[
H_F = \sum_{k \in \mathbb{Z}_{>0}^r, |k| = d + r} H^k_F.
\]

Let $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ be a solution of the linear Diophantine equation
\[
|k| = k_1 + \cdots + k_r = d + r.
\]
For any $r$ subsets $S_i \subset \Delta_i \cap M$ such that $|S_i| = k_i$ $(1 \leq i \leq r)$ we define the nonnegative integer $\nu(S_1, \ldots, S_r)$ as follows: choose an element $s_i$ in each $S_i$ $(1 \leq i \leq r)$, define $S$ to be the $d \times r$-matrix whose rows are all possible nonzero vectors $s - s_i$, where $s \in S_i$, $1 \leq i \leq r$, and set $\nu(S_1, \ldots, S_r) := (\det S)^2$. It is easy to see that up to sign $\det S$ does not depend on the choice of elements $s_i \in S_i$ and therefore $\nu(S_1, \ldots, S_r)$ is well defined.

**Proposition 3.6.** Let $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ be a positive integral solution of the linear Diophantine equation
\[
|k| = k_1 + \cdots + k_r = d + r.
\]
Then the mixed Hessian can be computed by the following formula
\[
H^k_F = t_{d+1}^{k_1} \cdots t_{d+r}^{k_r} \sum_{(S_1, \ldots, S_r)} \nu(S_1, \ldots, S_r) \prod_{i=1}^r \prod_{s_i \in S_i} a_{s_i}^{(i)} t^{s_i},
\]
where the sum runs over all $r$-tuples $(S_1, \ldots, S_r)$ of subsets $S_i \subset \Delta_i \cap M$ such that $|S_i| = k_i$ $(1 \leq i \leq r)$.

**Proof.** The formula for $H^k_F$ follows immediately from the formula in 2.4(b) applied to the Cayley polytope $\Delta$.

**Definition 3.7.** Let $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ be a positive integral solution of the equation
\[
|k| = k_1 + \cdots + k_r = d + r.
\]
Consider the toric residue
\[
\text{Res}_F : f^{(d+r)}_\Delta \to \mathbb{C}
\]
defined as a \(\mathbb{C}\)-linear map which vanishes on \(\langle F_1, \ldots, F_{d+r}\rangle I_\Delta\) and sends \(H_F\) to \(\text{Vol}(\tilde{\Delta}) = \text{Vol}(\Delta_1 \ast \cdots \ast \Delta_r)\). The restriction \(\text{Res}^k_F\) of \(\text{Res}_F\) to the \(k\)-th homogeneous component \(I^k_\Delta\):

\[
\text{Res}^k_F : I^k_\Delta \to \mathbb{C}
\]

will be called the \(k\)-mixed toric residue associated with \(f_1, \ldots, f_r\).

Since \(H^k_F\) is an element of \(I^k_\Delta\), it is natural to ask about the value of \(\text{Res}^k_F(H^k_F)\).

**Conjecture 3.8.** Let \(k = (k_1, \ldots, k_r) \in \mathbb{Z}_{>0}^r\) be a positive integral solution of \(k_1 + \cdots + k_r = d + r\). We set \(\bar{k} = (\bar{k}_1, \ldots, \bar{k}_r) := (k_1 - 1, \ldots, k_r - 1)\). Then

\[
\text{Res}^k_F(H^k_F) = V(\underbrace{\Delta_1 \ast \cdots \ast \Delta_1}_{\bar{k}_1}, \ldots, \underbrace{\Delta_r \ast \cdots \ast \Delta_r}_{\bar{k}_r}),
\]

where \(V(\Theta_1, \ldots, \Theta_d)\) denotes the mixed volume of convex polytopes \(\Theta_1, \ldots, \Theta_d\) multiplied by \((d + r - 1)\)!

Our conjecture agrees with a result of Danilov and Khovanskii:

**Proposition 3.9.** [DK, \S 6] The normalized volume of the Cayley polytope \(\tilde{\Delta} = \Delta_1 \ast \cdots \ast \Delta_r\) can be computed by the following formula:

\[
\text{Vol}(\Delta_1 \ast \cdots \ast \Delta_r) = \sum_{|\bar{k}| = d} V(\underbrace{\Delta_1 \ast \cdots \ast \Delta_1}_{\bar{k}_1}, \ldots, \underbrace{\Delta_r \ast \cdots \ast \Delta_r}_{\bar{k}_r}).
\]

**Remark 3.10.** Let \(r = d\) and \(k = (d + 1, 1, \ldots, 1)\). It follows from 3.6 and 2.4(b) that

\[
H^k_F = H_{t_{d+1}, f_1}(t_{d+2}, f_2) \cdots (t_{2d}, f_d),
\]

where

\[
H_{t_{d+1}, f_1} = \det \left( \begin{array}{cc}
 t_i & \frac{\partial}{\partial t_i} \\
 t_j & \frac{\partial}{\partial t_j}
\end{array} \right)_{1 \leq i, j \leq d+1}.
\]

On the other hand, we have

\[
V(\underbrace{\Delta_1 \ast \cdots \ast \Delta_1}_{\bar{k}_1}, \ldots, \underbrace{\Delta_r \ast \cdots \ast \Delta_r}_{\bar{k}_r}) = V(\underbrace{\Delta_1 \ast \cdots \ast \Delta_1}_{d}) = \text{Vol}(\Delta_1).
\]

Therefore, Conjecture 3.8 can be considered as a "generalization" of 2.5(ii).

It is easy to show that the cone \(F\) from 3.1 is a reflexive Gorenstein cone of index \(r\) if and only if \(\Delta = \Delta_1 + \cdots + \Delta_r\) is a reflexive polytope. In this situation, we have

\[
I_\Delta = t_{d+1} \cdots t_{d+r} S_\Delta.
\]

Therefore one has canonical isomorphisms:

\[
I^k_\Delta \cong S^k_\Delta, \quad \forall k \in \mathbb{Z}_{>0},
\]
where the monomial basis in $S^\mathbf{k}_\Delta$ can be identified with the set of all lattice points in \\
$\mathbf{k}_1\Delta_1 + \cdots + \mathbf{k}_r\Delta_r$. The $\mathbf{k}$-homogeneous component of corresponding toric residue map \\
\\n$$
\text{Res}^\mathbf{k}_\Delta : S^\mathbf{k}_\Delta \rightarrow \mathbb{C}.
$$
\\nwill be also called $\mathbf{k}$-mixed toric residue.

4. Toric Residue Mirror Conjecture

Let $M$ and $N = \text{Hom}(M, \mathbb{Z})$ be the dual to each other abelian groups of rank $d$,
$M_\mathbb{R}$ and $N_\mathbb{R}$ their $\mathbb{R}$-scalar extensions and $\Delta \subset M_\mathbb{R}$ a reflexive polytope with the
unique interior lattice point $0 \in M$. Denote by $\mathbb{P}_\Delta$ a Gorenstein toric Fano variety
associated with $\Delta$. Let $D_1, \ldots, D_s$ be the toric divisors on $\mathbb{P}_\Delta$ corresponding to
the codimension-1 faces $\Theta_1, \ldots, \Theta_s$ and $e_1, \ldots, e_s$ the vertices of the dual reflexive
polytope $\Delta^* \subset N_\mathbb{R}$ such that
\\n$$
\Delta = \{ x \in M_\mathbb{R} : \langle x, e_j \rangle \geq -1, \; j = 1, \ldots, s \},
$$
\\n$$
\Theta_j = \Delta \cap \{ x \in M_\mathbb{R} : \langle x, e_j \rangle = -1 \}, \; j = 1, \ldots, s.
$$
\\
Definition 4.1. A Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ is called a nef-partition
of the reflexive polytope $\Delta$ if all vertices of $\Delta_1, \ldots, \Delta_r$ belong to $M$, and
\\n$$
\min_{x \in \Delta_i} \langle x, e_j \rangle \in \{ 0, -1 \}, \; \forall 1 \leq i \leq r, \; \forall 1 \leq j \leq s.
$$
\\
Since $\min_{x \in \Delta} \langle x, e_j \rangle = -1$ for all $j \in \{ 1, \ldots, s \}$, the equality $\min_{x \in \Delta_i} \langle x, e_j \rangle = -1$
holds exactly for one index $i \in \{ 1, \ldots, r \}$ if we fix a vertex $e_j \in \Delta^*$. Therefore,
we can split the set of vertices $\{ e_1, \ldots, e_s \} \subset \Delta^*$ into a disjoint union of subsets
$B_1, \ldots, B_r$ where
\\n$$
B_i := \{ e_j : j \in \{ 1, \ldots, s \}, \; \min_{x \in \Delta_i} \langle x, e_j \rangle = -1 \}.
$$
\\
Now we can define $r$ nef Cartier divisors
\\n$$
E_i := \sum_{j : e_j \in B_i} D_j, \; i = 1, \ldots, r.
$$
\\
Therefore, a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ of polytopes induces a partition of
the anti-canonical divisor $-K_{\mathbb{P}_\Delta} = D_1 + \cdots + D_n$ of $\mathbb{P}_\Delta$ into a sum of $r$ nef Cartier divisors:
\\n$$
-K_{\mathbb{P}_\Delta} = E_1 + \cdots + E_r.
$$
\\
Now it is easy to see that the above definition of the nef-partition is equivalent to the definition given in [Bo].
Definition 4.2. If $\Delta = \Delta_1 + \cdots + \Delta_r$ is a nef-partition, then for any $i = 1, \ldots, r$ we denote

$$\nabla_i := \{ y \in N_\mathbb{R} : \langle x, y \rangle \geq -\delta_{ij}, \ x \in \Delta_j, \ j = 1, \ldots, r \}. $$

The lattice polytopes $\nabla_1, \ldots, \nabla_r$ define another nef-partition $\nabla := \nabla_1 + \cdots + \nabla_r$ of the reflexive polytope $\nabla \subset N_\mathbb{R}$ which is called dual nef-partition.

The lattice polytopes $\nabla_1, \ldots, \nabla_r$ can be also defined as

$$\nabla_j := \text{conv}(\{0\} \cup B_j) \subset M_\mathbb{R}, \quad j = 1, \ldots, r.$$  

Moreover, one has two dual reflexive polytopes

$$\Delta^* = \text{conv}(\nabla_1 \cup \cdots \cup \nabla_r) \subset N_\mathbb{R}$$  

$$\nabla^* = \text{conv}(\Delta_1 \cup \cdots \cup \Delta_r) \subset M_\mathbb{R}.$$  

Nef-partition $\Delta = \Delta_1 + \cdots + \Delta_r$ defines a family of $(d-r)$-dimensional Calabi-Yau complete intersections defined by vanishing of $r$ Laurent polynomials $f_1, \ldots, f_r$ with Newton polytopes $\Delta_1, \ldots, \Delta_r$. According to [Bo], the dual nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ defines the mirror dual family of Calabi-Yau complete intersections.

Define $A_j$ to be a subset in $\Delta_j \cap M$ containing all vertices of $\Delta_j$ and set $A_j := A_j \setminus \{0\}, \ j = 1, \ldots, r$. It is easy to see that $A_i \cap A_j = \emptyset$ for all $i \neq j$. We set $A_1 \cup \cdots \cup A_r := \{v_1, \ldots, v_n\}$ and define $a_1, \ldots, a_n \in \mathbb{C}$ to be the coefficients of the Laurent polynomials

$$f_j(t) := 1 - \sum_{i : v_i \in A_j} a_i t^{v_i}, \quad j = 1, \ldots, r.$$  

Let $A := \{0\} \cup A_1 \cup \cdots \cup A_r$ and $\tilde{\Delta} = \Delta_1 \ast \cdots \ast \Delta_r$ be the Cayley polytope. Denote by $\pi$ the injective mapping

$$A_1 \cup \cdots \cup A_r \to \tilde{\Delta} \cap \tilde{M}$$

which sends a nonzero lattice point $m \in A_j$ to $(m, b_j)$ ($1 \leq j \leq r$) and define

$$\tilde{A} := \pi(A_1 \cup \cdots \cup A_r) \cup \{(0, b_1), \ldots, (0, b_r)\}.$$  

We hold notations from [BM1, §4].

Definition 4.3. Choose a coherent triangulation $\mathcal{T} = \{\tau_1, \ldots, \tau_p\}$ of the reflexive polytope $\nabla^* = \text{conv}(\Delta_1 \cup \cdots \cup \Delta_r)$ associated with $A$ such that 0 is a vertex of all its $d$-dimensional simplices $\tau_1, \ldots, \tau_p$. Define a coherent triangulation $\tilde{\mathcal{T}} = \{\tilde{\tau}_1, \ldots, \tilde{\tau}_p\}$ of $\tilde{\Delta} = \Delta_1 \ast \cdots \ast \Delta_r$ associated with $\tilde{A}$ as follows: a $(d + r - 1)$-dimensional simplex $\tilde{\tau}_i \in \tilde{\mathcal{T}}$ is the convex hull of $\pi$-images of all nonzero vertices of $\tau$ and $\{(0, b_1), \ldots, (0, b_r)\}$. We call $\tilde{\mathcal{T}}$ the induced triangulation of $\tilde{\Delta}$. 
Let \( P := P_{\Sigma(T)} \) be the \( d \)-dimensional simplicial toric variety defined by the fan \( \Sigma(T) \subset M_\mathbb{R} \) (\( P \) is a partial crepant desingularization of the Gorenstein toric Fano variety \( P_V \)) and denote by \( P_\beta \) the Morrison-Plesser moduli space [BM1, Definition 3.3] corresponding to a lattice point
\[
\beta = (\beta_1, \ldots, \beta_n) \in R(\Sigma) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 v_1 + \cdots + x_n v_n = 0\}
\]
in the Mori cone \( K_{\text{eff}}(P) \). One has a canonical surjective homomorphism
\[
\psi_\beta : H^2(P, \mathbb{Q}) \to H^2(P_\beta, \mathbb{Q}).
\]

**Definition 4.4.** By abuse of notations, let us denote by \([D_j] \in H^2(P_\beta, \mathbb{Q})\) (\(1 \leq j \leq n\)) the image of \([D_j] \in H^2(P, \mathbb{Q})\) under \(\psi_\beta\). Using the multiplication in the cohomology ring \(H^\ast(P_\beta, \mathbb{Q})\), we define the intersection product
\[
\Phi_\beta := [E_1]^{(E_1, \beta)} \cdots [E_r]^{(E_r, \beta)} \prod_{i : (D_i, \beta) < 0} [D_i]^{-(D_i, \beta) - 1}
\]
considered as a cohomology class in \(H^{2(\dim P_\beta - d)}(P_\beta, \mathbb{Q})\) and call \(\Phi_\beta\) the **Morrison-Plesser class** corresponding to the nef-partition \(\Delta = \Delta_1 + \cdots + \Delta_r\).

**Definition 4.5.** Let \( k = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 0}^r \) be a positive integral solution of
\[
|k| = k_1 + \cdots + k_r = d + r.
\]

A polynomial \( P(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) is called \( k \)-homogeneous if it is homogeneous of degree \( k_i = k_i - 1 \) with respect to every group of \(|A_i|\) variables \( x_j \) \((v_j \in A_i)\) \((1 \leq i \leq r)\).

Now we are able to formulate a generalized Toric Residue Mirror Conjecture:

**Conjecture 4.6.** Let \( \Delta = \Delta_1 + \cdots + \Delta_r \) and \( \nabla = \nabla_1 + \cdots + \nabla_r \) be two arbitrary dual nef-partitions. Choose any coherent triangulation \( T = \{\tau_1, \ldots, \tau_p\} \) of \( \nabla^\ast \) associated with \( A \) such that 0 is a vertex of all the simplices \( \tau_1, \ldots, \tau_p \) as above. Then for any \( k \)-homogeneous polynomial \( P(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) of degree \( d \) the Laurent expansion of the \( k \)-mixed toric residue
\[
R_P(a) := (-1)^d \text{Res}^\mathbf{\bar{\Delta}}_{\mathbf{F}}(t_{d+1}^{k_1} \cdots t_{d+r}^{k_r} P(a_1 t_1^{\tau_1}, \ldots, a_n t_\tau^{\tau_p}))
\]
at the vertex \( v_{\bar{\tau}} \in \text{Sec}(\mathbf{\bar{\Delta}}) \) corresponding to the induced triangulation \( \bar{T} = \{\bar{\tau}_1, \ldots, \bar{\tau}_p\} \) coincides with the generating function of intersection numbers
\[
I_P(a) := \sum_{\beta \in K_{\text{eff}}(P)} I(P, \beta) a^\beta,
\]
where the sum runs over all integral points \( \beta = (\beta_1, \ldots, \beta_n) \) of the Mori cone \( K_{\text{eff}}(P) \), \( a^\beta := a_1^{\beta_1} \cdots a_n^{\beta_n} \),
\[
I(P, \beta) = \int_{P_\beta} P([D_1], \ldots, [D_n]) \Phi_\beta = \langle P([D_1], \ldots, [D_n]) \Phi_\beta \rangle_\beta,
\]
and \( \Phi_\beta \in H^{2(\dim \mathbb{P}_\beta - d)}(\mathbb{P}_\beta, \mathbb{Q}) \) is the Morrison-Plesser class of \( \mathbb{P}_\beta \). We assume 
\( I(P, \beta) \) to be zero if \( \mathbb{P}_\beta \) is empty.

5. Complete intersections in weighted projective spaces

Let \( \mathbb{P} = \mathbb{P}(w_1, \ldots, w_n) \) be a \( d \)-dimensional weighted projective space, \( n = d + 1 \). The fan \( \Sigma \) of \( \mathbb{P}(w_1, \ldots, w_n) \) is determined by \( n \) vectors \( v_1, \ldots, v_n \in M \cong \mathbb{Z}^d \) which generate \( M \) and satisfy the relation

\[ w_1v_1 + \cdots + w_nv_n = 0. \]

If we assume that \( \gcd(w_1, \ldots, w_n) = 1 \) and

\[ w_i(w_1 + \cdots + w_n), \quad i = 1, \ldots, n, \]
then \( \mathbb{P} \) is a Gorenstein toric Fano variety with the anticanonical divisor \( -K_\mathbb{P} = D_1 + \cdots + D_n \), where \( D_i \) is the toric divisor corresponding to the vector \( v_i \). These divisors are related modulo rational equivalence as

\[ \frac{[D_1]}{w_1} = \cdots = \frac{[D_n]}{w_n} =: [D_0]. \]

Consider a decomposition \( \{v_1, \ldots, v_n\} \) into a disjoint union of \( r \) nonempty subsets \( A_1, \ldots, A_r \) and define the divisors \( E_i := \sum_{j : v_j \in A_i} D_j \) on \( \mathbb{P} \) such that \([E_i] = d_i[D_0]\), where \( d_j = \sum_{i \in A_j} w_i \), \( j = 1, \ldots, r \). Note that the integers \( d_i \) satisfy \( d_1 + \cdots + d_r = w_1 + \cdots + w_n \). Let \( \Delta_i := \text{conv}(\{0\} \cup A_i) \) \( (1 \leq i \leq r) \). The polytopes \( \Delta_1, \ldots, \Delta_r \) define a nef-partition \( \Delta := \Delta_1 + \cdots + \Delta_r \) if and only if

\[ w_i|d_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, r. \]

The following result generalizes [BM1, Theorem 7.3]:

**Theorem 5.1.** Let \( P \in \mathbb{Q}[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( d \). Then the generating function of intersection numbers on the Morrison-Plesser moduli spaces has the form

\[ I_P(y) = \nu \cdot P(w_1, \ldots, w_n) \sum_{b \geq 0} \mu^b y^b = \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y}, \]

where

\[ \nu := \frac{1}{w_1 \cdots w_n}, \quad \mu := \frac{d_1^{w_1} \cdots d_r^{w_n}}{w_1^{w_1} \cdots w_n^{w_n}}, \quad y := a_1^{w_1} \cdots a_n^{w_n}. \]

**Proof.** The lattice points \( \beta \) in the Mori cone of \( \mathbb{P} \) correspond to the linear relations

\[ bw_1v_1 + \cdots + bw_nv_n = 0, \quad b \in \mathbb{Z}_{\geq 0}. \]

Therefore we set \( y := a_1^{w_1} \cdots a_n^{w_n} \).
The Morrison-Plesser moduli space \( \mathbb{P}_\beta \) is the \( (\sum_{i=1}^n w_i)b + d \)-dimensional weighted projective space:

\[
\mathbb{P}(w_1, \ldots, w_1, \ldots, w_n, \ldots, w_n).
\]

It is easy to see that the Morrison-Plesser class defined by the nef-partition is

\[
\Phi_\beta = (d_1[D_0])^{\frac{d_1}{b+1}} \cdots (d_r[D_0])^{\frac{d_r}{b+1}}.
\]

Using \( \langle [D_0]^{\dim \mathbb{P}_\beta} \rangle_\beta = 1/w_1^{w_1 b+1} \cdots w_n^{w_n b+1} \), we obtain

\[
I_P(y) = \sum_{b \geq 0} \langle P([D_1], \ldots, [D_n])(d_1[D_0])^{\frac{d_1}{b+1}} \cdots (d_r[D_0])^{\frac{d_r}{b+1}} \rangle_\beta y^b
\]

\[
= P(w_1, \ldots, w_n) \sum_{b \geq 0} (d_1^{d_1} \cdots d_r^{d_r})^b \langle [D_0]^{\dim \mathbb{P}_\beta} \rangle_\beta y^b
\]

\[
= P(w_1, \ldots, w_n) \sum_{b \geq 0} (d_1^{d_1} \cdots d_r^{d_r})^b \frac{1}{w_1^{w_1 b+1} \cdots w_n^{w_n b+1}} y^b
\]

\[
= \nu \cdot P(w_1, \ldots, w_n) \sum_{b \geq 0} \mu^b y^b
\]

\[
= \nu \cdot P(w_1, \ldots, w_n) \frac{1}{1 - \mu y}.
\]

\[\square\]

The convex hull of the vectors \( v_1, \ldots, v_n \) is a reflexive polytope \( \nabla^* \subset M_\mathbb{R} \cong \mathbb{R}^d \). Let \( \tilde{M} := M \oplus \mathbb{Z}^r \) be an extension of the lattice \( M \) and \( \{ b_1, \ldots, b_r \} \) the standard basis of \( \mathbb{Z}^r \). The \((d + r - 1)\)-dimensional Cayley polytope

\[
\tilde{\Delta} = \Delta_1 \ast \cdots \ast \Delta_r
\]

is the convex hull of \((d + r + 1)\) points: \((0, b_1), \ldots, (0, b_r)\) and \((v_k, b_j)\) \((k = 1, \ldots, d + 1)\), where \( v_k \in A_j \). We denote this set of points by \( \tilde{A} \). The points from \( \tilde{A} \) are affinely dependent, while any proper subset of \( \tilde{A} \) is affinely independent, i.e., defines a circuit (see [GKZ, Chapter 7]). It is easy to see that the only affine relation (up to a real multiple) between the points from \( A \) is

\[
d_1 e_1 + \cdots + d_r e_r - w_1 u_1 - \cdots - w_n u_n = 0.
\]

Thus by [GKZ, Chapter 7, Proposition 1.2], polytope \( \tilde{\Delta} \) has exactly two triangulations: the triangulation \( \mathcal{T} = \mathcal{T}_1 \) with the simplices \( \text{conv}(A \setminus \{e_i\}), \ i = 1, \ldots, r \), and the triangulation \( \mathcal{T}_2 \) with the simplices \( \text{conv}(A \setminus \{u_k\}), \ k = 1, \ldots, n \). Note that

(1) \[\text{Vol}(\text{conv}(A \setminus \{e_i\})) = d_i, \ i = 1, \ldots, r,\]

(2) \[\text{Vol}(\text{conv}(A \setminus \{v_k\})) = w_k, \ k = 1, \ldots, n.\]
Therefore \( \text{Vol}(\bar{\Delta}) = \sum_{i=1}^r d_i = \sum_{k=1}^n w_k. \)

Let 
\[
f_j(t) := 1 - \sum_{i : v_i \in A_j} a_i t^{v_i} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}], \quad j = 1, \ldots, r
\]
be generic Laurent polynomials. Denote by 
\[
F(t) = t_{d+1} f_1(t) + \cdots + t_{d+r} f_r(t)
\]
a Laurent polynomial whose support polytope is \( \bar{\Delta}. \)

The next statement follows directly from [GKZ, Chapter 9, Proposition 1.8] and from the equalities (1), (2).

**Proposition 5.2.** The \( A \)-discriminant of \( F \) is equal (up to sign) to the binomial
\[
D_A(F) = \prod_{k=1}^n w_k^{w_k} - \prod_{i=1}^r d_i \prod_{k=1}^n a_k^{w_k} = \prod_{k=1}^n w_k^{w_k}(1 - \mu y),
\]
where \( y = \prod_{k=1}^n a_k^{w_k} \) and the first summand in \( D_A(F) \) corresponds to the triangulation \( \mathcal{T} \).

**Theorem 5.3.** Let \( P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) be a \( \bar{k} \)-homogeneous polynomial with \( |\bar{k}| = d \). Then
\[
R_P(a) = (-1)^d \text{Res}_F^k \left( t_1^{v_1} \cdots t_d^{v_d} P(a_1 x_1, \ldots, a_n x_n) \right) = \frac{\nu \cdot P(w_1, \ldots, w_n)}{1 - \mu y},
\]
where \( y := a_1^{w_1} \cdots a_n^{w_n} \).

**Proof.** By Proposition 2.6 the toric residue \( R_P(a) \) is the following sum over the critical points \( \xi \) of the polynomial \( F_1(t, y) := f_1(t) + y_2 f_2(t) + \cdots + y_r f_r(t) \), where \( (t, y) \in (\mathbb{C}^*)^d \times (\mathbb{C}^*)^{r-1} \):
\[
R_P(a) = (-1)^d \sum_{\xi \in \mathcal{V}_1} \frac{P(a_1 \xi^{v_1}, \ldots, a_n \xi^{v_n})}{F_1(\xi)} H_{F_1}(\xi).
\]

We rewrite polynomial \( F_1 \) as
\[
F_1 = y_2 + \cdots + y_r + 1 - \sum_{i=1}^n c_i t^{v_i},
\]
where \( c_i = y_j \cdot a_i \) if \( v_i \in A_j \). Then at the critical point \( \xi \), we have
\[
\frac{\xi^{v_1}}{w_1} = \cdots = \frac{\xi^{v_n}}{w_n} = z
\]
and
\[
z^{w_1 + \cdots + w_n} = \left( \frac{c_1}{w_1} \right)^{w_1} \cdots \left( \frac{c_n}{w_n} \right)^{w_n}.
\]
Moreover, at the critical points one has:

\[ f_2(\xi) = \cdots = f_r(\xi) = 0, \]

which is equivalent to

\[
f_j(\xi) = 1 - \sum_{i: w_i \in A_j} a_i \xi^{w_i} = 1 - \left( \sum_{i: w_i \in A_j} w_i \right) \frac{z}{\eta_j} = 1 - \frac{d_j z}{\eta_j} = 0, \quad j = 2, \ldots, r,
\]

where \( \eta_j \) is the value of \( y_j \) at the critical point. Hence, it is easy to see that \( \eta_j = d_j z, \) \( (j = 2, \ldots, r) \) and \( F_1 = 1 - d_1 z, \) which implies

\[
z^{w_1 + \cdots + w_n} = \left( \frac{a_1}{w_1} \right)^{w_1} \cdots \left( \frac{a_n}{w_n} \right)^{w_n} d_2^{d_2} \cdots d_r^{d_r} z^{d_2 + \cdots + d_r},
\]

or, equivalently,

\[
z^{d_1} = \left( \frac{a_1}{w_1} \right)^{w_1} \cdots \left( \frac{a_n}{w_n} \right)^{w_n} d_2^{d_2} \cdots d_r^{d_r}.
\]

The value of the Hessian \( H^1_F \) at \( \xi \) equals

\[
H^1_F(\xi) = (-1)^d w_1 \cdots w_n d_1 \cdots d_r z^{d+r-1}.
\]

Since there are exactly \( w_1 + \cdots + w_n = d_1 + \cdots + d_r \) critical points of \( F, \) the summation over the critical points is equivalent to the summation over the roots of (3), we get

\[
R_P(y) = \sum_{x^{d_1} = \left( \frac{a_1}{w_1} \right)^{w_1} \cdots \left( \frac{a_n}{w_n} \right)^{w_n} d_2^{d_2} \cdots d_r^{d_r}} \frac{P(w_1, \ldots, w_n)}{w_1 \cdots w_n d_1 (1 - d_1 z)}
= \frac{P(w_1, \ldots, w_n)}{w_1 \cdots w_n} \sum_{b \geq 0} (d_1^{d_1} \cdots d_r^{d_r})^{b} \left( \frac{a_1^{w_1} \cdots a_n^{w_n}}{w_1^{w_1} \cdots w_n^{w_n}} \right)^{b}
= \nu \cdot P(w_1, \ldots, w_n).
\]

\[
6. \text{ Complete intersections in product of projective spaces}
\]

In this section, we check the Toric Residue Mirror Conjecture for nef-partitions corresponding to mirrors of complete intersections in product of projective spaces \( P = P^{d_1} \times \cdots \times P^{d_p} \) of dimension \( d = d_1 + \cdots + d_p. \) We set \( n_i := d_i + 1 \) and denote by \( N = (n_{ij}) \) an integral \( p \times r \)-matrix with non-negative elements having columns \( n_1, \ldots, n_r \in \mathbb{Z}_{\geq 0}^p. \) A complete intersection \( V \) of \( r \) hypersurfaces \( V_1, \ldots, V_r \) in \( P \) of
multidegrees \( n_1, \ldots, n_r \) is a Calabi-Yau \((d - r)\)-fold if and only if \( \sum_{j=1}^r n_{ij} = n_i \) \((i = 1, \ldots, p)\). We will use the standard notation

\[
\begin{pmatrix}
\mathbb{P}^{d_1} & n_{11} & \cdots & n_{1r} \\
\vdots & \vdots & & \vdots \\
\mathbb{P}^{d_p} & n_{p1} & \cdots & n_{pr}
\end{pmatrix}
\]

to denote this complete intersection.

The cone of effective curves \( K_{\text{eff}}(\mathbb{P}) \) is isomorphic to \( \mathbb{R}_{\geq 0}^p \) and its integral part \( K_{\text{eff}}(\mathbb{P})_{\mathbb{Z}} \) consists of the points \( \beta = (b_1, \ldots, b_p) \in \mathbb{Z}_{\geq 0}^p \). Thus, the Morrison-Plesser moduli spaces are the products of projective spaces: \( \mathbb{P}_\beta = \mathbb{P}^{n_1 b_1 + d_1} \times \cdots \times \mathbb{P}^{n_p b_p + d_p} \) and the generating function for intersection numbers may be written

\[
I_\beta(y) = \sum_{b_1, \ldots, b_p \geq 0} I(P, \beta) y_1^{b_1} \cdots y_p^{b_p}.
\]

**Theorem 6.1.** The generating function for intersection numbers associated with monomial \( x^k = x_1^{k_1} \cdots x_p^{k_p} \) can be written as the integral

\[
I_{x^k}(y) = \left( \frac{1}{2\pi i} \right)^p \int_\Gamma \frac{z_1^{k_1} \cdots z_p^{k_p} dz_1 \wedge \cdots \wedge dz_p}{G_1(z) \cdots G_p(z)},
\]

where the polynomials \( G_i \) have the form

\[
G_i = z_i^{n_i} - \prod_{j=1}^r (n_{ij} z_1 + \cdots + n_{pj} z_p)^{n_{ij}}, \quad i = 1, \ldots, p,
\]

and \( \Gamma \) is the compact cycle in \( \mathbb{CP}^1 \) defined by \( \Gamma = \{|G_1| = \cdots = |G_p| = \epsilon\} \) for small positive \( \epsilon \).

**Proof.** Let \([H_1]\) denotes the class of hyperplane section in \( \mathbb{P}^{d_1} \). The class of the divisor \( E_j \) defining hypersurface \( V_j \) equals

\[
[E_j] = n_{1j} [H_1] + \cdots + n_{pj} [H_p], \quad j = 1, \ldots, r.
\]

Hence, the coefficients of the series \( I_{x^k}(y) \) are

\[
([H_1]^{k_1} \cdots [H_p]^{k_p} \prod_{j=1}^r (n_{1j}[H_1] + \cdots + n_{pj}[H_p])^{n_{1j} b_1 + \cdots + n_{pj} b_p})_\beta.
\]

The lattice points \( \beta = (b_1, \ldots, b_p) \in \mathbb{Z}_{\geq 0}^r \) in the integral part of the Mori cone \( K_{\text{eff}}(\mathbb{P}) \) correspond to the \( p \) linear relations

\[
b_1 u_{i1} + \cdots + b_i u_{im_i} = 0, \quad i = 1, \ldots, p,
\]
where \(v_{j1}, \ldots, v_{jn_j}\) generate lattice \(M_j\) of rank \(d_j\) (\(1 \leq j \leq p\)). Therefore we set \(y_i := a_{i1} \cdots a_{i n_i}.\) Using the property of the integral
\[
\left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} z_1^{m_1-1} \cdots z_p^{m_p-1} \, dz = \begin{cases}
1, & m_1 = \cdots = m_p = 0, \\
0, & \text{otherwise},
\end{cases}
\]
where \(\gamma_\rho = \{ |z_1| = \cdots = |z_p| = \rho \}\) is the cycle winding around the origin (\(\rho > 0\) is small) and the fact that the intersection numbers on \(\mathbb{P}\) are
\[
\langle [H_1]^i \cdots [H_p]^j \rangle_\beta = \begin{cases}
1, & l_j = n_j b_j + d_j, \quad j = 1, \ldots, r, \\
0, & \text{otherwise},
\end{cases}
\]
we can represent the functions \(I(x^k, \beta)\) by integrals
\[
I(x^k, \beta) = \left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} z_1^{k_1} \cdots z_p^{k_p} \prod_{j=1}^r (n_{ij} z_1 + \cdots + n_{pj} z_p)^{n_{ij} b_j + \cdots + n_{pj} b_p} \, dz \\
\frac{z_1^{n_1 b_1 + d_1 + 1} \cdots z_p^{n_p b_p + d_p + 1}}{z_1^{n_1 b_1 + d_1 + 1} \cdots z_p^{n_p b_p + d_p + 1}}.
\]

Denote
\[
F_i(z) := \prod_{j=1}^r (n_{ij} z_1 + \cdots + n_{pj} z_p)^{n_{ij}}, \quad i = 1, \ldots, p.
\]
If \(z \in \gamma_\rho\) for some fixed \(\rho\), then the geometric series
\[
z_1^{k_1 - n_1} \cdots z_p^{k_p - n_p} \sum_{b_1, \ldots, b_p \geq 0} \left(\frac{F_1(z) y_1}{z_1^{n_1}}\right)^{b_1} \cdots \left(\frac{F_p(z) y_p}{z_p^{n_p}}\right)^{b_p} = \prod_{i=1}^p \left(\frac{z_1^{k_1} \cdots z_p^{k_p}}{z_i^{n_i} - F_i(z) y_i}\right)
\]
converges absolutely and uniformly for all \(y\) from the neighbourhood \(U_\varepsilon = \{ y : ||y|| < \varepsilon \}\), where \(0 < \varepsilon < \min_{i=1, \ldots, p} (\rho^i / M_i)\), \(M_i = \max_{z \in U_\rho} |F_i(z)|\). Integrating the last expression and changing the order of integration and summation, we get
\[
I_{x^k}(y) = \left(\frac{1}{2\pi i}\right)^p \int_{\gamma_\rho} z_1^{k_1} \cdots z_p^{k_p} \, dz_1 \wedge \cdots \wedge dz_p \\
\prod_{i=1}^p \left(\frac{z_i^{n_i} - F_i(z) y_i}{z_i^{n_i} - F_i(z) y_i}\right).
\]
The cycle \(\gamma_\rho\) for fixed \(y \in U_\varepsilon\) can be replaced by its homologous by Rouché's principle for residues (see [Ts, Chapter 2, §8] or [AY, Lemma 4.9])
\[
\gamma_\rho \sim \Gamma = \{ z : |z_1^{n_1} - F_1(z) y_1| = \cdots = |z_p^{n_p} - F_p(z) y_p| = \delta \}.
\]
Therefore, we have
\[
I_{x^k}(y) = \left(\frac{1}{2\pi i}\right)^p \int_{\Gamma} z_1^{k_1} \cdots z_p^{k_p} \, dz_1 \wedge \cdots \wedge dz_p \\
\prod_{i=1}^p \left(\frac{z_i^{n_i} - F_i(z) y_i}{z_i^{n_i} - F_i(z) y_i}\right)
\]
which finishes the proof. \(\Box\)

The Conjecture 4.6 follows now from a general result in [BM2] which identifies
\[
\left(\frac{1}{2\pi i}\right)^p \int_{\Gamma} z_1^{k_1} \cdots z_p^{k_p} \, dz_1 \wedge \cdots \wedge dz_p \\
\frac{G_1(z) \cdots G_p(z)}{z_1^{n_1} \cdots z_p^{n_p}}
\]
with the toric residue.

7. Computation of Yukawa \((d - r)\)-point functions

Let \(\Delta = \Delta_1 + \cdots + \Delta_r\) be a nef-partition of a reflexive polytope \(\Delta\), \(A_i \subset \partial \Delta_i \cap M\) a subset containing all nonzero vertices of \(\Delta_i\) \((1 \leq i \leq r)\). We set \(A_1 \cup \cdots \cup A_r := \{v_1, \ldots, v_n\}\) and consider a \(\Delta_1 \cdots \Delta_r\)-regular sequence of Laurent polynomials

\[
f_j(t) := 1 - \sum_{i, v_i \in A_j} a_i t^{v_i} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}], \quad j = 1, \ldots, r,
\]

which define \(r\) affine hypersurfaces

\[
Z_{f_j} := \{t \in \mathbb{T} \cong (\mathbb{C}^*)^d : f_j(t) = 0\}, \quad j = 1, \ldots, r,
\]

The compactification \(\overline{Z}_f\) in \(\mathbb{P}_\Delta\) of the affine complete intersection \(Z_f := Z_{f_1} \cap \cdots \cap Z_{f_r}\) is a \((d - r)\)-dimensional projective Calabi-Yau variety with at worst Gorenstein canonical singularities. Using the Poincaré residue mapping

\[
\text{Res} : H^d(\mathbb{T} \setminus Z_{f_1} \cup \cdots \cup Z_{f_r}) \to H^{d-r}(Z_{f_1} \cap \cdots \cap Z_{f_r})
\]

one can construct a nowhere vanishing section of the canonical bundle of \(\overline{Z}_f\) as

\[
\Omega := \text{Res} \left( \frac{1}{f_1 \cdots f_r} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d} \right).
\]

**Definition 7.1.** Let \(Q(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]\) be a homogeneous polynomial of degree \(d - r\). The \(Q\)-Yukawa \((d - r)\)-point function is defined by the formula

\[
Y_Q(a_1, \ldots, a_n) := \frac{(-1)^{(d-r)(d-r-1)}}{(2\pi i)^{d-r}} \int_{Z_f} \Omega \wedge Q \left( \frac{a_1}{\partial a_1}, \ldots, \frac{a_n}{\partial a_n} \right) \Omega,
\]

where the differential operators \(a_1 \partial/a_1, \ldots, a_n \partial/a_n\) are determined by the Gauß-Manin connection. If \(\tilde{k} = (k_1, \ldots, k_r)\) is a nonnegative integral vector with \(|\tilde{k}| = d - r\) and \(Q(x_1, \ldots, x_n)\) is a \(\tilde{k}\)-homogeneous polynomial (\(\text{deg} x_j = k_i \leftrightarrow v_j \in A_i\)), then

\[
Q \left( \frac{a_1}{\partial a_1}, \ldots, \frac{a_n}{\partial a_n} \right) \Omega = (-1)^{d-r} \text{Res} \left( \frac{Q(a_1 t^{v_1}, \ldots, a_n t^{v_n}) dt_1}{f_1^{k_1+1} \cdots f_r^{k_r+1} t_1 \cdots t_d} \right).
\]

**Theorem 7.2.** Let \(Q(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]\) be a \(\tilde{k}\)-homogeneous polynomial with \(|\tilde{k}| = d - r\). We define

\[
P(x_1, \ldots, x_n) := \prod_{j=1}^r \left( \sum_{v_i \in A_j} x_i \right) Q(x_1, \ldots, x_n).
\]
Then the Yukawa \((d - r)\)-point function is equal to the \(k\)-mixed toric residue

\[
Y_Q(a_1, \ldots, a_n) = (-1)^d \text{Res}_F^k \left( t_{d+1}^{x_{d+1}} \cdots t_{d+r}^{x_{d+r}} P(a_1 t^{v_1}, \ldots, a_n t^{v_n}) \right).
\]

**Proof.** We sketch only the idea of the proof. The hypersurface

\[
Z_F = \{ t_{d+1} f_1 + \cdots + t_{d+r} f_r = 1 \}
\]

in \((\mathbb{C}^*)^d \times \mathbb{C}^r\) is a \(\mathbb{C}^{r-1}\)-bundle over \((\mathbb{C}^*)^d \setminus (Z_{f_1} \cap \cdots \cap Z_{f_r})\). This fact allows to identify primitive parts of the cohomology groups \(H^{d-r}(Z_{f_1} \cap \cdots \cap Z_{f_r})\) and \(H^{d-1}(Z_F)\) together with their intersection forms. By the result of Mavlutyov [Mav], one can compute the intersection form on \(H^{d-1}(Z_F)\) using toric resides.

**Example 7.3.** Consider the mirror family \(V^*\) to Calabi-Yau complete intersections \(V\) of \(r\) hypersurfaces of degrees \(d_1, \ldots, d_r\), respectively in \(\mathbb{P}^d\), \(d_1 + \cdots + d_r = d + 1\). Its nef-partition can be constructed as follows. Let \(v_1, \ldots, v_d\) be a basis vectors of the lattice \(M\) and

\[
v_{d+1} := -v_1 - \cdots - v_d.
\]

We divide the set \(\{ v_1, \ldots, v_{d+1} \}\) into a disjoint union of \(r\) subsets \(A_1, \ldots, A_r\) such that \(|A_i| = d_i\). For \(j = 1, \ldots, r\), we define Laurent polynomials

\[
f_j(t) := 1 - \sum_{v_i \in A_j} a_i t^{v_i} \in \mathbb{C}[t_{d+1}^{\pm 1}, \ldots, t_d^{\pm 1}].
\]

Then the affine part of \(V^*\) is the complete intersection \(Z_f \subset T\) of hypersurfaces \(Z_{f_1}, \ldots, Z_{f_r} \subset T\) defined by polynomials \(f_1, \ldots, f_r\). The Yukawa coupling for \(V^*\) has been computed in [BvS, Proposition 5.1.2]:

\[
Y_Q(y) = \frac{d_1 \cdots d_r Q(1, \ldots, 1)}{1 - \mu y},
\]

where \(y = a_1 \cdots a_n\) and \(\mu = \prod_{i=1}^r d_i^{d_i}\).

**Example 7.4.** Consider Calabi-Yau varieties \(V\) obtained as complete intersection of hypersurfaces \(V_1, V_2, V_3\) in \(\mathbb{P}^3 \times \mathbb{P}^3\) of degrees \((3,0), (0,3)\) and \((1,1)\) respectively of type

\[
\begin{pmatrix}
\mathbb{P}^3 & 0 & 3 & 1 \\
0 & \mathbb{P}^3 & 3 & 1
\end{pmatrix}.
\]

Let \(M \cong \mathbb{Z}^6\) and \(\nabla^* = \text{conv}(\Delta_1 \cup \Delta_2 \cup \Delta_3) \subset M_{\mathbb{R}}\) be a reflexive polytope defined by the polytopes \(\Delta_1 := \text{conv}\{0, v_1, v_2, v_3\}, \Delta_2 := \text{conv}\{0, v_5, v_6, v_7\}\) and \(\Delta_3 := \text{conv}\{0, v_4, v_8\}\), where

\[
\begin{align*}
v_1 &= (1, 0, 0, 0, 0, 0), \quad v_2 = (0, 1, 0, 0, 0, 0), \quad v_3 = (0, 0, 1, 0, 0, 0), \\
v_4 &= (-1, -1, -1, 0, 0, 0), \quad v_5 = (0, 0, 1, 0, 0, 0), \quad v_6 = (0, 0, 0, 0, 1, 0), \\
v_7 &= (0, 0, 0, 0, 0, 1), \quad v_8 = (0, 0, 0, -1, -1, -1).
\end{align*}
\]
The nef-partition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ corresponds to mirrors $V^*$ of $V = V_1 \cap V_2 \cap V_3$. We define the disjoint sets: $A_1 := \{v_1, v_2, v_3\}$, $A_2 := \{v_5, v_6, v_7\}$, $A_3 := \{v_4, v_8\}$ and the Laurent polynomials

$$f_1(t) := 1 - \sum_{i : v_i \in A_1} a_i t^{v_i} = 1 - a_1 t_1 - a_2 t_2 - a_3 t_3,$$

$$f_2(t) := 1 - \sum_{i : v_i \in A_2} a_i t^{v_i} = 1 - a_5 t_4 - a_6 t_5 - a_7 t_6,$$

$$f_3(t) := 1 - \sum_{i : v_i \in A_3} a_i t^{v_i} = 1 - a_4^{t_1^{-1}} t_2^{-1} t_3^{-1} - a_8^{t_4^{1-1}} t_5^{1-1} t_6^{1-1}.$$

The complete intersection $Z_f := Z_{f_1} \cap Z_{f_2} \cap Z_{f_3}$ of the affine hypersurfaces

$$Z_{f_j} = \{ t \in (\mathbb{C}^*)^6 : f_j(t) = 0 \}, \quad j = 1, 2, 3$$

is an affine part of $V^*$.

Denote by $y_1 = 3^3 a_1 a_2 a_3 a_4$, $y_2 = 3^3 a_5 a_6 a_7 a_8$ the new variables and by $\theta_1 := y_1 \partial / \partial y_1$, $\theta_2 := y_2 \partial / \partial y_2$ the corresponding logarithmic partial derivations. Given a form-residue

$$\Omega := \text{Res} \left( \frac{1}{f_1 f_2 f_3} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_6}{t_6} \right) \in H^2(Z_f),$$

we compute the 2-parameter Yukawa couplings defined as integrals

$$Y^{(k_1, k_2)}(y_1, y_2) = \frac{-1}{(2\pi i)^3} \int_{Z_f} \Omega \wedge \theta_1^{k_1} \theta_2^{k_2} \Omega, \quad k_1 + k_2 = 3.$$

**Proposition 7.5.** The Yukawa couplings are

$$Y^{(3,0)}(y_1, y_2) = \frac{9 y_1}{(1 - y_1 - y_2)(1 - y_1)^2}, \quad Y^{(2,1)}(y_1, y_2) = \frac{9}{(1 - y_1 - y_2)(1 - y_1)^2},$$

$$Y^{(1,2)}(y_1, y_2) = \frac{9 y_2}{(1 - y_1 - y_2)(1 - y_2)^2}, \quad Y^{(0,3)}(y_1, y_2) = \frac{9 y_2}{(1 - y_1 - y_2)(1 - y_2)^2}.$$

**Remark 7.6.** Note that the functions in Proposition 7.5 are completely consistent with Yukawa couplings from [BvS, §8.3]. Indeed, if we put $K(y_1, y_2) = Y^{(3,0)} + 3Y^{(2,1)} + 3Y^{(1,2)} + Y^{(0,3)}$ and consider restriction to the diagonal subfamily $y = y_1 = y_2$, then we get the same expression as in [BvS]:

$$K(y, y) = \frac{18(3 - 2y)}{(1 - 2y)(1 - y)^2}.$$

Denote by $F(t) := t_1 f_1(t) + t_2 f_2(t) + t_3 f_3(t)$ the Cayley polynomial associated with Laurent polynomials $f_1, f_2, f_3$, and by $\tilde{\Delta} = \Delta_1 * \Delta_2 * \Delta_3 \subset \tilde{M}_R = M_R \oplus \mathbb{R}^3$ its supporting polytope which is the Cayley polytope associated with $\Delta_1, \Delta_2, \Delta_3$. 
Proposition 7.7. Let $\Delta := \Delta \cap \bar{M}$ and $F(t)$ be the Cayley polynomial as above. Then the principal $A$-determinant of $F$ has the form

$$E_A(F) = (a_1 \cdots a_8)^2 (1 - y_1)^3 (1 - y_1)^3 (1 - y_1 - y_2).$$

Remark 7.8. It is easy to see that the products of $Y^{(k_1, k_2)}$ by $E_A(F)$ are polynomials in $a_1, \ldots, a_8$.

Let us find the generating function $I_P(y)$ for the monomial $P(x) = x_1^{b_1} x_2^{b_2}$. There are two linear independent integral relations between $v_1, \ldots, v_8$:

$$v_1 + \cdots + v_4 = 0, \quad v_5 + \cdots + v_8 = 0.$$

Hence the Mori cone $K_{\text{eff}}(\mathbb{P})$ is spanned by the vectors

$$l^{(1)} = (1, 1, 1, 1, 0, 0, 0, 0), \quad l^{(2)} = (0, 0, 0, 0, 1, 1, 1, 1)$$

and the Morrison-Plesser moduli spaces are $\mathbb{P}_\beta = \mathbb{P}^{4b_1+3} \times \mathbb{P}^{4b_2+3}$ ($b_1, b_2 \in \mathbb{Z}_{\geq 0}$). The cohomology ring of $\mathbb{P}_\beta$ is generated by two hyperplane classes: $[H_1]$ and $[H_2]$. We set $E_1 := 3[H_1]$, $E_2 := 3[H_2]$ and $E_3 := [H_1] + [H_2]$. Then the nef-partition of the anticanonical divisor $-K_\mathbb{P}$ corresponding to the nef-partition $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ is defined by $-K_\mathbb{P} = E_1 + E_2 + E_3$. Therefore, the Morrison-Plesser cohomology class associated with the nef-partition of $-K_\mathbb{P}$ equals

$$\Phi_\beta = (3[H_1])^{3b_1} (3[H_2])^{3b_2} ([H_1] + [H_2])^{b_1+b_2}$$

and the generating function for intersection numbers can be written

$$I_P(y) = \sum_{b_1, b_2 \geq 0} \langle [H_1]^{b_1+3b_2+1} [H_2]^{b_2+3b_1+1} ([H_1] + [H_2])^{b_1+b_2+1} \rangle y_1^{b_1} y_2^{b_2}.$$

Intersection theory on $\mathbb{P}_\beta$ implies

$$I_P(y) = 9 \sum_{b_1, b_2 \geq 0} \frac{(b_1 + b_2 + 1)!}{(b_1 - k_1 + 2)!(b_2 - k_2 + 2)!} y_1^{b_1} y_2^{b_2}.$$

By Theorem 6.1 we can write $I_P(y)$ as the integral

$$I_P(y) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \frac{9 z_1^{b_1+1} z_2^{b_2+1} (z_1 + z_2) dz_1 \wedge dz_2}{(z_1^3 - z_2^3 (z_1 + z_2) y_1) (z_2^3 - z_1^3 (z_1 + z_2) y_2)}$$

with the cycle $\Gamma = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1^3 - z_2^3 (z_1 + z_2) y_1| = \varepsilon_1, |z_2^3 - z_1^3 (z_1 + z_2) y_2| = \varepsilon_2\}$, $\varepsilon_1, \varepsilon_2 > 0$. Computing the last integrals, we get the same rational functions as in Proposition 7.5.

Example 7.9. Consider an example of Calabi-Yau variety $V$ obtained as complete intersections of two hypersurfaces of degrees $(4, 0)$, $(1, 2)$ in $\mathbb{P} = \mathbb{P}^4 \times \mathbb{P}^1$ which corresponds to the configuration

$$\begin{pmatrix} \mathbb{P}^4 & \mid & 4 & 1 \\ \mathbb{P}^1 & \mid & 0 & 2 \end{pmatrix}.$$
This example was investigated in details by Hosono, Klemm, Theisen and Yau (cf. [HKTY]). The corresponding nef-partition $\Delta = \Delta_1 + \Delta_2 \subset M_\mathbb{R} \cong \mathbb{R}^5$ consists of polytopes $\Delta_1 := \text{conv}\{0, v_1, v_2, v_3, v_4\}$ and $\Delta_2 := \text{conv}\{0, v_5, v_6, v_7\}$, where

$v_1 = (1, 0, 0, 0, 0), \quad v_2 = (0, 1, 0, 0, 0), \quad v_3 = (0, 0, 1, 0, 0), \quad v_4 = (0, 0, 0, 1, 0),

v_5 = (-1, -1, -1, -1, 0), \quad v_6 = (0, 0, 0, 0, 1), \quad v_7 = (0, 0, 0, 0, -1).

We have two disjoint sets: $A_1 := \{v_1, v_2, v_3, v_4\}$ and $A_2 := \{v_5, v_6, v_7\}$ which are the vertices of the reflexive polytope $\nabla^*$ and define the Laurent polynomials

$$f_1(t) = 1 - \sum_{i : v_i \in A_1} a_i t^{v_i} = 1 - a_1 t_1 - a_2 t_2 - a_3 t_3 - a_4 t_4,$$

$$f_2(t) = 1 - \sum_{i : v_i \in A_2} a_i t^{v_i} = 1 - a_5 (t_1 t_2 t_3 t_4)^{-1} - a_6 t_5 - a_7 t_5^{-1}.$$

Denote by $y_1 := a_1 \cdots a_5, y_2 := a_6 a_7$ the new variables and $\theta_1 := y_1 \partial / \partial y_1, \theta_2 := y_2 \partial / \partial y_2$ the corresponding logarithmic partial derivations. Let $\Omega$ be a form defined by

$$\Omega := \text{Res} \left( \frac{1}{f_1 f_2} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \right) \in H^3(Z_{f_1} \cap Z_{f_2}).$$

Then the Yukawa coupling associated with $f_1, f_2$ is the integral

$$Y^{(k_1, k_2)}(y_1, y_2) := \frac{-1}{(2\pi i)^3} \int_{Z_f} \Omega \wedge \theta_1^{k_1} \theta_2^{k_2} \Omega, \quad k_1 + k_2 = 3,$$

where $Z_f := Z_{f_1} \cap Z_{f_2}$ is an affine Calabi-Yau complete intersection which compactification forms a mirror dual family to $V$.

**Proposition 7.10. [HKTY]** The Yukawa couplings $Y^{(k_1, k_2)}(y)$ are:

$$Y^{(3,0)}(y) = \frac{8}{D_0}, \quad Y^{(2,1)}(y) = \frac{4(1 - 256y_1 + 4y_2)}{D_0 D_1},$$

$$Y^{(1,2)}(y) = \frac{8y_2(3 - 512y_1 + 4y_2)}{D_0 D_1^2},$$

$$Y^{(0,3)}(y) = \frac{4y_2(1 - 256y_1 + 24y_2 - 3072y_1 y_2 + 16y_2^2)}{D_0 D_1^3}.$$

Let $F(t) := t_6 f_1(t) + t_7 f_2(t)$ be the Cayley polynomial associated with $f_1(t)$ and $f_2(t)$. Its support polytope is the Cayley polytope $\tilde{\Delta} = \Delta_1 \ast \Delta_2 \subset \bar{M}_\mathbb{R} = M_\mathbb{R} \oplus \mathbb{R}^2$ which is the convex hull of the vectors:

$u_1 = (0, 0, 0, 0, 1, 0), \quad u_2 = (1, 0, 0, 0, 0, 1, 0), \quad u_3 = (0, 1, 0, 0, 0, 1, 0),

u_4 = (0, 0, 1, 0, 0, 1, 0), \quad u_5 = (0, 0, 0, 1, 0, 1, 0), \quad u_6 = (0, 0, 0, 0, 0, 0, 1),

u_7 = (-1, -1, -1, -1, 0, 0, 1), \quad u_8 = (0, 0, 0, 0, 1, 0, 1), \quad u_9 = (0, 0, 0, 0, -1, 0, 1).$
Proposition 7.11. Let \( A := \{u_1, \ldots, u_9\} \subset \widetilde{M} \) and
\[
D_0 := (1 - 256y_1)^2 - 4y_2, \quad D_1 := 1 - 4y_2.
\]
Then the principal \( A \)-determinant of \( F(t) \) has the following form:
\[
E_A(F) = \begin{array}{l}
-640a_1^8a_2^8a_3^8a_4^8a_5^8a_6^8a_7^8 - 16777216a_1^{10}a_2^{10}a_3^{10}a_4^{10}a_5^{10}a_6^8a_7^8 + \\
160a_1^8a_2^8a_3^8a_4^8a_5^8a_6^8a_7^7 + 16777216a_1^{10}a_2^{10}a_3^{10}a_4^{10}a_5^{10}a_6^8a_7^8 + \\
65536a_1^{10}a_2^{10}a_3^{10}a_4^8a_5^{10}a_6^5a_7^5 + 6291456a_1^{10}a_2^{10}a_3^{10}a_4^{10}a_5^{10}a_6^7a_7^7 - \\
1048576a_1^{10}a_2^{10}a_3^{10}a_4^{10}a_5^{10}a_6^6a_7^6 - 1024a_1^8a_2^8a_3^8a_4^8a_5^8a_6^6a_7^6 + \\
1280a_1^8a_2^8a_3^8a_4^8a_5^8a_6^9a_7^9 - 512a_1^8a_2^8a_3^8a_4^8a_5^8a_6^5a_7^5 - \\
20a_1^8a_2^8a_3^8a_4^8a_5^8a_6^6a_7^6 + 131072a_1^8a_2^8a_3^8a_4^8a_5^8a_6^6a_7^8 + \\
a_1^8a_2^8a_3^8a_4^8a_5^8a_6^5a_7^5 + 8192a_1^8a_2^8a_3^8a_4^8a_5^8a_6^6a_7^6 - \\
49152a_1^8a_2^8a_3^8a_4^8a_5^8a_6^7a_7^7 - 131072a_1^8a_2^8a_3^8a_4^8a_5^8a_6^6a_7^9,
\end{array}
\]
where the terms corresponding to the vertices of Newton polytope of \( E_A(F) \) are underlined.

Proof. The principal \( A \)-determinant can be found by using the algorithm proposed by A. Dickenstein and B. Sturmfels [DS] via the computation of the corresponding Chow forms. \( \square \)

Remark 7.12. We note that \( D_0 \) is the principal component of the discriminant locus \( E_A(F) = 0 \) and the component \( D_1 \) corresponds to the edge \( \Gamma \) of \( \widetilde{M} \) with
\[
\Gamma \cap \widetilde{M} = \{u_6, u_8, u_9\} = \{(0, 0, 0, 0, 0; 0, 1), (0, 0, 0, 1; 0, 1), (0, 0, 0, -1; 0, 1)\}.
\]

The Newton polytope of \( E_A(F) \) is the secondary polytope \( \text{Sec}(A) \) depicted in Figure 1. The vertices of \( \text{Sec}(A) \) are in one-to-one correspondence with coherent triangulations \( T_1, \ldots, T_4 \) of \( \widetilde{\Delta} \) which are:

\[
T_1 = \{(u_1, u_3, u_4, u_5, u_5, u_7, u_9), (u_1, u_2, u_4, u_5, u_6, u_7, u_8), (u_1, u_2, u_3, u_5, u_6, u_7, u_8), (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8), (u_2, u_3, u_4, u_5, u_6, u_7, u_8)\},
\]

\[
T_2 = \{(u_1, u_3, u_4, u_5, u_7, u_8, u_9), (u_1, u_2, u_4, u_5, u_7, u_8, u_9), (u_1, u_2, u_3, u_5, u_7, u_8, u_9), (u_1, u_2, u_3, u_4, u_5, u_7, u_8, u_9)\},
\]

\[
T_3 = \{(u_1, u_2, u_4, u_5, u_7, u_8, u_9), (u_1, u_2, u_3, u_4, u_5, u_7, u_8, u_9)\},
\]

\[
T_4 = \{(u_1, u_2, u_3, u_4, u_5, u_7, u_9), (u_1, u_2, u_3, u_4, u_5, u_7, u_9), (u_2, u_3, u_4, u_5, u_6, u_7, u_9)\}.
\]
THEOREM 1. Secondary polytope and coherent triangulations

The generating function $I_P(y)$ for intersection numbers corresponding to the monomial $P(x) = x_1^{b_1}x_2^{b_2}$ can be computed from the intersection theory on the Morrison-Plesser moduli spaces. Using two independent integral relations between $v_1, \ldots, v_7$

$$v_1 + \cdots + v_5 = 0, \quad v_6 + v_7 = 0,$$

we see that the Mori cone $K_{\text{eff}}(\mathbb{P})$ is spanned by two vectors

$$l^{(1)} = (1, 1, 1, 1, 0, 0, 0), \quad l^{(2)} = (0, 0, 0, 0, 0, 1, 1).$$

The Morrison-Plesser moduli spaces are $\mathbb{P}_\beta = \mathbb{P}^{b_1+4} \times \mathbb{P}^{2b_2+1}$ $(b_1, b_2 \in \mathbb{Z}_{\geq 0})$. The cohomology of $\mathbb{P}_\beta$ are generated by the hyperplane classes $[H_1]$ and $[H_2]$. Let $E_1 := [H_1] + 2[H_2]$ and $E_2 := 4[H_1]$. Then the nef-partition $\Delta = \Delta_1 + \Delta_2$ of polytopes induces the nef-partition of the anticanonical divisor

$$-K_\mathbb{P} = E_1 + E_2 = ([H_1] + 2[H_2]) + (4[H_1]).$$

It is straightforward to see that the corresponding Morrison-Plesser class is

$$\Phi_\beta = ([H_1] + 2[H_2])^{b_1+2b_2}(4[H_1])^{4b_1}.$$ 

So we get

$$I_P(y) = \sum_{b_1,b_2 \geq 0} \langle [H_1]^{b_1}[H_2]^{b_2}([H_1] + 2[H_2])^{b_1+2b_2+1}(4[H_1])^{4b_1+1} \rangle_\beta y_1^{b_1}y_2^{b_2}.$$ 

Using the intersection theory on $\mathbb{P}_\beta$, we obtain

$$I_P(y) = \sum_{b_1,b_2 \geq 0} 2^{8b_1+2b_2-k_2+3} \frac{(b_1 + 2b_2 + 1)!}{(b_1 - k_1 + 3)(2b_2 - k_2 + 1)!} y_1^{b_1}y_2^{b_2}.$$ 

By Theorem 6.1 the function $I_P(y)$ admits the integral representation:

$$I_P(y) = \frac{1}{(2\pi i)^2} \int \frac{4z_1^{k_1+1}z_2^{k_2} (z_1 + 2z_2) dz_1 \wedge dz_2}{(z_1^5 - (z_1 + 2z_2)(4z_1)^4y_1)(z_2^2 - (z_1 + 2z_2)^2y_2)}$$
with the cycle

\[ \Gamma = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1^5 - (z_1 + 2z_2) (4z_1)^4 y_1| = \varepsilon_1, |z_2^2 - (z_1 + 2z_2)^2 y_2| = \varepsilon_2\}, \]

where \(\varepsilon_1, \varepsilon_2\) are positive. These integrals can be easily computed and yield the same rational functions as in Proposition 7.10.

REFERENCES


[Bat] V.V. Batyrev, Quantum cohomology rings of toric manifolds, Asterisque 218 (1993), 9–34.


[Bo] L.A. Borisov, Towards the mirror symmetry for Calabi-Yau complete intersections in Gorenstein toric Fano varieties.


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