Non-commutative geometry and new stable structures

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This paper grew out of an observation that some new stable structures discovered in the 1990’s as counterexamples to well-known conjectures in pure model theory might be related to non-commutative geometry.

The general meaning of the conjectures was that “very good”, or more technically, very stable structures must be in a certain way reducible to algebraic geometry over algebraically closed fields or to linear structures (Trichotomy conjecture and Algebraicity conjecture for groups, see [Z0]). This proved to be true to some extent (see [HZ]) but still two types of counterexamples signal the necessity to reconsider the connection between model theoretic classification principles and classical mathematics.

The first class of counterexamples shows that nonlinear one-dimensional Zariski geometries are not necessarily algebraic curves. Given a smooth algebraic curve $C$ with big enough group of regular automorphisms one can produce a “smooth” Zariski curve $\tilde{C}$ along with a finite cover $p: \tilde{C} \to C$. $\tilde{C}$ can not be identified with any algebraic curve because the construction produces an unusual subgroup of the group of regular automorphisms of $\tilde{C}$ ([HZ, section 10]. The main theorem of [HZ] states that it is the biggest deviation from an algebraic curve that can happen to a Zariski curve. Typical example of an unusual subgroup of such $\tilde{C}$ is the nilpotent group of two generators $U$ and $V$ with the central commutator $\epsilon = [U, V]$ of finite order $N$. So, the defining relations are

$$UV = \epsilon VU, \quad \epsilon^N = 1.$$  

This, of course, hints towards the known structure of non-commutative geometry, the non-commutative (quantum) torus at the $N$th root of unity. We call this example $T_N$.

The other example is of a different nature. B.Poizat constructed in [P] a multiplicative subgroup $G$ of an algebraically closed field (we may assume this to be the field $\mathbb{C}$ of complex numbers) such that $(\mathbb{C}, +, \cdot, G)$ has $\omega$-stable theory of rank half of that of $\mathbb{C}$ (so called “bad field”, related to the Algebraicity conjecture). The present author has shown in [Z2] that, assuming Schanuel’s conjecture, one can construct $G$ by means of real analytic geometry. More specifically one can consider $G$ of the form $G = \exp(\alpha Z) \cdot \exp(\beta \mathbb{R})$, $\alpha$ and $\beta$ linearly independent over $\mathbb{R}$, $\beta \notin \mathbb{R} \cup i\mathbb{R}$, and see that $(\mathbb{C}, +, \cdot, G)$ is superstable of dimension half of that of $\mathbb{C}$. We then note that the structure on the quotient $\mathbb{C}^*/G$ is geometrically the same as what one gets in the quotient

$$T^2_h = (S \times S)/L.$$  

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of the square of the unit circle $S \subset \mathbb{C}^*$ by a Kronecker foliation $L_h$ (set-wise this is the same as the group $\mathbb{R}/(1, h)$). This is a basic example and motivation of A. Connes [C] for introducing non-commutative geometry.

Of course, one of the biggest challenges in relating non-commutative geometry to model theory comes from the difference in the way objects are represented in each of the approaches. Geometers tend to replace a structure $M$ by the dual object, the algebra $\mathbb{C}[M]$ of functions on $M$, or even more abstract non-commutative algebra of “observables” which take the role of the algebra of functions. Generally, non-commutative geometry does not assume that one has a reverse procedure of getting a structure back from the algebra of observables. Yet it is desirable to have a manifold-kind structure underlying a given algebra of observables. Yu. Manin makes this point in [Man] I.1.4 as a foundational problem.

In the present paper we undertake a thorough study of both classes of examples. We try to give answers to the following questions:

1. What are the “algebras of functions” for $T_N$ and $T_h^2$? Can these structures be identified as objects of non-commutative geometry?

2. What is the structure that non-commutative geometry “sees” on $T_N$ and $T_h^2$?

3. Is there a uniform representation of both types of structures?

By virtue of construction the algebra of Zariski continuous (regular) functions $T_N \rightarrow \mathbb{C}$ is the same as that of $\mathbb{C}^*$, that is $\mathbb{C}[t, t^{-1}]$, so does not reflect enough of the structure $T_N$. We show that specifically to the structures under question one can introduce the algebra of semi-definable functions. These are not uniquely defined but the commutative algebra $\mathcal{H}$ they generate is determined uniquely up to isomorphism. Moreover, uniquely determined is the action on $\mathcal{H}$ of certain linear operators related to the “hidden” structure of $T_N$. Algebra of these linear operators is the same as that of non-commutative torus at root of unity known to geometers.

One of the semi-definable functions plays a special role in the construction of $U$ and $V$, this is the angular function

$$\text{ang}_N : \mathbb{C}^* \rightarrow \mathbb{C}[N], \quad N\text{-roots of } 1,$$

satisfying certain conditions. Answering the second question above we show that $T_N$ can be identified with a space of linear functionals $\mathcal{H} \rightarrow \mathbb{C}$ of a positive orientation. We introduce the orientation in terms of the angular
function. Alternatively but equivalently $T_N$ can be identified with the space of $N$-dimensional irreducible modules of positive orientation over the coordinate algebra.

Then we look for a similar construction that can play a role of the limit structure $T_N$ as $N$ tends to $\infty$. The usual model-theoretic limit (the ultraproduct) does not quite work here, for the same reasons as the universal cover $\exp: \mathbb{C} \to \mathbb{C}^*$ can not be obtained as the ultraproduct of finite covers $x \mapsto x^N, \mathbb{C}^* \to \mathbb{C}^*$. We find a natural construction in terms of the structure of real and complex numbers, dependent on a real parameter $h$, the \textit{analytic Zariski} structure $T_h$, which we show to behave as the limit structure in many respects. In particular the irreducible modules are of countable dimension with $U$-eigenvalues of the form $q^m \mu, \ m \in \mathbb{Z}, q = \exp 2\pi i h, \mu \in \mathbb{C}^*$, one $\mu$ for each module. The corresponding space $\mathcal{H}$ of semi-definable functions on $T_h$ together with the action of $U$ and $V$ on it turns out to be a close analogue of the space with an action corresponding to Connes’ quantum torus $T^2_h$. The correspondent angular function gets the form of a function

$$(\mathbb{C}, +) \to \exp(2\pi i h \mathbb{Z}),$$

behaving similarly to the function $z \mapsto \exp(i \text{Re } z)$.

At this point we don’t have a full analogy yet, since setwise the space of our irreducible positively oriented modules is $\mathbb{C}/\langle 1, h \rangle$ rather than $\mathbb{R}/\langle 1, h \rangle$. Connes specifies, using his $\mathbb{C}^*$-algebras language, that $U$ and $V$ must be \textit{unitary} operators. This immediately translates into the fact that the eigenvalues $q^m \mu$ above must lie on the unit circle and so he gets $\mathbb{S}/\langle q \rangle$ while we have $\mathbb{C}^*/\langle \overline{q} \rangle$. Instead of using the (unstable) $\mathbb{C}^*$-algebras language we note that the \textit{group of regular automorphisms} of $T_h$ (commuting with $U$ and $V$) is exactly the above group $\mathcal{G} = \exp(2\pi i h \mathbb{Z} + \beta \mathbb{R})$. This implies that the action of $U$ and $V$ is well-defined on the quotient $\mathbb{C}^*/\mathcal{G}$ which is definable in our $T_h$ and is representing Connes’ $T^2_h$.

We hence found a way to represent uniformly our $T_N$’s together with Connes’ $T_h$. Moreover, we can see that there exists a \textit{universal object $U$} in this uniform representation. Namely, for each $N \in \mathbb{N} \cup \{h\}$ there is a surjective map $e_N: U \to T_N$

which also gives an interpretation of $T_N$ in terms of $U$.

It is important to mention that the above description of the structures can not be complete without giving a detailed description of the languages
involved. In fact there are at least two levels of languages. The basic language is the language of the example in [HZ], and we prove that $T_h$ is superstable in this language (probably is analytic Zariski of dimension 1 see [PZ] and [Z1]).

We also discuss the language which allows the angular function $ang$. The conditions defining $ang$ do not constitute a complete theory, so it is natural to choose a complete extension which axiomatises the existentially closed structures. In fact such a choice amounts to choosing $ang$ in a uniformly random way. We conjecture that under this choice the theory is supersimple. This has been proven by D.Evans in a basic case. It seems both feasible and mathematically meaningful to undertake a detailed analysis of the structure of definable sets in the theory, and develop a probabilistic measure theory on the sets.

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1 Non-algebraic Zariski geometries

1.1 Recall the following theorem C of [HZ].

**Theorem** There exist irreducible pre-smooth Zariski structures (in particular of dimension 1) which are not interpretable in an algebraically closed field.

**The construction**

Let \( M \) be an irreducible pre-smooth Zariski structure, \( G \leq \text{ZAut} M \) acting freely on \( M \) and for some \( \tilde{G} \) with finite \( H \):

\[
1 \to H \to \tilde{G} \to \tilde{G}/H \to 1.
\]

Consider a set \( S \subseteq M \) of representatives of \( G \)-orbits: for each \( a \in M \), \( G \cdot a \cap S \) is a singleton.

Consider the formal set

\[
M(\tilde{G}) = \tilde{M} = \tilde{G} \times S
\]

and the projection map

\[
p : (g, s) \mapsto p_0(g) \cdot s.
\]

Consider also, for each \( f \in \tilde{G} \) the function

\[
f : (g, s) \mapsto (fg, s).
\]

Claim 1. The structure

\[(\tilde{M}, \{f\}_{f \in \tilde{G}}; p^{-1}(\text{Zariski relations on } M))\]

is an irreducible pre-smooth Zariski structure, its isomorphism type is determined by \( M \) and \( \tilde{G} \) only and \( \dim \tilde{M} = \dim M \).

Proof. One can use obvious automorphisms of the structure to prove quantifier elimination. The statement of the claim then follows by checking the definitions. The detailed proof is given in [HZ] Proposition 10.1.
Claim 2. Suppose $H$ does not split, for every proper $G_0 < \tilde{G}$

$$G_0 \cdot H \neq \tilde{G}.$$  

Then, every equidimensional Zariski expansion $\tilde{M}'$ of $\tilde{M}$ is irreducible.

Indeed. Let $C = \tilde{M}'$ is an $|H|$-cover of the variety $M$, so $\dim C = \dim M$ and $C$ has at most $|H|$ distinct irreducible components, say $C_i$, $1 \leq i \leq n$. For generic $y \in M$ the fiber $p^{-1}(y)$ intersects every $C_i$ (otherwise $p^{-1}(M)$ is not equal to $C$).

Hence $H$ acts transitively on the set of irreducible components. So, $\tilde{G}$ acts transitively on the set of irreducible components, so the setwise stabiliser $G_0$ of $C_1$ in $\tilde{G}$ is of index $n$ in $\tilde{G}$ and also $H \cap G_0$ is of index $n$ in $H$. Hence, 

$$\tilde{G} = G_0 \cdot H, \qquad H \not\subset G_0$$

contradicting our assumptions. Claim proved.

Claim 3. $\tilde{G} \leq \text{ZAut } \tilde{M}$, that is $\tilde{G}$ is a subgroup of the group $\text{ZAut } M$ of Zariski-continuous bijections of $M$.

Immediate by the construction.

Lemma. Suppose $M$ is a rational or elliptic curve (over an algebraically closed field $F$ of characteristic zero), $H$ does not split, $\tilde{G}$ is nilpotent and for some big enough integer $\mu$ there is a non-abelian subgroup $G_0$

$$|\tilde{G} : G_0| \geq \mu.$$  

Then $\tilde{M}$ is not interpretable in an algebraically closed field.

**Proof** First we show.

Claim 4. Without loss of generality we may assume that $\tilde{G}$ is infinite.

Recall that $G$ is a subgroup of the group $\text{ZAut } M$ of rational (Zariski) automorphisms of $M$. Every algebraic curve is birationally equivalent to a smooth one, so $G$ embeds into the group of birational transformations of a smooth rational curve or an elliptic curve. Now remember that any birational transformation of a smooth algebraic curve is biregular. If $M$ is rational then the group $\text{ZAut } M$ is $\text{PGL}(2, F)$. Choose a semisimple (diagonal) $s \in \text{PGL}(2, F)$ be an automorphism of infinite order such that $\langle s \rangle \cap G = 1$
and $G$ commutes with $s$. Then we can replace $G$ by $G' = \langle G, s \rangle$ and $\tilde{G}$ by $\tilde{G}' = \langle \tilde{G}, s \rangle$ with the trivial action of $s$ on $H$. One can easily see from the construction that the $\tilde{M}'$ corresponding to $\tilde{G}'$ is the same as $\tilde{M}$, except for the new definable bijection corresponding to $s$.

We can use the same argument when $M$ is an elliptic curve, in which case the group of automorphisms of the curve is given as a semidirect product of a finitely generated abelian group (complex multiplication) acting on the group on the elliptic curve $E(F)$.

Now, assuming that $\tilde{M}$ is definable in an algebraically closed field $F'$ we will have that $F$ is definable in $F'$. It is known to imply that $F'$ is definably isomorphic to $F$, so we may assume that $F' = F$.

Also, since $\dim \tilde{M} = \dim M = 1$, it follows that $\tilde{M}$ up to finitely many points is in a bijective definable correspondence with a smooth algebraic curve, say $C = C(F)$.

$\tilde{G}$ then by the argument above is embedded into the group of rational automorphisms of $C$.

The automorphism group is finite if genus of the curve is 2 or higher, so by Claim 4 we can have only rational or elliptic curve for $C$.

Consider first the case when $C$ is rational. The automorphism group then is $\text{PGL}(2, F)$. Since $\tilde{G}$ is nilpotent its Zariski closure in $\text{PGL}(2, F)$ is an infinite nilpotent group $U$. Let $U^0$ be the connected component of $U$, which is a normal subgroup of finite index. By Malcev's Theorem (see [Merzliakov], 45.1) there is a number $\mu$ (dependent only on the size of the matrix group in question but not on $U$) such that some normal subgroup $U^0$ of $U$ of index at most $\mu$ is a subgroup of the unipotent group

\[
\begin{pmatrix}
1 & z \\
0 & 1
\end{pmatrix}
\]

this is Abelian, contradicting the assumption that $\tilde{G}$ has no abelian subgroups of index less than $\mu$.

In case $C$ is an elliptic curve the group of automorphisms is a semidirect product of a finitely generated abelian group (complex multiplication) acting freely on the abelian group of the elliptic curve. This group has no nilpotent non-abelian subgroups. This finishes the proof of the Lemma and of the theorem. $\square$
In general it is harder to analyse the situation when \( \dim M > 1 \) since the group of birational automorphisms is not so immediately reducible to the group of biregular automorphisms of a smooth variety in higher dimensions. But nevertheless the same method can prove the useful fact that the construction produces examples essentially of non algebro-geometric nature.

**Proposition** (i) Suppose \( M \) is an abelian variety, \( H \) does not split and \( \tilde{G} \) is nilpotent not abelian. Then \( \tilde{M} \) can not be an algebraic variety with \( p : \tilde{M} \to M \) a regular map.

(ii) Suppose \( M \) is the (semi-abelian) variety \( (F^\times)^n \). Suppose also that \( \tilde{G} \) is nilpotent and for some big enough integer \( \mu = \mu(n) \) has no abelian subgroup \( G_0 \) of index bigger than \( \mu \). Then \( \tilde{M} \) can not be an algebraic variety with \( p : \tilde{M} \to M \) a regular map.

**Proof** (i) If \( M \) is an abelian variety and \( \tilde{M} \) were algebraic, the map \( p : \tilde{M} \to M \) has to be unramified since all its fibers are of the same order (equal to \( |H| \)). Hence \( \tilde{M} \) being a finite unramified cover must have the same universal cover as \( M \) has. So, \( \tilde{M} \) must be an abelian variety as well. The group of automorphisms of an abelian variety \( A \) without complex multiplication is the abelian group \( A(F) \). The contradiction.

(ii) Same argument as in (i) proves that \( \tilde{M} \) has to be isomorphic to \( (F^\times)^n \). The Malcev theorem cited above finishes the proof. 

**Proposition.** Suppose \( M \) is an \( F \)-variety and, in the construction of \( \tilde{M} \), the group \( G \) is finite. Then \( \tilde{M} \) is definable in any expansion of the field \( F \) by a total linear order.

In particular, if \( M \) is a complex variety, \( \tilde{M} \) is definable in the reals.

**Proof** Extend the ordering of \( F \) to a linear order of \( M \) and define

\[
S := \{ s \in M : s = \min G \cdot s \}.
\]

The rest of the construction of \( \tilde{M} \) is definable. 

**Remark** In other known examples of non-algebraic \( \tilde{M} \) (with \( G \) infinite) \( \tilde{M} \) is still definable in any expansion of the field \( F \) by a total linear order.
Problem (i) Classify Zariski structures definable in the reals.

(ii) Classify Zariski structures definable in the reals as a smooth real manifold.

(iii) Find new Zariski structures definable in $\mathbb{R}_{an}$ as a smooth real manifold.
2 A non-algebraic Zariski curve and its coordinate algebra

2.1 Let $F$ be an algebraically closed field of characteristic 0 and $N$ a positive integer. Consider the groups given by generators and defining relations,

$$G = \langle u, v : uv = vu \rangle,$$


Let $a, b \in F^*$ multiplicatively independent.

$G$ acts on $F^x$:

$$u \cdot x = ax, \quad v \cdot x = bx.$$  

Taking $M$ to be $F^x$ this determines, by 1.1, a presmooth non-algebraic Zariski curve $\tilde{M}$ which from now on we denote $T_N$.

Since $[U, V]$ is a central element, in every representation of $\tilde{G}$ one can replace $[U, V]$ by an $\epsilon \in F$, a primitive root of unity of order $N$. So, the defining relation for $\tilde{G}$ becomes just

$$VU = \epsilon UV,$$

or

$$VUV^{-1}U^{-1} = \epsilon.$$  

The correspondent definition for the covering map $p : \tilde{M} \to M$ then gives us

$$p(Ut) = ap(t), \quad p(Vt) = bp(t). \quad (1)$$

2.2 Semi-definable functions.

Lemma Given $\alpha, \beta$ such that $\alpha^N = a, \beta^N = b$, one can define bijections

$$x_k : T_N \to F^* \quad k = 0, \ldots, N - 1$$

so that for any $t \in T_N$ the following functional equations are satisfied,

$$x_k(t)^N = p(t) \quad (2)$$

$$x_k(Ut) = \alpha^k x_k(t), \quad (3)$$
\[ x_k(\mathbf{V}t) = \beta x_{k+1}(t), \text{ where } x_N = x_0, \quad (4) \]
\[ \frac{x_{k+1}(t)}{x_k(t)} = \frac{x_k(t)}{x_{k-1}(t)}. \quad (5) \]

**Proof** First, notice that (3),(4) imply
\[ x_k([\mathbf{U}, \mathbf{V}]^{-1}t) = \epsilon x_k(t), \quad (6) \]
where \([\mathbf{U}, \mathbf{V}]^{-1} = \mathbf{U}^{-1}\mathbf{V}^{-1}\mathbf{U}\mathbf{V}.

To construct the \(x_k\) choose randomly an injection \(\mathcal{N} : F^* \to F^*\) with the property
\[ (\sqrt[N]{w})^N = w. \]
For any \(s \in S\) and \(t \in \mathcal{G} \cdot s\) of the form \(t = \mathbf{U}^m\mathbf{V}^n[\mathbf{U}, \mathbf{V}]^l \cdot s\), set
\[ x_k(\mathbf{U}^m\mathbf{V}^n[\mathbf{U}, \mathbf{V}]^l \cdot s) := \alpha^m \beta^n \epsilon^{mk-l} \sqrt[N]{s}. \]
This satisfies (2)-(5).

To see that each \(x_k\) is injective consider \(t, t' \in T_N\) such that \(x_k(t) = x_k(t')\).
We then have, by (2), that \(p(t) = p(t')\). Hence \(t' = ht\) for some \(h \in H\), that is for \(h = [\mathbf{U}, \mathbf{V}]^j\), some \(j \in \{0, \ldots, N - 1\}\). By (6) this is possible only if \(j = 0\), that is \(t = t'\).

In order to prove that \(x_k\) is surjective we need to solve the equation
\[ x_k(t) = \mu \]
for any given \(\mu \in F^*\). Since \(p\) is surjective we can find \(t' \in T\) such that \(p(t') = \mu^N\), and so by (2) we have \(x_k(t') = \epsilon^l \mu\), for some \(l \in \mathbb{Z}\). Take now \(t = [\mathbf{U}, \mathbf{V}]^l t'\) and by (6) this solves the equation. □

2.3 Define the **angular function** on \(F^*\) as a function \(\text{ang} : F^* \to F[N]\), roots of unity of order \(N\).

Set for \(\lambda \in F^*\),
\[
\text{ang}(\lambda) = \frac{x_1(t)}{x_0(t)}, \text{ if } \lambda = x_0(t).
\]
This is well-defined since \(x_0\) is a bijection.

Acting by \(\mathbf{U}\) on \(t\) and using (3) we have
\[
\text{ang} \alpha \lambda = \epsilon \text{ang} \lambda \quad (7)
\]
We also have
\[ \text{ang} \epsilon \lambda = \text{ang} \lambda. \tag{8} \]
since by (6)
\[ x_0([U, V]^{-1}t) = \epsilon x_0(t) = \epsilon \lambda, \]
and at the same time
\[ \text{ang}(\epsilon \lambda) = \frac{x_1([U, V]^{-1}t)}{x_0([U, V]^{-1}t)} = \frac{x_1(t)}{x_0(t)} = \text{ang} \lambda. \]

Finally, suppose \( x_1(t) = \lambda \). Then \( x_0(Vt) = \beta \lambda \), by (4), and \( x_1(Vt) = \beta x_2(t) = \beta \lambda \cdot \text{ang} \lambda \), by (5). Since \( \text{ang} \beta \lambda = x_1(Vt) : x_0(Vt) \), we have
\[ \text{ang} \beta \lambda = \text{ang} \lambda. \tag{9} \]

Now we consider the structure
\[(F, +, \cdot, \text{ang}).\]
It is clear that \( F \) is partitioned into \( N \) ‘sectors’ using the angular function:
\[ P_\delta = \{ \mu \in F^* : \text{ang} \mu = \delta \}. \]

**Proposition** \( T_N \) is definable in \( (F, +, \cdot, \text{ang}) \) using parameters \( \alpha \) and \( \beta \). Moreover, \( x_0, \ldots, x_{N-1} \) are definable in the structure as well.

**Proof** Define \( T = F^\times \) as a set, and for any \( t \in F^\times \) set
\[ p(t) = t^N, \quad U t = \alpha t, \quad V t = \beta \text{ang}(t) t. \]
We then have
\[ t \to^U \alpha t \to^V \alpha \beta \text{ang}(\alpha t) t = \alpha \beta \text{ang}(t) \epsilon t \to^{U^{-1}} \beta \text{ang}(t) \epsilon t \to^{V^{-1}} \epsilon t. \]
That is
\[ V^{-1}U^{-1}VUt = \epsilon t \]
so, the group \( \hat{G} \) acts on the \( T \) freely.
It is also clear that
\[ p(Ut) = ap(t), \quad p(Vt) = bp(t), \quad p^{-1}(p(t)) = \{ [U, V]^{-l}t : l = 0, \ldots, N - 1 \} \]
as required by the description of $T_N$.

Finally, set $x_k(t) := (\text{ang } t)^k \cdot t$. □

From now on we use notation

$$\bar{T}_N := (F, +, \cdot, \text{ang}).$$

The interpretation of $T_N$ in the proof of the above proposition we will consider canonical, with respect to $\alpha$ and $\beta$.

**Remark 1** The isomorphism type of $T_N$ defined by means of $\bar{T}_N$ depends on the isomorphism type (so of the cardinality) of the field $F$ with parameters $\alpha, \beta, \epsilon$ only, and not on the choice of the angular function (equivalently $P_\delta$) since by the construction in 1.1 any two structures $M$ with the same $\bar{G}$ are isomorphic over $M$.

**Corollary** Assuming that $F = \mathbb{C}$ and $a, b \in \epsilon \cdot \mathbb{R}_{>0}$, $\epsilon = \exp 2\pi i/N$, we have that $T_N$ is definable in the reals using parameters $\alpha, \beta \in \mathbb{R}$ and $\epsilon$ such that $\alpha^N = a, \beta^N = b$.

**Proof** It is enough to define an angular function with respect to the chosen parameters. Consider

$$P = \{z \in \mathbb{C}^\times : \frac{2\pi}{N} > \arg z \geq 0\}.$$

Define

$$P_{\epsilon^k} := \epsilon^k P, \quad k = 0, \ldots, N - 1$$

and

$$\text{ang } \lambda := \epsilon^k \text{ iff } \lambda^N \in \epsilon^k P.$$

This satisfies (7)-(9) by our assumptions. □

**2.4 Question** Consider a structure $\bar{T}_N$ which is existentially closed in the class of structures satisfying (7) - (9). What is the model-theoretic status of the theory of this structure? Is it supersimple?
Remark  Before this paper has been finished D.Evans answered this question in positive.

The fact that $\mathcal{T}_N$ is supersimple has certain methodological significance. There is a common, albeit informal, understanding that simple structures (theories) come basically from stable structures by introducing a ‘random noise’. So, one may think of $\mathcal{T}_N$ as an algebraic curve with a random angular function.

Problem  Study the structure of definable subsets on $\mathcal{T}_N$. Is there a good probabilistic measure theory on $\mathcal{T}_N$?
2.5 The space of semi-definable functions.

Let $\mathcal{H}$ be the $F$-algebra of semi-definable functions on $T_N$ generated by $x_0, \ldots, x_{N-1}, x_0^{-1}, \ldots, x_{N-1}^{-1}$.

**Remark** $\mathcal{H}$ is determined as a commutative $F$-algebra uniquely up to isomorphism by its generators $x_0, \ldots, x_{N-1}$ satisfying the relations (2).

We may also regard it as an $F$-vector space with some linear operators on them.

We define linear operators $U^*$ and $V^*$ on $\mathcal{H}$:

$$
U^* : \psi(t) \mapsto \psi(Ut), \\
V^* : \psi(t) \mapsto \psi(Vt).
$$

(10)

Obviously these operators are invertible, so $U^{*-1}$, $V^{*-1}$ are the inverses. Denote $\tilde{G}^*$ the group generated by the operators $U^*$, $V^*$, $U^{*-1}$, $V^{*-1}$.

$\mathcal{H}$ with the action of $\tilde{G}^*$ on it is determined uniquely up to isomorphism by the defining relation (2)-(6) and so is independent on the arbitrariness in the choices of $x_0, \ldots, x_{N-1}$.

Finally we notice

**Lemma** The correspondence $U \mapsto U^*$, $V \mapsto V^*$ generates the anti-isomorphism $\tilde{G} \to \tilde{G}^*$ satisfying the property

$$(g_1g_2)^* = g_2^*g_1^*, \text{ for any } g_1, g_2 \in \tilde{G}.$$ 

**Proof** It can easily be seen if we define the pairing

$$\mathcal{H} \times T \to F, \ (\psi, t) \mapsto \psi(t).$$

This allows to consider the adjoint action of any $g \in \tilde{G}$ on $\mathcal{H}$ setting $g^* \psi$ as the unique element of $\mathcal{H}$ such that

$$(g^* \psi, t) = (\psi, gt), \text{ for all } t \in T.$$ 

We can immediately identify that this definition extends (10). The desired formula follows.\square
2.6 Let Max(\mathcal{H}) be the space of maximal ideals of the commutative algebra \mathcal{H}.

**Lemma 1** Max(\mathcal{H}) consists of ideals \( I_\bar{\mu}, \bar{\mu} = (\mu_0, \ldots, \mu_{N-1}), \mu_0^N = \cdots = \mu_{N-1}^N, \)
\( I_\bar{\mu} = \langle (x_0 - \mu_0), \ldots, (x_{N-1} - \mu_{N-1}) \rangle. \)

**Proof** This is a standard fact of commutative algebra. □

Assuming \( F \) is endowed with an angular function \( \text{ang} : F^\times \to F[N] \) we call \( \bar{\mu} \) as above oriented positively if \( \mu_k = \text{ang}(\mu_0)^k \cdot \mu_0. \) Correspondingly, we call an ideal \( I_\bar{\mu} \), oriented positively if \( \bar{\mu} \) is.

Max^+(\mathcal{H}) will denote the subspace of Max(\mathcal{H}) consisting of positively oriented ideals \( I. \)

**Lemma 2** \( \bar{\mu} \) is positively oriented if and only if
\[ \langle \mu_0, \ldots, \mu_{N-1} \rangle = \langle x_0(t), \ldots, x_{N-1}(t) \rangle, \]
for some \( t \in T. \)

**Proof** Indeed, since \( x_0 \) is a bijection, there is \( t \in T \) such that \( x_0(t) = \mu_0. \) Now apply the definition of natural angular function of 2.3. □

2.7 **Lemma**

(i) There is a bijective correspondence \( \Xi : \text{Max}^+(\mathcal{H}) \to T_N \) between the space of positively oriented maximal ideals and \( T_N. \)

(ii) The action (10) of \( \hat{G}^* \) on \( \mathcal{H} \) induces an action on Max(\mathcal{H}) and leaves Max^+(\mathcal{H}) setwise invariant.

(iii) The action of \( \hat{g}^* \in \hat{G}^* \) on Max(\mathcal{H})(and so on \( T_N \)) can be identified as
\[ g^* : I_{(x_0, \ldots, x_{N-1}(t))} \mapsto I_{(x_0(g^{-1}t), \ldots, x_{N-1}(g^{-1}t))}. \]

**Proof** (i). We set
\[ \Xi(t) := I_{\bar{\mu}}, \text{ for } \bar{\mu} = (x_0(t), \ldots, x_{N-1}(t)). \]
Then \( \Xi(t) \) is positively oriented by Remark 2 in 2.6.
Notice that by definition $\bar{\mu}$ is determined uniquely by $\mu_0$. But $x_0 : T_N \to F^\times$ is bijective, so $\Xi$ is bijective.

(ii)-(iii). For a given $g \in \tilde{G}$, the map $\psi \to g^* \psi$ is an automorphism of the commutative $F$-algebra $\mathcal{H}$, since $g^* \psi(t) = \psi(gt)$. So, it sends maximal ideals to maximal ideals, namely

$$g : \langle (x_0 - \mu_0), \ldots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (x_0^g - \mu_0), \ldots, (x_{N-1}^g - \mu_{N-1}) \rangle.$$ 

Notice that, for the unique $t_\mu \in T_N$ such that $x_0(t_\mu) = \mu_0, \ldots, x_{N-1}(t_\mu) = \mu_{N-1}$

$$\langle x_0(U^{-1}t_\mu), \ldots, x_{N-1}(U^{-1}t_\mu) \rangle = \langle \alpha^{-1}\mu_0, \ldots, \alpha^{-1}\epsilon^{1-N}\mu_{N-1} \rangle,$$

by (3). Analogously, by (4)

$$\langle x_0(V^{-1}t_\mu), \ldots, x_{N-1}(V^{-1}t_\mu) \rangle = \langle \beta^{-1}\mu_{N-1}, \beta^{-1}\mu_0, \ldots, \beta^{-1}\mu_{N-2} \rangle.$$

So, by Lemma 2.6.2 both tuples on the right-hand side are positively oriented.

Now notice that by (3) and (4)

$$U : \langle (x_0 - \mu_0), \ldots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (\alpha x_0 - \mu_0), \ldots, (\alpha \epsilon^{N-1} x_{N-1} - \mu_{N-1}) \rangle = \langle (x_0 - \alpha^{-1}\mu_0), \ldots, (x_{N-1} - \alpha^{-1}\epsilon^{1-N}\mu_{N-1}) \rangle = \langle (x_0 - x_0(U^{-1}t)), \ldots, (x_{N-1} - x_{N-1}(U^{-1}t)) \rangle$$

and

$$V : \langle (x_0 - \mu_0), \ldots, (x_{N-1} - \mu_{N-1}) \rangle \mapsto \langle (\beta x_1 - \mu_0), \ldots, (\beta x_0 - \mu_{N-1}) \rangle = \langle (x_0 - \beta^{-1}\mu_{N-1}), \ldots, (x_{N-1} - \beta^{-1}\mu_{N-2}) \rangle = \langle (x_0 - x_0(V^{-1}t)), \ldots, (x_{N-1} - x_{N-1}(V^{-1}t)) \rangle.$$ 

This proves that the image of positive $I_{\bar{\mu}}$ under $U$ and $V$ is positive. Hence the image under the action of any $g \in \tilde{G}$ is positive, and we have (ii). The above also shows that the action induced by $\Xi$ is anti-isomorphic to the original action and so proves (iii). □
2.8 We may also treat $T$ as the space of $F$-linear functionals $\mathcal{H} \to F$ defined by the pairing of 2.5,

$$\mathcal{H}_T = \{ F_t : \psi \mapsto (\psi, t), \ t \in T \}.$$  

Obviously, the kernel of a nonzero functional is a maximal ideal. Moreover,

$$\ker F_t = \{ \phi \in \mathcal{H} : (\phi, t) = 0 \} = I_{(x_0(t), \ldots, x_{N-1}(t))}.$$  

We also denote $\ker F_t := I^t$.

We call a linear functional $F$ on $\mathcal{H}$ positive if $\ker F$ is a positive maximal ideal.

**Proposition**

(i) The correspondence 

$$t \mapsto F_t$$

between $T$ and the space $\mathcal{H}_+^*$ of positive linear functionals on $\mathcal{H}$ is bijective.

(ii) The correspondence transfers isomorphically the natural action of $\tilde{G}$ on $T$ to a natural action of $\tilde{G}$ on $\mathcal{H}_+^*$.

(iii) Consider also the commutative algebra $\mathcal{H}_0$ generated by $p(t)$ and, for each linear functional $F_t$ its restriction $F^0_t$ on $\mathcal{H}_0$. Then, for any $t_1, t_2 \in T$,

$$F^0_{t_1} = F^0_{t_2} \text{ iff } p(t_1) = p(t_2) \text{ iff } F_{t_1} = e^j F_{t_2}, \text{ for some } j \in \{0, \ldots, N-1\},$$

and the correspondence

$$F^0_t \mapsto p(t)$$

is a bijection between the space $\mathcal{H}_0^*$ of all linear functionals of the form $F^0_t$ and $\mathcal{F}^\times$.

**Proof** Let $I \in \text{Max}(\mathcal{H})$. To any such $I$ canonically corresponds the functional

$$F^I : \psi \mapsto \lambda \in F, \text{ such that } (\psi - \lambda) \in I.$$

We write

$$F(\psi) := \{ F, \psi \}.$$  

Now, in case $I = I^t = I_{(x_0(t), \ldots, x_{N-1}(t))}$, we see that

$$\{ F^I, \psi \} = \psi(t) = (\psi, t). \quad (11)$$

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The latter establishes the required bijection between $\mathcal{H}_T^*$ and $T_N$. On the other hand, since functionals of $\mathcal{H}_T^*$ are in bijective correspondence with positive ideals, by Lemma 2.6.2, $\mathcal{H}_T^* = \mathcal{H}_T^+$, the set of all positive functionals. This proves (i).

(ii). Given $F \in \mathcal{H}^*$ and $f \in \hat{G}^*$ define $f^*F$ as the unique functional such that

$$\{f^*F, \psi\} = (F, f\psi).$$

Then by dualities we have the isomorphism of group with actions on $T$ and $\mathcal{H}^+$ correspondingly

$$g \in \hat{G} \mapsto g^{**} \in \hat{G}^{***},$$

$$(\psi, gt) = (g^*_\psi, t) = \{F^l, g^*\psi\} = \{g^{**}F^l, \psi\}.$$ 

(iii). It is immediate from definitions that if $F^l$ evaluates $x_0$ as $\mu \in F^x$, then the function $p$ (as an element of $\mathcal{H}$) is evaluated as $\mu^N$. The statement follows.

2.9 We give here an alternative representation of the algebra $\mathcal{F}[U^*, V^*, U^{*-1}, V^{*-1}]$ as an algebra of linear operators on $\mathcal{H}$.

Lemma Given $\beta$ such that $\beta^N = b$, one can define functions

$$y_k : T_N \to F^* \quad k = 0, \ldots, N - 1$$

so that for any $t \in T_N$ the following functional equations are satisfied,

$$y_k(Ut) = e^ky_k(t),$$

$$y_k(Vt) = \beta y_{k+1}(t), \text{ where } y_N = x_0,$$

$$\frac{y_{k+1}(t)}{y_k(t)} = \frac{y_k(t)}{y_{k-1}(t)}.$$

Proof Similarly to 2.5 set

$$y_k(U^mV^n[U, V]^l \cdot s) := \beta^m e^{mk-l} \sqrt{s}.$$
This satisfies (12)-(14). □

Notice that we have the well-defined function

$$\xi(t) = \frac{y_{k+1}(t)}{y_k(t)}.$$  

Denote $y := y_1$. Note that by (14) $y_{k+1} = \xi^k y$, for $k = 0, \ldots, N - 1$.

Let $\mathcal{H}$ be the commutative $F$-algebra generated by $y$ and $\xi$. Any element of $\mathcal{H}$ is of the form

$$\sum_{m \in \mathbb{Z}} \sum_{k=0}^{N-1} a_{mk} \xi^k y^m$$

with finitely many nonzero $a_{mk} \in F$. Define operators on $\mathcal{H}$,

$$\hat{U} : \psi \mapsto U^* \psi, \quad (U^* \psi(t) = \psi(U t))$$

$$\hat{V} : \psi \mapsto \xi \cdot \psi.$$  

Using (12) we get the defining relation

$$\hat{U} \hat{V} = \varepsilon \hat{V} \hat{U}.$$  

Hence, this defines the algebra isomorphic to $F[U^*, V^*, U^{-1}, V^{-1}]$ acting on $\mathcal{H}$.

Consider, for a fixed $t \in T$, the ideal $J_t = \mathcal{H} \cdot (y_N^N - p(t))$ of the commutative algebra $\mathcal{H}$. This is a linear subspace of $\mathcal{H}$ which is invariant under $\hat{U}$ and $\hat{V}$. So the linear quotient-space

$$\mathcal{H}_t = \mathcal{H}/J_t$$

is a $F[\hat{U}, \hat{V}, \hat{U}^{-1}, \hat{V}^{-1}]$-module.

Since any element of $\mathcal{H}_t$ is of the form

$$\sum_{m=0}^{N-1} \sum_{k=0}^{N-1} a_{mk} \xi^k y^m$$

with $a_{mk} \in F$, $y^k = y^k + J_t$, the module is of dimension $N^2$ and can be decomposed into the direct sum

$$\mathcal{H}_t = M_t \oplus \xi M_t \cdots \oplus \xi^{N-1} M_t,$$
where
\[ \xi^k M_t = \{ \sum_{m=0}^{N-1} a_{mk} \xi^k y^m : a_{0,k}, \ldots, a_{N-1,k} \in F \} \]
are \( F[\mathcal{U}, \mathcal{V}, \mathcal{U}^{-1}, \mathcal{V}^{-1}] \)-modules. Moreover, each of the \( N \) modules is of dimension \( N \) and is irreducible.

Choose \( \mu \) so that \( \mu^N = p(t) \) and denote and \( \bar{\mu} = \mu^{-N} \bar{x}^N \). The \( \mathcal{V} \)-eigenvectors of \( \xi^k M_t \) are
\[ e^k_i = \bar{\mu} + \epsilon^{i+k} \mu^{-1} \xi^k y + \cdots + (\epsilon^{i+k} \mu^{-1})^N (\xi^k y)^{N-1}, \quad i = 0, \ldots, N - 1, \]
with eigenvalues \( \epsilon^{-(i+k)} \mu \), correspondingly. Hence, \( \{e^k_0, \ldots, e^k_{N-1}\} \) forms a basis of \( \xi^k M_t \). Obviously,
\[ \mathcal{U} e^k_i = e^k_{i+1}, \quad e^k_N = e^k_0, \quad e^k_{N+1} = \epsilon^0, \ldots \] (15)
Hence for exactly one value of \( k \in \{0, \ldots, N - 1\} \) we have the eigenvalues satisfying \( \text{ang} \mu = \epsilon^k \) (and remember \( \text{ang} \mu = \text{ang} \epsilon \mu \)), which corresponds to the sequence \( \langle \mu, \epsilon^k \mu, \ldots, \epsilon^{k(N-1)} \mu \rangle \) corresponding to the positively oriented functional \( F_t \) of 2.8, for \( \mu = x_0(t) \). We call such an irreducible module \( \xi^k M_t \) a positively oriented module. Notice that the sequence \( \langle \epsilon \mu, \epsilon^k \mu, \ldots, \epsilon^{k(N-1)} \epsilon \mu \rangle \) is again positively oriented and corresponds to \( [\mathcal{U}, \mathcal{V}]t \), as \( \epsilon \mu = x_0([\mathcal{U}, \mathcal{V}]t) \). The reordering of the sequence corresponds to the choice of the first \( \mathcal{V} \)-eigenvector \( e^k_0 \), and the consecutive are determined by (15). We call the module with a fixed choice of a first \( \mathcal{V} \)-eigenvector polarised. So we have proven

**Proposition** There is a bijective correspondence between
(a) positively oriented functionals
(b) positively oriented polarised irreducible modules
(c) points of \( T_N \).

Two points \( t, t' \in T_N \) correspond to isomorphic modules (without polarisation) if and only if \( t' = [\mathcal{U}, \mathcal{V}]^m t \), for some \( m \in \mathbb{Z} \). Correspondingly,
\[ F_{t'}(\psi) = \epsilon^m F_t. \]

**Remark** Another advantage of interpreting points of \( t \) as representations of \( \mathcal{U} \) and \( \mathcal{V} \) as linear operators is in the fact that one can impose some
relevant external conditions on the representations. Typically, when \( F = \mathbb{C} \) the modules can be considered with an inner product on them and the conditions are:

- \( U \) and \( V \) are unitary operators;
- or
- \( U \) and \( V \) are self-adjoint operators.

In the first case this will have as the consequence that all eigenvalues \( \mu \) above belong to the unit circle \( S \) of the complex plane. Under the second condition \( \mu \) has to be real, which contradicts the requirement that \( \epsilon \mu \) must be an eigenvector along with \( \mu \).

### 2.10 Comments

1. The spaces \( \mathcal{H} \) and \( \mathcal{H}^\dagger \) are analogues of the space \( S(\mathbb{R}^2, \mathbb{C}) \) of all Schwartz functions \( \mathbb{R}^2 \rightarrow \mathbb{C} \) decaying at infinity along with all its derivatives faster than \( \frac{1}{|x|^n} \), any \( n \) (see A.Connes).

2. In mathematical physics linear functionals on certain Hilbert spaces are called states.
   Assume for a moment that \( \mathcal{H} \) is an inner product space. Then any \( F \in \mathcal{H}^\dagger \) can be identified with the orthogonal complement \( I^\perp \) of the maximal ideal corresponding to \( F \). This is a one-dimensional subspace of \( \mathcal{H} \). This provides another version of the notion of states.

3. Even though the present definition of \( \mathcal{H} \) considers it a finitely generated commutative ring, it can not treat it as the coordinate ring of an algebraic variety since we consider positively oriented ideals only.

We used in 2.8 the natural pairing \( \mathcal{H} \times T \rightarrow F \) and the existence of enough functionals on the linear space \( \mathcal{H} \).

4. Despite the fact that \( T \) is in a bijective correspondence with a subset \( \mathcal{H}^\times_+ \) of the space of functionals we can not induce the additive structure on \( T \) since \( \mathcal{H}^\times_+ \) is not closed under addition.
3 The limit case

We introduce and study here a structure $\tilde{T}_N$ which can be seen as the limit version of $T_N$. It would be important in our view to formulate (and prove) the exact meaning of the transition $N \to \infty$ but we only draw here parts of the possible picture towards this aim.

3.1 Let $\alpha, \beta \in \mathbb{C}^\times$, $\alpha \mathbb{R} + \beta \mathbb{C} = \mathbb{C}$. Set, for $w \in \mathbb{C}$, the $\alpha\beta$-decomposition to be the uniquely determined decomposition

$$w = w_a \alpha + w_b \beta, \quad w_a, w_b \in \mathbb{R}.$$

Let $i_a, i_b \in \mathbb{R}$ be the coordinates of the decomposition

$$i = i_a \alpha + i_b \beta, \quad \text{here and below } i^2 = -1.$$

We also choose a real number $h$ and assume that $1, 2\pi i_a$ and $2\pi i_a h$ are linearly independent over $\mathbb{Q}$.

We define an additive $\alpha\beta$-version of the angular function, which we call band

$$bd_h : \mathbb{C} \to 2\pi i h \mathbb{Z}, \text{ fixed } h \in \mathbb{R} \setminus \mathbb{Q}$$
as follows.

First we define the function $r \mapsto [r]_h$ from $\mathbb{R}$ to $\mathbb{Z}$, the pseudo-integer part of $r$ with the properties, for all $r \in \mathbb{R}$,

$$[0]_h = 0, \quad [r + 1]_h = [r]_h + 1, \quad \{16\}$$

$$[r + 2\pi i_a]_h = [r]_h, \quad \{17\}$$

$$[r + 2\pi i_a h]_h = [r]_h \quad \{18\}$$

**Example** Consider a direct sum decomposition

$$\mathbb{R} = \mathbb{R}' + 2\pi i_a h \mathbb{Q} + 2\pi i_a h \mathbb{Q}, \text{ some subgroup } \mathbb{Q} < \mathbb{R}' < \mathbb{R},$$

and set, for all $r' \in \mathbb{R}'$, $c \in \mathbb{Q}$,

$$[r' + c_1 \cdot 2\pi i_a + c_2 \cdot 2\pi i_a h]_h := [r' + (c_1 - [c_1]) \cdot 2\pi i_a + (c_2 - [c_2]) \cdot 2\pi i_a h],$$

$[\cdot]$ the usual integer part of a real number. This satisfies (16)-(18),
Set
\[ \text{bd}_h w := 2\pi i h [w_h]. \]
We have then, by definition,
\[ \text{bd}_h (r \beta + w) = \text{bd}_h w, \text{ for every } r \in \mathbb{R}; \]  
(19)
\[ \text{bd}_h (w + 2\pi i) = \text{bd}_h (w); \]  
(20)
\[ \text{bd}_h (w + 2\pi i h) = \text{bd}_h w. \]  
(21)
By (16),
\[ \text{bd}_h (\alpha + w) = 2\pi i h + \text{bd}_h w. \]  
(22)
Set,
\[ \tilde{U} : w \mapsto \alpha + w, \]
\[ \tilde{V} : w \mapsto \beta + w + \text{bd}_h w. \]
We have
\[ w \mapsto^U \alpha + w \mapsto^V \alpha + \beta + w + \text{bd}_h (\alpha + w) = \alpha + \beta + w + \text{bd}_h w + 2\pi i h \mapsto^{U^{-1}} \]
\[ \mapsto^{U^{-1}} \beta + w + 2\pi i h + \text{bd}_h w \mapsto^{V^{-1}} 2\pi i h + w. \]
That is
\[ \tilde{V}^{-1} \tilde{U}^{-1} \tilde{V} \tilde{U} w = w + 2\pi i h, \]  
(23)
3.2 Define the additive subgroup of \( \mathbb{C} \)
\[ \mathcal{A}_h = \beta \mathbb{R} + 2\pi i h \mathbb{Z} + 2\pi i \mathbb{Z}. \]

**Proposition**  (i) \( \mathcal{A}_h \) is the subgroup of all **periods** of \( \text{bd}_h \), that is \( a \in \mathbb{C} \) such that \( \text{bd}_h (a + w) = \text{bd}_h w \).

(ii) \( \mathcal{A}_h \) is exactly the subgroup of shifts \( w \mapsto a + w \) of \( \mathbb{C} \) which are automorphisms of \( (\mathbb{C}, \tilde{U}, \tilde{V}) \).

(iii) \( \mathcal{A}_h \) is definable in \( (\mathbb{C}, +, \text{bd}_h) \).

**Proof** (i). Immediate from (19)- (21). For (ii) notice that \( \tilde{U} (a + w) = a + \tilde{U} w \), for all \( a \in \mathbb{C} \) and
\[ \tilde{V} (a + w) = a + \tilde{V} w \iff a \in \mathcal{A}_h. \]
(iv) Immediate by definitions. □

3.3 We consider here the two-sorted structures
\[((\mathbb{C}, +, \text{bd}_h), \exp, \mathbb{C}^\times)\] and \[((\mathbb{C}, +, \mathcal{A}_h), \exp, \mathbb{C}^\times)\]
where the second sort \(\mathbb{C}^\times\) on the nonzero complex numbers comes with the usual language of all Zariski closed relations.

Obviously the functions \(U\) and \(V\) are definable in \((\mathbb{C}, +, \text{bd}_h)\). Conversely, \(\text{bd}_h\) is definable in \((\mathbb{C}, +, \hat{V})\) using parameter \(\beta\).

**Proposition 1** The theory of \(((\mathbb{C}, +, \mathcal{A}_h), \exp, \mathbb{C}^\times)\) is superstable, provided the Schanuel conjecture is true.

**Proof** It is easy to see that the statement follows if the expansion of \(\mathbb{C}^\times\) with the unary predicate for the subgroup \(\mathcal{G}_h = \exp(\mathcal{A}_h) = \exp(2\pi i \mathbb{Z} + \beta \mathbb{R})\) is superstable. A stronger theorem, stating \(\omega\)-stability of the theory, for \(\mathcal{G} = \exp(\beta \mathbb{R} + \delta \mathbb{Q}), \beta \in \mathbb{C} \setminus (\mathbb{R} \cup i \mathbb{R}), \delta \in \mathbb{R} \setminus 2\pi i \mathbb{Q}\), was proved in [Z2]. The same proof describes the elementary theory of the structure and yields superstability for the present theory. See also [Z3]. □

**Notation** \(\mathcal{G}_h\) will stand for the subgroup \(\exp(\mathcal{A}_h)\) of \(\mathbb{C}^\times\).

On the other hand \(((\mathbb{C}, +, \text{bd}_h), \exp, \mathbb{C}^\times)\) defines the following unstable structure on the sort \(\mathbb{C}^\times\).

Denote, for \(\mathbf{t} = \exp w\),
\[\text{ang}_h \mathbf{t} := \exp \text{bd}_h w,\]
By (20) this is well-defined, and by (19),(21) we have analogues of (7)-(9), where \(q = \exp 2\pi i h\),
\[\text{ang}_h q \mathbf{t} = \text{ang}_h \mathbf{t},\]
\[\text{ang}_h e^{\beta} \mathbf{t} = \text{ang}_h \mathbf{t},\]
\[\text{ang}_h e^{\alpha} \mathbf{t} = q \cdot \text{ang}_h \mathbf{t}.\]

Hence, defining
\[\textbf{U} : \mathbf{t} \mapsto e^\alpha \cdot \mathbf{t}, \quad \textbf{V} : \mathbf{t} \mapsto e^{\beta} \cdot \mathbf{t} \cdot \text{ang}_h \mathbf{t},\]
we get
\[ \mathbf{VU}t = q \mathbf{UV}t, \text{ for all } t \in \mathbb{C}^\times. \]

It is easy to see that also
\[ \mathbf{U} \exp w = \exp \mathbf{U}w, \quad \mathbf{V} \exp w = \exp \mathbf{V}w. \]

We define
\[ \tilde{T}_h := (\mathbb{C}, +, \cdot, \text{ang}_h). \]
This is an obvious analogue of $\tilde{T}_N$ defined in 2.3.

Note that the group $\Gamma_h = \exp 2\pi i \mathbb{Z} = \text{ang}_h(\mathbb{C}^\times)$ is definable in $\tilde{T}_h$.

The full analogy with $\tilde{T}_N$ of 2.3 requires also a definition of $p_h$. We define
\[ p_h : \mathbb{C}^\times \to \mathbb{C}^\times / \Gamma_h, \]
the canonical homomorphism. This agrees with 2.3, moreover in the finite case $\mathbb{C}^\times / \langle e \rangle$ can be definably identified with $\mathbb{C}^\times$ in the full Zariski language, in particular the whole construction is a Zariski structure (obviously, of finite Morley rank).

We also define the maps $u$ and $v$ on $\mathbb{C}^\times / \Gamma_h$ by
\[ u_p(t) := p_h(Ut), \quad v_p(t) := p_h(Vt), \]
that is
\[ u : t \cdot \Gamma_h \mapsto e^\alpha \cdot t \cdot \Gamma_h, \quad v : t \cdot \Gamma_h \mapsto e^\beta \cdot t \cdot \Gamma_h. \]
This is obviously well-defined.

**Proposition 2** The group of shifts $t \mapsto gt$ on $\mathbb{C}^\times$ commuting with $\text{ang}_h$ (and so with $\mathbf{U}$ and $\mathbf{V}$) is $\mathcal{G}_h$. This group is definable in $\tilde{T}_h$. The theory of the structure $(\mathbb{C}, +, \cdot, \mathcal{G}_h, \Gamma_h)$ is superstable.

**Proof** Essentially the same argument as for Proposition 1. The superstability of the weaker structure $(\mathbb{C}, +, \cdot, \Gamma_h)$ is well-known and follows from the Lang property of $\Gamma_h$. □

**Problems** 1. Fix the theory $\mathcal{T}_h^\varphi$ of structures of the form $(F, +, \cdot, \text{ang}, e_a)$, $(e_a \text{ a constant})$ saying that
\[ (F, +, \cdot, \text{Aut}(\text{ang}), \text{ang}(F^\times), e_a) \equiv (\mathbb{C}, +, \cdot, \mathcal{G}_h, \Gamma_h, e^\circ) \]

(where Aut(\text{ang}) is the group of shifts of \( F^\times \) commuting with \text{ang}, and \text{ang}(F^\times) is the image under \text{ang})

and

\[ \forall t \in F^\times \text{ ang } g \cdot t = q \cdot \text{ ang } t \text{ iff } g^{-1}e_a \in \text{Aut}(\text{ang}). \]

Consider the class \( \mathcal{T}_G^h \) of existentially closed models of \( T_{Gh} \). What is the stability status of completions of \( \mathcal{T}_G^h \)? Are they supersimple?

2. Is \( \mathcal{T}_h \) above based on the band function \( \text{bd}_h \) given in the Example in 3.1 existentially closed in \( \mathcal{T}_G^h \)? Is it supersimple?

3.4 We notice here that in \((\mathbb{C}, +, \text{bd}_h, 2\pi i_a, \cdot, \text{exp}, \mathbb{C}^\times)\) (2\pi i_a and \cdot are unary operations here) one can definably construct an inverse to the usual exponentiation \( \text{exp} : \mathbb{C} \to \mathbb{C}^\times \).

Define the function

\[ \ln_0 : \mathbb{C}^\times \to \mathbb{C} \]

by setting, for \( t = \text{exp} w \),

\[ \ln_0 t = w - h^{-1}\text{bd}_h(w/2\pi i_a). \]

It is immediate that

\[ \text{exp}(\ln_0 t) = t. \]

Claim \( \ln_0 t \) is well-defined and is injective.

Indeed, if also \( \text{exp} w' = t \), \( w' = w + 2\pi ik \), some \( k \in \mathbb{Z} \), then

\[ \text{bd}_h(w'/2\pi i_a) = \text{bd}_k\left(\frac{w + 2\pi ik}{2\pi i_a}\right) = \text{bd}\left(\frac{w}{2\pi i_a}\right) + 2\pi ik, \text{ by (22)}. \]

Hence,

\[ w' - h^{-1}\text{bd}_h(w'/2\pi i_a) = w - h^{-1}\text{bd}_h(w/2\pi i_a), \]

as required.

In more detail,

\[ \ln_0 t = w - 2\pi i\left[\frac{w_a}{2\pi i_a}\right]_h. \]

(24)
So,
\[
\ln_0 t = \ln_0 t' \text{ iff } w - 2\pi i \lfloor \frac{w_a}{2\pi i a} \rfloor_h = w' - 2\pi i \lfloor \frac{w'_a}{2\pi i a} \rfloor_h,
\]
whence \( w - w' \in 2\pi i \mathbb{Z} \) and \( t = t' \),
hence \( \ln_0 \) is injective.

3.5 Now we redefine \( \bar{T}_N \) in a way compatible both with 2.3 and 3.3.

Define, for each positive \( N \in \mathbb{N} \) the map
\[
e_{Nh} : \mathbb{C} \to \mathbb{C}^x; \quad e_{Nh}(w) = \exp(N^{-1}h^{-1}w).
\]
It is convenient to distinguish the copies of \( \mathbb{C}^x \) which are images of \( e_{Nh} \) for different \( N \) as \( T_N \).

Set, for \( t = e_{Nh}(w) \in T_N \),
\[
U_N t := e_{Nh}(\bar{U}w), \quad V_N t := e_{Nh}(\bar{V}w).
\]
It follows,
\[
U_N t := e_{Nh}(\alpha) \cdot t, \quad V_N t := e_{Nh}(\beta) \cdot t \cdot \exp \left( \frac{2\pi i}{N} [w_a]_h \right).
\]
Denote
\[
\ang_N(t) := \exp \left( \frac{2\pi i}{N} [w_a]_h \right).
\]
This is well-defined. Indeed, any other representation of \( t \) would be of the form \( t = e_{Nh}(w + 2\pi i hNk) \), \( k \in \mathbb{Z} \). But \( (w + 2\pi i hNk)_a = w_a + 2\pi i_a hNk \), and \( [w_a + hNk]_h = [w_a]_h \) by (18).

Similarly one checks that \( \ang_N \) satisfies (7)-(9) with \( \epsilon = \exp \frac{2\pi i}{N} \) and corresponding parameters for \( \alpha, \beta \). So we get, by 2.3
\[
V_N U_N t = \epsilon U_N V_N t. \tag{25}
\]
Define
\[
\bar{T}_N = (\mathbb{C}, +, \cdot, \ang_N)
\]
This is the same definition as 2.3 except here we specified our choice of the angular function.
Proposition  The group of periods of $\ang_N$, that is $g \in \mathbb{C}^\times$ such that $\ang_N(g \cdot t) = \ang_N t$ is equal to

$$\mathcal{G}_{N^{-1} h^{-1}, ah^{-1}} \cdot \mathbb{C}[N] = \exp(2\pi i N^{-1} h^{-1} + \alpha h^{-1} \mathbb{Z} + \beta \mathbb{R}) \cdot \mathbb{C}[N].$$

In particular, this group is definable in the above $\hat{T}_N$ and the theory of

$$(\mathbb{C}, +, \cdot, \mathcal{G}_{N^{-1} h^{-1}, ah^{-1}})$$

is superstable.

Proof  By calculation: for $t = \exp N^{-1} h^{-1} w$ and $g = \exp N^{-1} h^{-1} u$, by definition,

$$\ang_N(gt) = \exp \frac{2\pi i}{N} [w_a + u_a]_h,$$

so $g$ is a period if and only if

$$\forall r \in \mathbb{R} \ [r + u_a]_h \equiv [r]_h \mod N\mathbb{Z},$$

iff $u_a \in 2\pi i_a \mathbb{Z} + 2\pi i_a h \mathbb{Z} + N\mathbb{Z}$ iff

$$g \in \exp(2\pi i_a h^{-1} N^{-1} + 2\pi i_a \alpha N^{-1} \mathbb{Z} + \alpha h^{-1} \mathbb{Z} + \beta \mathbb{R}) = \exp(2\pi i N^{-1} h^{-1} \mathbb{Z} + 2\pi i N^{-1} \mathbb{Z} + \alpha h^{-1} \mathbb{Z} + \beta \mathbb{R}).$$

The superstability follows by the same argument as in 3.3. $\square$

Problem  Is the theory of $\hat{T}_N$ as given by the present construction, supersimple?

3.6 Denote

$$\mathcal{U} = (\mathbb{C}, +, bd_h, h \cdot).$$

By the construction in 3.3 and 3.5 $\hat{T}_N$ is definable in $(\mathcal{U}, \exp, \mathbb{C}^\times)$, for all $N \in \mathbb{N} \cup \{h\}$.

The resulting picture is as follows, with the arrows showing definable surjections.
where $e_1(w) := \exp w$. 
4 Quantum torus

Our aim here is to connect the construction of $\tilde{T}_h$ to the well-known definition of the noncommutative (quantum) torus usually denoted $T^2_h$.

4.1 Following the pattern of 2.2 and 2.3 we introduce the algebra $\mathcal{H}$ generated by functions

$$x_k : \mathbb{C}^\times \to \mathbb{C}^\times, \quad k \in \mathbb{Z},$$

where $x_0 = x$ is the identity function and

$$x_k = \xi^k \cdot x, \quad \xi(t) = \text{ang}_h t.$$

We have by 3.3,

$$x_k(Ut) = e^\alpha q^k \cdot x_k(t),$$
$$x_k(Vt) = e^\beta x_{k+1}(w),$$
$$\xi(Ut) = q \cdot \xi(t), \quad \xi(Vt) = \xi(t).$$

As in 2.9 we normalise the operators $U^*$ and $V^*$ on functions by defining operators on $\mathcal{H},$

$$\tilde{U} : \psi \mapsto U^*\psi, \quad U^*\psi(w) = \psi(Uw);$$
$$\tilde{V} : \psi \mapsto \xi \cdot \psi.$$

Using the identities above we get immediately the usual

$$\tilde{U}\tilde{V} = q\tilde{V}\tilde{U}.$$

4.2 We can introduce an isomorphic space with operators in an alternative but closely connected way.

Let $z$ and $\zeta$ be the functions $\mathbb{C} \to \mathbb{C}^\times$ given by

$$z(w) = \exp w, \quad \zeta(w) = \exp \text{bd}_h w.$$

Denote $\tilde{\mathcal{H}}$ the commutative F-algebra generated by $z$ and $\zeta$, and denote $z_k = \zeta^k z$.

We have, using identities for $\text{bd}_h$,

$$z(\tilde{U}w) = e^\alpha \cdot z(w), \quad \zeta(\tilde{U}w) = q \cdot \zeta(w),$$
$$z(\tilde{V}w) = e^\beta \zeta(w)z(w), \quad \zeta(\tilde{V}w) = \zeta(w).$$
Again, we define operators on \( \hat{\mathcal{H}} \):

\[
\hat{U} : \psi \mapsto \hat{U}^* \psi, \\
\hat{V} : \psi \mapsto \zeta \cdot \psi.
\]

The space \( \hat{\mathcal{H}} \) is an analogue of the space \( S(\mathbb{R}^2, \mathbb{C}) \) of all Schwartz functions \( \mathbb{R}^2 \to \mathbb{C} \) decaying at infinity along with all its derivatives faster than \( \frac{1}{|x|^n} \), any \( n \) (see [C]), or \( S(\mathbb{Z}^2, \mathbb{C}) \) the Hilbert space of Schwartz sequences, that is complex valued sequences \( (c_{m,n}) \) decaying faster than any polynomial of \( m, n \).

In [C] with each leaf of the Kronecker foliation \( L_a = \{ (r,s) \in \mathbb{R}^2 : s + \theta r = a \} \), one associates the \( \mathbb{C}[\hat{U}, \hat{V}, \hat{U}^{-1}, \hat{V}^{-1}] \)-module \( \mathcal{H}_a \) obtained by restricting functions of \( S(\mathbb{R}^2, \mathbb{C}) \) to \( L_a \) and defining operators \( \hat{U} \) and \( \hat{V} \). Namely, the operator \( \hat{U} \) is defined by exactly the same formula as here and \( \hat{V} \) sends \( \psi(r,s) \) (function of two real variables \( r \) and \( s \)) to \( \exp(is) \cdot \psi(r,s) \) (notice that extra to these data there is a linear dependence between \( r \) and \( s \)). So, \( \xi \) is a good analogue of the function \( \exp(is) \) taking values in the unit circle.

Notice that \( \hat{U} \) and \( \hat{V} \) are unitary operators if we see \( \mathcal{H}_a \) as a Hilbert space. This makes the completion of \( \mathbb{C}[\hat{U}, \hat{V}, \hat{U}^{-1}, \hat{V}^{-1}] \) a \( \mathbb{C}^* \)-algebra.

By A.Connes the quantum torus \( T_\theta^2 \) is the space of all the modules \( \mathcal{H}_a \) on the correspondent \( L_a \).

**Remark** Consider again the algebra of functions \( \hat{\mathcal{H}} \) and denote, for \( a \in \mathbb{C}, \mathcal{H}_a \) the algebra obtained by restricting functions from \( \hat{\mathcal{H}} \) to the coset \( a + \mathcal{A}_h \). It follows from Proposition 3.2(ii) that the action of \( \hat{U} \) and \( \hat{V} \) on \( \hat{\mathcal{H}} \) induces a well-defined action on \( \mathcal{H}_a \), so this is a \( \mathbb{C}[\hat{U}, \hat{V}, \hat{U}^{-1}, \hat{V}^{-1}] \)-module for any \( a \in \mathbb{C} \).

### 4.3

To understand further relations of Connes’ construction to our \( T_h \) we prove the following.

**Claim 1.** There is a natural bijective correspondence

\[
\phi : \mathbb{C}/\mathcal{A}_h \to T_\theta^2,
\]

for \( \theta = h \), where \( T_\theta^2 \) is seen as the space of leaves of the Kronecker foliation.
Indeed, we have the decomposition of $\mathbb{C}$ into two real lines

$$
\mathbb{C} = i\mathbb{R} + \alpha\mathbb{R}, \quad \text{for any } z \in \mathbb{C} \text{ if } z = x + y\alpha, \ x, y \in \mathbb{R}.
$$

Rescale the real coordinates

$$
r := h^{-1}x, \quad s := 2\pi(2\pi i\alpha)^{-1}y
$$

and consider the mapping onto the direct product of two unit circles

$$
z \mapsto (x, y) \mapsto (r, s) \mapsto (\exp ir, \exp is).
$$

Under the map

$$
2\pi ih\mathbb{Z} + 2\pi i\alpha\mathbb{Z} \to \langle 2\pi h\mathbb{Z}, 2\pi i\alpha\mathbb{Z} \rangle \to \langle 2\pi \mathbb{Z}, 2\pi \mathbb{Z} \rangle \to 1,
$$

and since $2\pi i - 2\pi i\alpha \in \beta\mathbb{R}$,

$$
\beta\mathbb{R} \to \langle 2\pi, -2\pi i\alpha \rangle \mathbb{R} \to \langle 2\pi h^{-1}, -2\pi \rangle \mathbb{R} \to L_0.
$$

This establishes the bijection between the cosets of $A_h$ and the leaves $L_a$ of the foliation.

Claim 2. There is a bijective correspondence

$$
\tilde{p}_h : \mathbb{C}/A_h \to \mathbb{C}^\times/G_h,
$$

induced by $p_h$. Moreover, the action of $\tilde{U}$ and $\tilde{V}$ on $\mathbb{C}$ induces a well-defined action on $\mathbb{C}/A_h$ and correspondingly the action on $\mathbb{C}^\times/G_h$. The latter action coincides with the one induced by $u$ and $v$ on the cosets of $G_h$.

This is the direct consequence of Proposition 3.2(iii) and the definition of $p_h$.

**Corollary** $\tilde{p}_h \circ \phi^{-1}$ identifies $T^2_h$ with $\mathbb{C}^\times/G_h$, with all the structure on the latter induced from $T_h$. 

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