Local scale-invariances in the bosonic contact and pair-contact processes

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Abstract. Local scale-invariance for ageing systems without detailed balance is tested through studying the dynamical symmetries of the critical bosonic contact process and the critical bosonic pair-contact process. Their field-theoretical actions can be split into a Schrödinger-invariant term and a pure noise term. It is shown that the two-time response and correlation functions are reducible to certain multipoint response functions which depend only on the Schrödinger-invariant part of the action. For the bosonic contact process, the representation of the Schrödinger group can be derived from the free diffusion equation, whereas for the bosonic pair-contact process, a new representation of the Schrödinger group related to a non-linear Schrödinger equation with dimensionful couplings is constructed. The resulting predictions of local scale-invariance for the two-time responses and correlators are completely consistent with the exactly-known results in both models.

PACS numbers: 05.10-a, 05.40-a, 64.60.Ht, 02.20.Qs


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1. Introduction

The concept of scale-invariance is central to the modern understanding of critical phenomena in and out of equilibrium. Its exploitation through the renormalization group has in particular led to the recognition of universal critical exponents and scaling functions which describe the behaviour of physical observables, see e.g. [1] and references therein. Here we are concerned with the slow dynamics of systems brought rapidly to their critical point and/or into a phase with more than one thermodynamically stable state. Such a kind of behaviour is typical for glassy systems but also occurs in simple magnets with a purely relaxational dynamics which were quenched from a disordered state to a final temperature \( T \leq T_c \), where \( T_c > 0 \) is the critical temperature. For the latter, it is now understood that the dynamics is governed by a single time-dependent length-scale \( L = L(t) \sim t^{1/z} \) for \( t \) sufficiently large and where \( z \) is the dynamical exponent [2]. As an example, consider simple magnets relaxing towards equilibrium. For phase-ordering kinetics \( (T < T_c) \), the Bray-Rutenberg theory shows that dynamical scaling together with the assumption of a Porod law for the time-dependent structure factor predicts the value of \( z \) [3]; whereas for \( T = T_c \) the value of \( z \) is computed from critical (equilibrium) dynamics. More recently, it has been understood that the study to two-time observables provides further and deeper insight, in particular the ageing behaviour is made explicit through the breaking of time-translation invariance. The challenge is now to find the values of the associated non-equilibrium (ageing) exponents and also the form of the scaling functions, see below for the precise definitions.

A common way to study this problem is through a Langevin equation which should describe the dynamics of a coarse-grained order-parameter. This may be turned into a field-theory and renormalization-group methods then allow to extract values of these exponents, in quite good but not perfect agreement with the results of direct numerical simulations [4, 5]. On the other hand, the resulting predictions for the scaling functions appear to be far from the numerical results, see [6, 7]. An alternative approach seeks to extend dynamical scaling to a larger group of ‘local’ scale-transformations [8], see [9] for a recent review. In the framework of a de Dominicis-Janssen type theory [10, 11], the effective action \( S = S[\phi, \tilde{\phi}] \) is given in term of the order-parameter field \( \phi \) and its associated response field \( \tilde{\phi} \). Furthermore, for systems in contact with a thermal bath such that detailed balance holds one always has the decomposition \( S[\phi, \tilde{\phi}] = S_0[\phi, \tilde{\phi}] + S_b[\tilde{\phi}] \) into a ‘deterministic’ part \( S_0 \) which can be derived from the Langevin equation when all noise terms are dropped and the ‘noise’ term \( S_b[\tilde{\phi}] \) which depends only on the response function [12]. The form of response functions can then be found from the requirement of covariance under the group of local scale-transformations. As we shall explain below, correlation functions can be reduced to certain integrals of higher, multipoint response functions [12]. This approach yields the form of the scaling functions whereas the exponents are treated as parameters whose values have to be
supplied† and reproduce perfectly the results of both analytical and simulative studies of many common spin systems undergoing phase-ordering kinetics where \( z = 2.\)

It is an established fact that the basic Langevin equation for the order-parameter does not admit any symmetries beyond dilatations and (space-)translations, see [14] for a recent discussion. However, it has been shown that at least for simple magnets it is enough to concentrate on the dynamical symmetries of the \textit{deterministic} part of the Langevin equation only, as given by the action \( S_0.\) In particular, Schrödinger-invariance of that deterministic part is sufficient to be able to derive the two-time correlations \( C(t, s)\) and two-time response functions \( R(t, s)\) explicitly [12]. There is an exact agreement for systems such as the spherical model, the XY model in spin-wave approximation or the voter model which are all described by a linear Langevin equation. Good agreement with simulations of Ising, Potts and XY models was found as well [7, 15, 16].

In this paper, we extend the treatment of local scale-invariance to ageing systems with a dynamical exponent \( z = 2 \) but without detailed balance. Working with a de Dominicis-Janssen type theory, we find again a decomposition \( S[\phi, \bar{\phi}] = S_0[\phi, \bar{\phi}] + S_b[\phi, \bar{\phi}]\) into a ‘deterministic’, Schrödinger-invariant term \( S_0\) and ‘noise’ terms, each of which contains at least one response field more than order-parameter fields (explicit expressions will be given in sections 2 and 3). Then the Bargman superselection rules which follow from the Galilei-invariance of \( S_0\) are enough to establish that again the two-time response function is noise-independent and the two-time correlation function can be reduced to a finite sum of response functions the form of whom is strongly constrained again by the requirement of their Schrödinger-covariance. These developments provide further evidence for a \textit{hidden non-trivial local scale-invariance in ageing systems} which manifests itself directly in the ‘deterministic’ part (see [17] for the construction of Schrödinger-invariant semi-linear kinetic equations) but which strongly constrains the full noisy correlations and responses.

We test the present framework of local scale-invariance in two exactly solvable systems with a non-linear coarse-grained Langevin equation. A convenient set of models with a non-trivial ageing behaviour is furnished by the bosonic contact [18] and pair-contact processes [19], both at criticality. These systems are defined as follows. Consider a set of particles of a single species \( A\) which move on the sites of a hypercubic lattice in \( d\) dimensions. On any site one may have an arbitrary (non-negative) number of particles.\(\)†

† This is close in spirit to the treatment of \textit{equilibrium} phase transitions through conformal invariance, which fixes the form of the \( n\)-point correlators in terms of the scaling dimensions of the scaling fields [1]. Furthermore, those exponents can be determined exactly in 2\(D\) from symmetry considerations (i.e. representation theory of the Virasoro algebra) alone since the conformal symmetry is infinite-dimensional in that case.

‡ For non-equilibrium critical dynamics where \( z \neq 2 \) in general one also has a good match with numerical data for the response functions in direct space but systematic differences may appear in momentum-space calculations, e.g. in the 2\(D\) Ising model for \( t/s \lesssim 10\) [13].

§ This property distinguishes the models at hand from the conventional (‘fermionic’) contact and pair-
Single particles may hop to a nearest-neighbour site with unit rate and in addition, the following single-site creation and annihilation processes are admitted

\[ mA \xrightarrow{\mu} (m+1)A , \quad pA \xrightarrow{\lambda} (p-\ell)A \; \text{with rates } \mu \text{ and } \lambda \]  

(1)

where \( \ell \) is a positive integer such that \(|\ell| \leq p\). We are interested in the following special cases:

(i) **critical bosonic contact process**: \( p = m = 1 \). Here only \( \ell = 1 \) is possible. Furthermore the creation and annihilation rates are set equal \( \mu = \lambda \).

(ii) **critical bosonic pair-contact process**: \( p = m = 2 \). We fix \( \ell = 2 \), set \( 2\lambda = \mu \) and define the control parameter \( \alpha := \frac{3\mu}{2D} \)

(2)

The dynamics is described in terms of a master equation which may be written in a hamiltonian form \( \partial_t |P(t)\rangle = -H|P(t)\rangle \) where \( |P(t)\rangle \) is the time-dependent state vector and the hamiltonian \( H \) can be expressed in terms of creation and annihilation operators \( a(x,t) \) and \( a(x,t) \) \([20,21,22]\). It is well-known that these models are critical in the sense that their relaxation towards the steady-state is algebraically slow \([18,19,24]\). In particular, the local particle-density is \( \rho(x,t) := \langle a(x,t) \rangle \). Its spatial average remains constant in time

\[ \int \text{d}x \rho(x,t) = \int \text{d}x \langle a(x,t) \rangle = \rho_0 \]  

(3)

where \( \rho_0 \) is the initial mean particle-density. We are interested in the two-time connected correlation function

\[ G(r,t; s) := \langle a(x,t)a(x+r,s) \rangle - \rho_0^2 \]  

(4)

and take an uncorrelated initial state, hence \( G(r; 0,0) = 0 \). The linear two-time response function is found by adding a particle-creation term \( \sum_x h(x,t)(a^\dagger(x,t)-1) \) to the quantum hamiltonian \( H \) and taking the functional derivative

\[ R(r; t; s) := \left. \frac{\delta \langle a(x+r,t) \rangle}{\delta h(x,s)} \right|_{h=0} \]  

(5)

We have previously analyzed these quantities in the scaling limit where both \( t, s \) as well as \( t - s \) become large with respect to some microscopic reference time. The results are as follows \([24]\): consider the autocorrelation and autoresponse functions, which satisfy the scaling forms

\[ G(t, s) := G(0; t, s) = s^{-b}f_G(t/s) \]  

(6)

\[ R(t, s) := R(0; t, s) = s^{-1-a}f_R(t/s) \]  

(7)

contact processes whose critical behaviour is completely different.

|| If instead we would treat a coagulation process \( 2A \rightarrow A \), where \( \ell = 1 \), the results presented in the text are recovered by setting \( \lambda = \mu \) and \( \alpha = \mu/D \).
Table 1. Ageing exponents of the critical bosonic contact and pair-contact processes in the different regimes. The results for the bosonic contact process hold for an arbitrary dimension $d$, but for the bosonic pair-contact process they only apply if $d > 2$, since $\alpha_c = 0$ for $d \leq 2$.

Table 2. Scaling functions (up to normalization) of the autoresponse and autocorrelation of the critical bosonic contact and bosonic pair-contact processes.

where the values of the exponents $a$ and $b$ are listed in table 1. Here the critical value $\alpha_c$ for the pair-contact process is explicitly given by [19]

$$\frac{1}{\alpha_c} = 2 \int_0^\infty \, \text{du} \left( e^{-4u} I_0(4u) \right)^d$$

(8)

where $I_0$ is a modified Bessel function. The dynamical behaviour of the contact process is independent of $\alpha$. For the critical bosonic pair-contact process, there is a clustering transition between a spatially homogeneous state for $\alpha < \alpha_c$ and a highly inhomogeneous one for $\alpha > \alpha_c$ where dynamical scaling does not hold. These two transitions are separated by a multicritical point at $\alpha = \alpha_c$. Since our models do not satisfy detailed balance, there is no reason why the exponents $a$ and $b$ should coincide and our result $a \neq b$ for the bosonic pair-contact process is perfectly natural.

While the scaling function $f_R(y) = (y - 1)^{-d/2}$ has a very simple form, the autocorrelator scaling function has an integral representation

$$f_G(y) = G_0 \int_0^1 \, \text{d}\theta \theta^{a-b} (y + 1 - 2\theta)^{-d/2}$$

(9)

where the values for $a$ and $b$ are given in table 1 and $G_0$ is a known normalization constant. The explicit scaling functions are listed up to normalization in table 2 [24]. In this paper, we shall study to what extent their form can be understood from local scale-invariance.
This paper is organized as follows. In section 2 we treat the bosonic contact process in its field-theoretical formulation. The action is split into a Schrödinger-invariant term $S_0$ and a noise term $S_b$ and we show how the response and correlation functions can be exactly reduced to certain noiseless three- and four-point response functions. In this reduction the Bargman superselection rules which follow from the Schrödinger-invariance of $S_0$ play a central rôle. These tools allow us to predict the response-and correlation functions which will be compared to the exact results of table 2. In section 3 the same programme is carried out for the bosonic pair-contact process but as we shall see, the Schrödinger-invariant term $S_0$ of its action is now related to a non-linear Schrödinger equation. The treatment of this requires an extension of the usual representation of the Schrödinger Lie-algebra which now includes a dimensionful coupling constant. The construction is carried out in appendix A. The required $n$-point correlation functions coming from this new representation are derived in appendices B and C. Finally, in section 4 we conclude.

2. The contact process

2.1. Field-theoretical description

The master equation which describes the critical bosonic contact process as defined in section 1 can be turned into a field-theory in a standard fashion through an operator formalism which uses a particle annihilation operator $a(r, t)$ and its conjugate $a^\dagger(r, t)$, see for instance [20, 22] for detailed discussion of the technique. Since we shall be interested in the connected correlator, we consider the shifted field and furthermore introduce the shifted response field

$$
\phi(r, t) := a(r, t) - \rho_0 \\
\tilde{\phi}(r, t) := a(r, t) = a^\dagger(r, t) - 1
$$

such that $\langle \phi(r, t) \rangle = 0$ (our notation implies a mapping between operators and quantum fields, using the known equivalence between the operator formalism and the path-integral formulation [23, 22]). As we shall see, these fields $\phi$ and $\tilde{\phi}$ will become the natural quasiprimary fields from the point of view of local scale-invariance. We remark that the response function is not affected by this shift, since

$$
R(r, r'; t, s) = \frac{\delta \langle a(r, t) \rangle}{\delta h(r', s)} = \frac{\delta \langle \phi(r, t) \rangle}{\delta h(r', s)}
$$

Then the field-theory action reads, where $\mu$ is the reaction rate [25]

$$
S[\phi, \tilde{\phi}] = \int dR \int du \left[ \tilde{\phi}(2M \partial_u - \nabla^2)\phi - \mu \tilde{\phi}^2(\phi + \rho_0) \right] = S_0[\phi, \tilde{\phi}] + S_b[\phi, \tilde{\phi}]
$$

To keep expressions shorter, we have supressed the arguments of $\phi(R, u)$ and $\tilde{\phi}(R, u)$ under the integrals and we shall also do so often in what follows, if no ambiguity arises.
The diffusion constant $D$ is related to the ‘mass’ $M$ through $D = (2M)^{-1}$. We have decomposed the action as follows:

$$ S_0[\phi, \tilde{\phi}] := \int dR \int du \left[ \tilde{\phi}(2M\partial_u - \nabla^2)\phi \right] $$

(13)

describes the deterministic,\footnote{This terminology is used since the equation of motion of $\phi$ following from $S_0$ is a partial differential equation and not a stochastic Langevin equation.} noiseless part whereas the noise is described by

$$ S_b[\phi, \tilde{\phi}] := -\mu \int dR \int du \left[ \tilde{\phi}^2(\phi + \rho_0) \right]. $$

(14)

quite analogously to what happens in the kinetics of simple magnets, see [12] for details.

In principle, an initial correlator $G(r; 0, 0)$ could be assumed and will lead to a further contribution $S_{\text{ini}}$ to the action. For critical systems, one usually employs a term of the form $S_{\text{ini, st}} = -\frac{\mu}{2} \int dR \left( \phi(R, 0) - \langle \phi(R, 0) \rangle \right)^2$, see e.g. [11, 5] but this would have for us the disadvantage that it explicitly breaks Galilei-invariance. We shall rather make use of the Galilei-invariance of the noiseless action $S_0[\phi, \tilde{\phi}]$ and use as an initial term [4, 12]

$$ S_{\text{ini}}[\tilde{\phi}] = -\frac{1}{2} \int dR dR' \tilde{\phi}(R, 0) G(R - R'; 0, 0) \tilde{\phi}(R', 0). $$

(15)

Because of the initial condition $G(r; 0, 0) = 0$, however, $S_{\text{ini}}[\tilde{\phi}] = 0$ and we shall not need to consider it any further.

From the action (12), $n$-point functions can then be computed as usual

$$ \langle \phi_1(r_1, t_1) \cdots \phi_n(r_n, t_n) \rangle = \int D\phi D\tilde{\phi} \phi_1(r_1, t_1) \cdots \phi_n(r_n, t_n) \exp \left( -S[\phi, \tilde{\phi}] \right) $$

(16)

which through the decomposition (12) can be written as an average of the noiseless theory

$$ \langle \phi_1(r_1, t_1) \cdots \phi_n(r_n, t_n) \rangle = \left\langle \phi_1(r_1, t_1) \cdots \phi_n(r_n, t_n) \exp \left( -S_b[\phi, \tilde{\phi}] \right) \right\rangle_0 $$

(17)

where $\langle \ldots \rangle_0$ denotes the expectation value with respect to the noiseless theory.

2.2. Symmetries of the noiseless theory

In what follows, we shall need some symmetry properties of the noiseless part described by the action $S_0[\phi, \tilde{\phi}]$ which we now briefly recall. The noiseless equation of motion for the field $\phi$ is a free diffusion-equation $2M\partial_t\phi(x, t) = \nabla^2\phi(x, t)$. Its dynamical symmetry group is the well-known Schrödinger-group $Sch(d)$ [26, 27] which acts on space-time coordinates $(r, t)$ as $(r, t) \mapsto (r', t') = g(r, t)$ where

$$ t \rightarrow t' = \alpha t + \beta \gamma t + \delta, \quad r \rightarrow r' = \frac{R r + \nu t + a}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1 $$

(18)
and where $\mathcal{R}$ is a rotation matrix. Solutions $\phi$ of the free diffusion equation are carried to other solutions of the same equation and $\phi$ transforms as

$$
\phi(\mathbf{r}, t) \longrightarrow (T_g \phi)(\mathbf{r}, t) = f_g[g^{-1}(\mathbf{r}, t)]\phi[g^{-1}(\mathbf{r}, t)]
$$

(19)

where the companion function $f_g$ is known explicitly and contains the so-called ‘mass’ $\mathcal{M} = (2D)^{-1}$ [27, 28]. We list the generators of the Lie algebra $\mathfrak{sch}_1 = \text{Lie}(\text{Sch}(1))$ in one spatial dimension [29]

$$
\begin{align*}
X_{-1} &= -\partial_t \\
X_0 &= -t \partial_t - \frac{1}{2} r \partial_r - \frac{x}{2} \\
X_1 &= -t^2 \partial_t - tr \partial_r - xt - \frac{\mathcal{M}}{2} r^2 \\
Y_{-\frac{1}{2}} &= -\partial_r \\
Y_{\frac{1}{2}} &= -t \partial_r - \mathcal{M} r \\
M_0 &= -\mathcal{M}
\end{align*}
$$

(20)

Fields transforming under $\text{Sch}(d)$ are characterized by a scaling dimension and a mass. We list in table 3 some fields which we shall use below. We remark that for free fields one has

$$
\tilde{x}_2 = 2\tilde{x} , \quad x_\Upsilon = 2\tilde{x} + x , \quad x_\Sigma = 3\tilde{x} + x , \quad x_\Gamma = 3\tilde{x} + 2x
$$

(21)

but these relations need no longer hold for interacting fields. On the other hand, from the Bargman superselection rules (see [30] and below) we expect that the masses of the composite fields as given in table 3 should remain valid for interacting fields as well.

Throughout this paper, we shall make the important assumption that the fields $\phi$ and $\phi$ transform covariantly according to (19) under the Schrödinger group. By analogy with conformal invariance, such fields are called quasiprimary [8]. For quasiprimary fields the so-called Bargman superselection rules [30] holds true which state that

$$
\langle \phi \ldots \phi \tilde{\phi} \ldots \tilde{\phi} \rangle_0 = 0 \quad \text{unless } n = m
$$

(22)

<table>
<thead>
<tr>
<th>field</th>
<th>scaling dimension</th>
<th>mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$x$</td>
<td>$\mathcal{M}$</td>
</tr>
<tr>
<td>$\tilde{\phi}$</td>
<td>$\tilde{x}$</td>
<td>$-\mathcal{M}$</td>
</tr>
<tr>
<td>$\tilde{\phi}^2$</td>
<td>$\tilde{x}_2$</td>
<td>$-2\mathcal{M}$</td>
</tr>
<tr>
<td>$\Upsilon := \tilde{\phi}^2$</td>
<td>$x_\Upsilon$</td>
<td>$-\mathcal{M}$</td>
</tr>
<tr>
<td>$\Sigma := \tilde{\phi}^3$</td>
<td>$x_\Sigma$</td>
<td>$-2\mathcal{M}$</td>
</tr>
<tr>
<td>$\Gamma := \tilde{\phi}^3 \phi^2$</td>
<td>$x_\Gamma$</td>
<td>$-\mathcal{M}$</td>
</tr>
</tbody>
</table>

Table 3. Scaling dimensions and masses of some composite fields.
We recall the proof of these in appendix B. Before we consider the consequences of (22), we recall the well-known result on the form of noise-less $n$-point functions in ageing systems.

Since in ageing phenomena, time-translation invariance is broken, we must consider the subalgebra $\mathfrak{age}_1 \subset \mathfrak{sch}_1$ obtained by leaving out the generator of time-translations $X_{-1}$ [31]. Then the $n$-point function of quasiprimary fields $\phi_i, i = 1, \ldots n$ has to satisfy the covariance conditions [29, 8]

\begin{align}
\left( \sum_{i=1}^{n} X_k^{(i)} \right) \langle \varphi_1(r_1, t_1) \ldots \varphi_n(r_n, t_n) \rangle_0 &= 0 \quad ; \quad k \in \{0, 1\} \quad (23) \\
\left( \sum_{i=1}^{n} Y_m^{(i)} \right) \langle \varphi_1(r_1, t_1) \ldots \varphi_n(r_n, t_n) \rangle_0 &= 0 \quad ; \quad m \in \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \quad (24)
\end{align}

where $\varphi_i$ stands either for a quasiprimary field $\phi_i$ or a quasiprimary response field $\tilde{\phi}_i$. The $\varphi_i$ are characterized by their scaling dimension $x_i$ and their mass $M_i$. The generators $X_k$ are then the extension of (20) to $n$-body operators and the superscript $(i)$ refers to $\varphi_i$. The $n$-point function is zero unless the sum of all masses vanishes

$$
\sum_{i=1}^{n} M_i = 0 \quad (25)
$$

which reproduces the Bargman superselection rule (22). It is well-known [29, 8] that the noiseless two-point function $R_0(r, r'; t, s) = \langle \varphi_1(r, t) \varphi_2(r, s) \rangle_0$ is completely determined by the equations (23) and (24) up to a normalization constant.

$$
R_0(r, r'; t, s) = R_0(t, s) \exp \left( -\frac{M_1 (r - r')^2}{2 (t - s)} \right) \delta(M_1 + M_2) \quad (26)
$$

where the autoresponse function is given by

$$
R_0(t, s) = r_0(t - s)^{-\frac{1}{2}(x_1 + x_2)} \left( \frac{t}{s} \right)^{-\frac{1}{2}(x_1 - x_2)} \quad (27)
$$

This reproduces the expected scaling form (7) together with the scaling function $f_R(y)$ as given in table 2 if we identify

$$
x = x_1 = x_2, \quad \text{and} \quad x = a + 1 \quad (28)
$$

For the critical bosonic contact process, we read off from table 1 that $a = \frac{d}{2} - 1$. Hence one recovers $x = \frac{d}{2}$, as expected for a free-field theory.

2.3. Reduction formulæ

We now show that the Bargman superselection rule (22) implies a reduction of the $n$-point function of the full theory to certain correlators of the noiseless theory, which is described by $S_0$ only. This can be done generalizing the arguments of [12].
First, for the computation of the response function, we add the term \( \int dR \int du \phi(R, u) h(R, u) \) to the action. As usual the response function is

\[
R(r, r'; t, s) = \left\langle \phi(r, t) \tilde{\phi}(r', s) \right\rangle
\]

\[
= \left\langle \phi(r, t) \tilde{\phi}(r', s) \exp \left( -\mu \int dR \int du \tilde{\phi}^2(R, u)(\phi(R, u) + \rho_0) \right) \right\rangle_0
\]

\[
= \left\langle \phi(r, t) \tilde{\phi}(r', s) \right\rangle_0 = R_0(r, r'; t, s)
\]

where we expanded the exponential and applied the Bargman superselection rule. Indeed, the two-time response is just given by the response of the (gaussian) noise-less theory. We have therefore reproduced the exact result of table 2 for the response function of the critical bosonic contact process.

Second, we have for the correlator

\[
G(r, r', t, s) = \left\langle \phi(r, t) \phi(r' s) \exp \left( -\mu \int dR \int du \tilde{\phi}^2(R, u) \phi(R, u) \right) \right\rangle_0
\]

Expanding both exponentials

\[
\exp \left( -\mu \int dR \int du \tilde{\phi}^2(R, u) \phi(R, u) \right) = \sum_{n=0}^{\infty} \frac{(-\mu)^n}{n!} \left( \int dR \int du \tilde{\phi}^2(R, u) \phi(R, u) \right)^n
\]

\[
\exp \left( -\mu \rho_0 \int dR \int du \tilde{\phi}^2(R, u) \right) = \sum_{m=0}^{\infty} \frac{(-\rho_0 \mu)^m}{m!} \left( \int dR \int du \tilde{\phi}^2(R, u) \right)^m
\]

and using the Bargman superselection rule (22), non-vanishing terms only arise if

\[2n + 2m = n + 2 \text{ or else}\]

\[n + 2m = 2 \text{ (31)}\]

This can only be satisfied for \( n = 0 \) and \( m = 1 \) or for \( n = 2 \) and \( m = 0 \). The full noisy correlator hence is the sum of only two terms

\[
G(r, r'; t, s) = G_1(r, r'; t, s) + G_2(r, r'; t, s)
\]

where the first contribution involves a three-point function of the composite field \( \tilde{\phi}^2 \) of scaling dimension \( \tilde{x}_2 \) (see table 3)

\[
G_1(r, r'; t, s) = -\mu \rho_0 \int dR \int du \left\langle \phi(r, t) \phi(r', s) \tilde{\phi}^2(R, u) \right\rangle_0
\]

whereas the second contribution comes from a four-point function and involves the composite field \( \Upsilon \) (see table 3)

\[
G_2(r, r'; t, s) = \frac{\mu^2}{2} \int dR dR' \int du du' \left\langle \phi(r, t) \phi(r', s) \Upsilon(R, u) \Upsilon(R', u') \right\rangle_0
\]
We see that the connected correlator is determined by three- and four-point functions of the noiseless theory. We now use the symmetries of that noise-less theory to determine the two-, three-, and four-point functions as far as possible.

2.4. Correlator with noise

We consider $G_1(r, r', t, s)$ first. The appropriate three-point function is given in appendix B, equation (B24):

$$
\langle \phi(r, t)\phi(r', s)\phi^2(R, u) \rangle_0 = (t-s)^{x-\frac{d}{2}}(t-u)^{-\frac{d}{2}}(s-u)^{-\frac{d}{2}}
$$

$$
\times \exp \left( -\frac{M(R-R)^2}{2} - \frac{M(R'-R)^2}{2} \right) \Psi_3(u_1, v_1)\Theta(t-u)\Theta(s-u)
$$

(35)

with

$$
u_1 = \frac{u}{t} \cdot \frac{[(s-u)(r-R) - (t-u)(r'-R)]^2}{(t-u)(s-u)^2}
$$

$$
v_1 = \frac{u}{s} \cdot \frac{[(s-u)(r-R) - (t-u)(r'-R)]^2}{(t-u)^2(s-u)}
$$

(36)

and an undetermined scaling function $\Psi_3$. The $\Theta$-functions have been introduced by hand because of causality but this could be justified through a more elaborate argument along the lines of [31]. Introduced into (33), this gives the general form for the contribution $G_1(r, r'; t, s)$. We concentrate here on the autocorrelator, i.e. $r = r'$ and find, with $y = t/s$

$$
G_1(t, s) = -\mu R_0 s^{-x-\frac{d}{2}}(y-1)^{-(x-\frac{d}{2})}
$$

$$
\times \int_0^1 d\theta (y-\theta)^{-\frac{d}{2}x}(1-\theta)^{-\frac{d}{2}} \int_R dR \exp \left( -\frac{M R^2}{2} \frac{y+1-2\theta}{(y-\theta)(1-\theta)} \right)
$$

$$
\times H \left( \frac{\theta}{y}, \frac{R^2(y-1)^2}{(y-\theta)(1-\theta)^2}, \frac{R^2(y-1)^2}{(y-\theta)^2(1-\theta)} \right)
$$

(37)

where $H$ is an undetermined scaling function. Very much in the same way, we find for $G_2(t, s)$

$$
G_2(t, s) = \mu^2 \frac{s^{-x-\frac{d}{2}+2}}{2} \cdot (y-1)^{-(x-\frac{d}{2})} \int_0^1 d\theta \int_0^1 d\theta' (y-\theta)^{-\frac{d}{2}x}(1-\theta)^{-\frac{d}{2}x'}
$$

$$
\times (y-\theta')^{-\frac{d}{2}x'}(1-\theta')^{-\frac{d}{2}} \int_{R^{2d}} dRdR' \exp \left( -\frac{M R^2}{2} - \frac{M R'^2}{2} \right)
$$

$$
\times \Psi_4(\tilde{u}_3(R, \theta, R', \theta'), \tilde{u}_4(R, \theta, R', \theta'), \tilde{v}_3(R, \theta, R', \theta'), \tilde{v}_4(R, \theta, R', \theta'))
$$

(38)

where $\Psi_4$ is another undetermined function and the functions $\tilde{u}_3, \tilde{u}_4, \tilde{v}_3, \tilde{v}_4$ can be worked out from the appropriate expressions (B26) in the appendix B by the replacements $r_3 - r_2 \rightarrow R, r_4 - r_2 \rightarrow R', t_2 \rightarrow 1, t_1 \rightarrow y, t_3 \rightarrow \theta, t_4 \rightarrow \theta'$ (remember that $r_1 = r_2$)

As we have a free-field theory for the critical bosonic contact process, we expect from table 1 and eq. (28) that $x = \tilde{x} = d/2$ and hence the following scaling dimensions
for the composite fields
\[ \tilde{x}_2 = d, \quad x_T = \frac{3}{2} d \]  

(39)

Consequently, the autocorrelator takes the general form
\[ G(t, s) = s^{1-d/2} g_1(t/s) + s^{2-d} g_2(t/s) \]  

(40)

For \( d \) larger than the lower critical dimension \( d_\ast = 2 \), the second term merely furnishes a finite-time correction. On the other hand, for \( d < d_\ast = 2 \), it would be the dominant one and we can only achieve agreement with the known exact result if we assume \( \Psi_4 = 0 \). In what follows, we shall discard the scaling function \( g_2 \) and shall concentrate on showing that our expressions for \( g_1 \) are compatible with the exact results given in table 2.

In order to do so, we choose the following special form for the function \( \Psi_3 \)
\[ \Psi_3(u_1, v_1) = \Xi \left( \frac{1}{u_1} - \frac{1}{v_1} \right) \]  

(41)

where \( \Xi \) remains an arbitrary function. Then we are back in the case already treated in [12]. We find
\[ G_1(t, s) = -\mu \rho_0 s^{\tilde{x}_2+1-x-\tilde{x}_2} (y - 1)^{\tilde{x}_2-x-\frac{d}{2}} \times \int_0^1 d\theta \left( (y - \theta)(1 - \theta) \right)^{\frac{d}{2} - \tilde{x}_2} \phi_1 \left( \frac{y + 1 - 2\theta}{y - 1} \right) \]  

(42)

where the function \( \phi_1 \) is defined by
\[ \phi_1(w) = \int d\mathbf{R} \exp \left( -\frac{Mw}{2} \mathbf{R}^2 \right) \Xi(\mathbf{R}^2) \]  

(43)

As in [12] we choose
\[ \phi_1(w) = \phi_{0,c} w^{-1-a}. \]  

(44)

This form for \( \phi_1(w) \) guarantees that the three-point response function
\[ \langle \phi(r, t) \phi(r, s) \phi^2(r', u) \rangle_0 \] is nonsingular for \( t = s \). We have thus
\[ G(t, s) = G_1(t, s) = s^{-b} f_G \left( \frac{t}{s} \right) \]  

(45)

with
\[ f_G(y) = -\mu \rho_0 \phi_{0,c} \int_0^1 d\theta \left( y + 1 - 2\theta \right)^{-\frac{d}{2}} \]
\[ = \frac{2\mu \rho_0 \phi_{0,c}}{d} \left( (y - 1)^{-\frac{d}{2}+1} - (y + 1)^{-\frac{d}{2}+1} \right) \]  

(46)

and we have reproduced the corresponding entry in table 2 for the critical bosonic contact process. +

+ We remark that for \( 2 < d < 4 \), the same form of the autocorrelation function is also found in the critical voter-model [32].
3. The pair-contact process

3.1. Field-theoretical description and reduction formula

For the pair-contact process we have two different cases, namely the case $\alpha < \alpha_c$ and the case at criticality $\alpha = \alpha_c$. The following considerations apply to both cases and we shall for the moment leave the value of $\alpha$ arbitrary and only fix it at a later state.

The action for the pair-contact process on the critical line is [25, eq. (30)]

$$S[a, \bar{a}] = \int dR \int du \left[ \bar{a}(2M\partial_t - \nabla^2)a - \alpha \bar{a}^2a^2 - \mu \bar{a}^3a^2 \right]$$

As before, see eq. (10), we switch to the quasiprimary fields $\phi(r, t) = a(r, t) - \rho_0$ and $\tilde{\phi}(r, t) = \bar{a}(r, t)$. Then the action becomes

$$S[\phi, \tilde{\phi}] = \int dR \int du \left[ \tilde{\phi}(2M\partial_t - \nabla^2)\phi - \alpha \tilde{\phi}^2\phi^2 - \alpha \rho_0^2 \tilde{\phi}^2 - 2\alpha \rho_0 \bar{\phi}^2 \phi - \mu \bar{\phi}^3 \phi^2 - 2\mu \rho_0 \bar{\phi}^3 \phi - \mu \rho_0^2 \bar{\phi}^3 \right]$$

$$= S_0[\phi, \tilde{\phi}] + S_0[\phi, \tilde{\phi}]$$

Also in this model, similarly to the treatment of section 2, a decomposition of the action into a first term with a non-trivial dynamic symmetry and a remaining noise term is sought such that the correlators and responses can be reexpressed in terms of certain $n$-point functions which only depend on $S_0$. The first term reads

$$S_0[\phi, \tilde{\phi}] := \int dr \int dt \left[ \tilde{\phi}(2M\partial_t - \nabla^2)\phi - \alpha \tilde{\phi}^2\phi^2 \right] .$$

and we derive its Schrödinger-invariance in appendix A. The remaining part is the noise-term which reads

$$S_0[\phi, \tilde{\phi}] = \int dR \int du \left[ -\alpha \rho_0^2 \tilde{\phi}^2 - 2\alpha \rho_0 \bar{\phi}^2 \phi - \mu \bar{\phi}^3 \phi^2 - 2\mu \rho_0 \bar{\phi}^3 \phi - \rho_0^2 \tilde{\phi}^3 \right]$$

Also in this case the Bargman superselection rule (22) holds true. This means that we can proceed now in a very similar way as before.* First we have to check which $n$-point functions contribute to the response and correlation function. We rewrite $\exp(-S_0[\phi, \tilde{\phi}])$ as a product of five exponentials and expand each factor. The indices of the sums are denoted by $k_i$ for the $i$-th term in (50), for instance for the first term

$$\exp \left( - \int dR \int du \alpha \rho_0^2 \tilde{\phi}^2(R, u) \right) = \sum_{k_1=0}^{\infty} \frac{1}{k_1!} \left( - \int dR \int du \alpha \rho_0^2 \tilde{\phi}^2(R, u) \right)^{k_1}$$

For the response function again only the first term of each sum contributes, that is

$$R(r, r'; t, s) = R_0(r, r'; t, s)$$

* This argument works provided each term in $S_0$ contains at least one response field $\tilde{\phi}$ more than order-parameter fields $\phi$. 
is noise-independent. For the correlation function, we have the condition $2k_1 + 2k_2 + 3k_3 + 3k_4 + 3k_5 = 2 + k_2 + 2k_3 + k_4$ or simply

$$2k_1 + k_2 + 3k_3 + 2k_4 + 3k_5 = 2$$

which implies immediately that

$$k_5 = 0.$$  

In table 4 we list the five different contributions to the correlation function. We denote also the form of the composite field, its scaling dimension and whether it is a three- or four-point function which contributes. A short inspection of the general form of the $n$-points function given in the appendix B shows that the contributions have the form (with $y = t/s$)

$$G_1(t, s) = s^{-x - \frac{1}{2}d_2 + \frac{d}{4} + 1} f_1(y), \quad G_4(t, s) = s^{-x - \frac{1}{2}d_2 + \frac{d}{4} + 1} f_4(y)$$

for the 3-point functions and

$$G_2(t, s) = s^{-x - d + 2} f_2(y), \quad G_3(t, s) = s^{-x - d + 2} f_3(y)$$

$$G_5(t, s) = s^{-x - \frac{1}{2}d_2 + \frac{d}{4} + 1} f_5(y)$$

for the four-point functions. The scaling functions $f_i(y)$ involve an arbitrary functions $\tilde{\Psi}_i$ which are not fixed by the symmetries (see appendix B for details). As we do not have a free-field theory in this case we can not make any assumptions about the value of the scaling dimensions of the composite fields. Therefore we do not know which terms will be the leading ones in the scaling regime. However, it turns out that the term $G_1(t, s)$ alone can reproduce our result correctly. Thus we set the scaling functions $f_n = 0$ with $n = 2, \ldots, 5$ analogously to the last section. We now concentrate on

$$G_1(t, s) = \alpha \rho^2 \int dR \int du \left< \phi(r, t) \phi(r, s) \tilde{\phi}^2(R, u) \right>_0$$
3.2. Symmetries of the noiseless theory

As in the last chapter, we require for the calculation of the two- and three-point functions the symmetries of the following non-linear ‘Schrödinger equation’ obtained from (49)

\[ 2\mathcal{M}\partial_t \phi(x, t) = \nabla^2 \phi(x, t) + \mathcal{F}(\phi, \bar{\phi}) \]

(57)

with a nonlinear potential

\[ \mathcal{F}(\phi, \bar{\phi}) = -g\phi^2(x, t)\bar{\phi}(x, t) \]

(58)

While for a constant \( g \) the symmetries of this equation are well-known, it was pointed out recently that \( g \) rather should be considered as a dimensionful quantity and hence should transform under local scale-transformations as well [17]. This requires an extension of the generators used so far and we shall give this in appendix A. The computation of the \( n \)-point functions covariant with respect to these new generators is given in the appendices B and C. In doing so, we have for technical simplicity assumed that to each field \( \varphi_i \) there is one associated coupling constant \( g_i \) and only at the end, we let

\[ g_1 = \ldots = g_n =: g \]

(59)

Therefore, from eq. (52) we find for the response function (see (C12))

\[ R_0(r, r'; t, s) = (t - s)^{-\frac{1}{2}(x_1 + x_2)} \left( \frac{t}{s} \right)^{-\frac{1}{2}(x_1 - x_2)} \]

\[ \times \text{exp} \left( -\frac{\mathcal{M} (r - r')^2}{2(t - s)} \right) \bar{\Psi}_2 \left( \frac{t}{s}, \frac{t - s}{g^{1/\alpha}}, \frac{g}{(t - s)^{\alpha}} \right) \]

(60)

with an undetermined scaling function \( \bar{\Psi}_2 \). This form is clearly consistent with our results in table 2 if we identify

\[ x := x_1 = x_2 = a + 1 = \frac{d}{2} \], \( \bar{\Psi}_2 = \text{const.} \)

(61)

This holds true for both \( \alpha < \alpha_c \) and \( \alpha = \alpha_c \). In distinction with the bosonic contact process, the symmetries of the noiseless part \( S_0 \) do not fix the response function completely but leave a certain degree of flexibility in form of the scaling function \( \bar{\Psi}_2 \).

For the calculation of the correlator we need from eq. (56) the following three-point function

\[ \left\langle \phi(r, t)\phi(r', s)\bar{\phi}^2(R, u) \right\rangle_0 = (t - s)^{\frac{1}{2} - \frac{d}{2} + (t - u)^{-\frac{1}{2} - \frac{d}{2}}(s - u)^{-\frac{1}{2} - \frac{d}{2}}} \]

\[ \times \text{exp} \left( -\frac{\mathcal{M} (r - R)^2}{2(t - u)} - \frac{\mathcal{M} (r' - R)^2}{2(s - u)} \right) \bar{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) \]

(62)

with

\[ u_1 = \frac{u}{t} \cdot \frac{((s - u)(r - R) - (t - u)(r' - R))^2}{(t - u)(s - u)^2} \]

(63)
\[ v_1 = \frac{u}{s} \cdot \frac{[(s-u)(r-R) - (t-u)(r'-R)]^2}{(t-u)^2(s-u)} \]  
\[ \beta_1 = \frac{1}{s_2} \cdot \frac{\alpha^{1/y}}{(t-u)^2}, \quad \beta_2 = \frac{1}{s_2} \cdot \frac{\alpha^{1/y}}{(s-u)^2}, \quad \beta_3 = \alpha^{1/y} s_2 \] 
\[ s_2 = \frac{1}{t-u} + \frac{1}{u} \]

We choose the following realisation for \( \tilde{\Psi}_3 \)

\[ \tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) = \Xi \left( \frac{1}{u_1} - \frac{1}{v_1} \right) \left[ -\frac{(\sqrt{\beta_1} - \sqrt{\beta_2})\sqrt{\beta_3}}{\beta_3 - \beta_2 \beta_3} \right]^{(a-b)} \]

where the scaling function \( \Xi \) was already encountered in eq. (41) for the bosonic contact process. We now have to distinguish the two different cases \( \alpha < \alpha_c \) and \( \alpha = \alpha_c \). For the first case \( \alpha < \alpha_c \), we have \( a - b = 0 \) so that the last factor in (67) disappears and we simply return to the expressions already found for the bosonic contact process, in agreement with the known exact results. However, at the multicritical point \( \alpha = \alpha_c \) we have \( a - b \neq 0 \) and the last factor becomes important. We point out that only the presence or absence of this factor distinguishes the cases \( \alpha < \alpha_c \) and \( \alpha = \alpha_c \).

If we substitute the values for \( \beta_1, \beta_2 \) and \( \beta_3 \), \( \tilde{\Psi}_3 \) becomes

\[ \tilde{\Psi}_3(u_1, v_1, \beta_1, \beta_2, \beta_3) = \Xi \left( \frac{1}{u_1} - \frac{1}{v_1} \right) \left[ -\frac{\theta(y-1)}{(y-\theta)(1-\theta)} \right]^{(a-b)} \]

This factor does not involve \( R \) so that we obtain in a similar way as before

\[ G_1(t, s) = s^{-b}(y-1)^{(b-a)-a-1} \int_0^1 d\theta \frac{1}{(y-\theta)(1-\theta)} \phi_1 \left( \frac{y + 1 - 2\theta}{y - 1} \right) \left[ \frac{\theta(y-1)}{(y-\theta)(1-\theta)} \right]^{a-b} \]

where we have identified

\[ \tilde{x}_2 = 2(b-a) + d \]

\( G_1(t, s) \) reduces to the expression (9) if we choose the same expression for \( \phi_1(w) \) as before. We have thus reproduced all scaling functions correctly.

4. Conclusions

The objective of our investigation has been to test further the recent proposal of using the non-trivial dynamical symmetries of a part of the Langevin equation in order to derive properties of the full stochastic non-equilibrium model. To this end, we have compared the known exact results for the two-time autoresponse and autocorrelation functions in two specific models, see table 2, with the expressions derived from the standard field-theoretical actions which are habitually used to describe these systems.
This is achieved through a decomposition of the action into two parts \( S = S_0 + S_b \) such that (i) \( S_0 \) is Schrödinger-invariant and the Bargman superselection rules hold for the averages calculated with \( S_0 \) only and (ii) the remaining terms contained in \( S_b \) are such that a perturbative expansion terminates at a finite order, again due to the Bargman superselection rules. The two models we considered, namely the bosonic variants of the critical contact and pair-contact processes, satisfy these requirements and are clearly in agreement with the predictions of local scale-invariance (LSI). In particular, our identification eq. (10) of the correct quasi-primary order-parameter and response fields is likely to be useful in more general systems.

Specifically, we have seen the following.

(i) In the bosonic contact process, the symmetries of the noiseless part \( S_0 \) of the action is described in terms of the representation of the Schrödinger-group relevant for the free diffusion equation. In consequence, the form of the two-time response function is completely fixed by LSI and in agreement with the known exact result. The connected autocorrelator is exactly reducible to certain noiseless three- and four-point functions. Schrödinger-invariance alone cannot determine these but the remaining free scaling functions can be chosen such that the known exact results can be reproduced.

(ii) For the bosonic pair-contact process, the symmetries of the partial action \( S_0 \) are described in terms of a new representation pertinent to a non-linear Schrödinger equation. This new representation, which we have explicitly constructed, involves a dimensionful coupling constant \( g \). Therefore even the response function is no longer fully determined. As for the autocorrelation function, which again can be exactly reduced to certain three- and four-point functions calculable from the action \( S_0 \), the remaining free scaling functions can be chosen as to fully reproduce the known exact results.

The consistency of the predictions of LSI with the exact results of these models furnishes further evidence in favour of an extension of the well-known dynamical scaling towards a (hidden) local scale-invariance which influences the long-time behaviour of slowly relaxing systems. An essential ingredient were the Bargman superselection rules which at present can only be derived for a dynamical exponent \( z = 2 \). An extension of our method to models with \( z \neq 2 \) would first of all require a way to generalize the Bargman superselection rules. We hope to return to this open problem elsewhere.

**Acknowledgements:** F.B. acknowledges the support by the Deutsche Forschungsgemeinschaft through grant no PL 323/2. S.S. was supported by the EU Research Training Network HPRN-CT-2002-00279. M.H. thanks the Isaac Newton Institute and the Universität des Saarlandes for warm hospitality, where this work was finished. This work was also supported by the Bayerisch-Französisches Hochschulzentrum (BFHZ).
Appendix A. Representations of $\text{age}_1$ and $\text{sch}_1$ for semi-linear Schrödinger equations

We discuss the Schrödinger-invariance of semi-linear Schrödinger equations of the form (57) and especially with non-linearities of the form (58). With respect to the well-known Schrödinger-invariance of the linear Schrödinger equation, the main difference comes from the presence of a dimensionful coupling constant $g$ of the non-linear term.

It is enough to consider explicitly the one-dimensional case which simplifies the notation. In one spatial dimension, the Schrödinger algebra $\text{sch}_1$ is spanned by the following generators

\begin{equation}
\text{sch}_1 = \left\langle X_{-1}, X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0 \right\rangle
\end{equation}

while its subalgebra $\text{age}_1$ is spanned by

\begin{equation}
\text{age}_1 = \left\langle X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0 \right\rangle
\end{equation}

These generators for $g = 0$ are listed explicitly in eq. (20) and the non-vanishing commutators can be written compactly

\begin{align}
[X_n, X_{n'}] &= (n - n')X_{n+n'} \\
[X_n, Y_m] &= (n/2 - m)Y_{n+m} \\
[Y_{1/2}, Y_{-1/2}] &= M_0
\end{align}

where $n, n' \in \{\pm 1, 0\}$ and $m \in \{\pm \frac{1}{2}\}$ (see [8] for generalizations to $d > 1$).

Following the procedure given in [17], we now construct new representations of $\text{age}_1$ and of $\text{sch}_1$ which takes into account a dimensionful coupling $g$ with scaling dimension $\hat{y}$ as follows.

(i) The generator of space-translations reads simply

\begin{equation}
Y_{-\frac{1}{2}} = -\partial_r.
\end{equation}

(ii) The generator of scaling transformations is assumed to take the form

\begin{equation}
X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \hat{y}g\partial_g - \frac{x}{2}
\end{equation}

where $\hat{y}$ is the scaling dimension of the coupling $g$.

(iii) For $\text{sch}_1$ we also keep the usual generator of time-translations

\begin{equation}
X_{-1} = -\partial_t.
\end{equation}

(iv) The remaining generators we write in the most general form adding a possible $g$-dependence through yet unknown functions $L, Q, P$.

\begin{align}
M_0 &= -\mathcal{M} - L(t, r, g)\partial_g \\
Y_{\frac{1}{2}} &= -t\partial_t - \mathcal{M}r - Q(t, r, g)\partial_g \\
X_1 &= -t^2\partial_t - tr\partial_r - \frac{\mathcal{M}}{2}r^2 - xt - P(t, r, g)\partial_g
\end{align}
The representation given by eqs. (A4,A5,A6,A7) must satisfy the commutation relations (A3) for $\text{age}_1$ or $\text{sch}_1$. From these conditions the undetermined functions $L,Q$ and $P$ are derived. A straightforward but slightly longish calculation along the lines of [17] shows that for $\text{age}_1$, one has

$$L = 0, \quad Q = 0, \quad P = p_0(M) t^{\hat{y}+1} m(t/g)$$

(A8)

Here, $m(v)$ is an arbitrary differentiable function and $p_0(M)$ a $M$-dependent constant. We shall use the shorthand $v = t^{\hat{y}}/g$ in what follows.

In consequence, for $\text{age}_1$ only the generator $X_1$ is modified with respect to the representation eq. (20) and this is described in by the function $m(v)$ and the constant $p_0(M)$.

On the other hand, for $\text{sch}_1$ the additional condition $[X_1, X_{-1}] = 2X_0$ leads to $p_0 = 2\hat{y}$, $m(v) = v^{-1}$.

Hence, the new representations are still given by eq. (20) with the only exception of $X_1$ which reads

$$\text{age}_1 : \quad X_1 = -t^2 \partial_t - tr \partial_r - p_0(M) t^{\hat{y}+1} m(t^{\hat{y}}/g) \partial_g - \frac{Mr^2}{2} - xt$$

$$\text{sch}_1 : \quad X_1 = -t^2 \partial_t - tr \partial_r - 2\hat{y}tg \partial_g - \frac{Mr^2}{2} - xt$$

(A9)

We require in addition the invariance of linear Schrödinger equation $(2M \partial_t - \partial_r^2)\phi = 0$ with respect to this new representation. In terms of the Schrödinger operator $\hat{S}$ this means

$$[\hat{S}, X] = \lambda \hat{S} \quad \text{where} \quad \hat{S} := 2M_0 X_{-1} - Y_{-1}^2$$

(A10)

and $X$ is one of the generators of $\text{age}_1$ eq. (A2) or of $\text{sch}_1$ eq. (A1). Obviously, $\lambda = 0$ if $X \in \langle X_{-1}, Y_{\pm 1}, M_0 \rangle$ and $\lambda = -1$ if $X = X_0$. Finally, for $X_1$ we have from the definition of the Schrödinger operator $\hat{S}$

$$[\hat{S}, X_1] = -4M_0 X_0 + (Y_{1/2} Y_{-1/2} + Y_{-1/2} Y_{1/2})$$

$$= -2t \hat{S} + M (1 - 2x - 4\hat{y}g \partial_g)$$

(A11)

where in the second line the explicit forms eqs. (A4,A5,A6,A7) were used. This also holds for all those representations of $\text{age}_1$ for which there exists an operator $X_{-1} \not\in \text{age}_1$ such that $[X_1, X_{-1}] = 2X_0$ and we shall restrict our attention to those in what follows.

On the other hand, the direct calculation of the same commutator with the explicit form (A9) gives for $\text{age}_1$

$$[\hat{S}, X_1] = -2t \hat{S} + M (1 - 2x) - 2M p_0(M) t^{\hat{y}} [(\hat{y} + 1)m(v) + \hat{y}vm'(v)] \partial_g$$

(A12)

Besides $\lambda = -2t$, the consistency between these two implies for $m(v)$ the equation

$$v \left( (\hat{y} + 1)m(v) + \hat{y}v \frac{dm(v)}{dv} \right) = \frac{2\hat{y}}{p_0}$$

(A13)
with the general solution
\[ m(v) = \frac{2\hat{y} 1}{p_0 v} + \frac{m_0}{p_0} v^{-1-1/\hat{y}} \]  
(A14)

where \( m_0 = m_0(M) \) is an arbitrary constant. The larger algebra \( \text{sch}_1 \) is recovered from this if we set \( p_0 = 2\hat{y} \) and \( m_0 = 0 \). Hence the final form for the generator \( X_1 \) in the special class of representations of the algebra \( \text{age}_1 \) defined above is
\[ X_1 = -t^2 \partial_t - t r \partial_t - 2\hat{y} t g \partial_g - m_0 g^{1+1/\hat{y}} \partial_g - \frac{Mr^2}{2} - xt \]  
(A15)

Summarizing, this class of representations of \( \text{age}_1 \) we constructed is characterized by the triplet \((x, M, m_0)\), whereas for \( \text{sch}_1 \), the same triplet is \((x, M, 0)\).

Finally, to make \( X_1 \) a dynamical symmetry on the solutions \( \Phi = \Phi_g(t, r) \) of the Schrödinger equation \( \hat{S}\Phi_g = 0 \) we must impose the auxiliary condition \((1 - 2x - 4\hat{y} g \partial_g )\Phi_g = 0 \) which leads to
\[ \Phi_g(t, r) = g^{(1-2x)/(4\hat{y})} \Phi(t, r) \]  
(A16)

In particular, we see that if \( x = 1/2 \), we have a representation of \( \text{age}_1 \) without any further auxiliary condition.

We now look for those semi-linear Schrödinger equations of the form \( \hat{S}\Phi = F(t, r, g, \Phi, \Phi^*) \) for which the representations of \( \text{age}_1 \) or \( \text{sch}_1 \) as given by eqs. (A4,A5,A6,A7) and with \( X_1 \) as in (A15) act as a dynamical symmetry. The non-linear potential \( F \) is known to satisfy certain differential equations which can be found using standard methods, see [33, 17, eq. (2.8)]. In our case these equations read
\[ X_{-1} : \partial_t F = 0 \]  
(A17)
\[ Y_{-\frac{1}{2}} : \partial_r F = 0 \]  
(A18)
\[ M_0 : (\Phi \partial_\Phi - \Phi^* \partial_{\Phi^*} - 1) F = 0 \]  
(A19)
\[ Y_{\frac{1}{2}} : [t \partial_t F - M r (\Phi \partial_\Phi - \Phi^* \partial_{\Phi^*} - 1)] F = 0 \]  
(A20)
\[ X_0 : \left[ t^2 \partial_t + \frac{1}{2} r \partial_r + \hat{y} g \partial_g + 1 - \frac{x}{2} (\Phi \partial_\Phi + \Phi^* \partial_{\Phi^*} - 1) \right] F = 0 \]  
(A21)
\[ X_1 : \left[ t^2 \partial_t - tr \partial_r + 2t (\hat{y} g \partial_g + 1) + m_0 g^{1+1/\hat{y}} \partial_g - \frac{Mr^2}{2} (\Phi \partial_\Phi - \Phi^* \partial_{\Phi^*} - 1) - xt (\Phi \partial_\Phi + \Phi^* \partial_{\Phi^*} - 1) \right] F = 0 \]  
(A22)

We first solve these for \( \text{sch}_1 \). From the conditions eqs. (A17,A18,A19,A20,A21) we easily find
\[ F = \Phi (\Phi \Phi^*)^{1/x} f \left( g^{\hat{y}} (\Phi \Phi^*)^{\hat{y}} \right) \]  
(A23)

where \( f \) is an arbitrary differentiable function. Two comments are in order:
(i) For a dimensionless coupling $g$, that is $\hat{y} = 0$, we have $x = 1/2$. Then the scaling function reduces to a $g$-dependent constant and we recover the standard form for the non-linear potential $F$ as quoted ubiquitously in the mathematical literature, see e.g. [34].

(ii) Taking into account the generator $X_1$ from eq. (A22) as well does not change the result. Hence in this case translation-, dilatation- and Galilei-invariance are indeed sufficient for the special Schrödinger-invariance generated by $X_1$, see also [31]. We point out that traditionally an analogous assertion holds for conformal field-theory, see e.g. [1], but counterexamples are known where in local theories scale- and translation-invariance are not sufficient for conformal invariance [35, 36].

Second, we now consider the representation of $\mathfrak{age}_1$ where $X_1$ is given by (A15). We have the conditions eqs. (A18,A19,A20,A21,A22). We write $F = \Phi F(\omega, t, g)$ with $\omega = \Phi \Phi^*$ and the remaining equations coming from $X_0$ and $X_1$ are

$$
(t \partial_t + \hat{y} g \partial_y - xu \partial_u + 1) F = 0 \\
(t^2 \partial_t + m_0 g^{1+1/\hat{y}} \partial_y) F = 0
$$

(A24)

with the final result

$$
F = \Phi (\Phi \Phi^*)^{1/x} f \left( (\Phi \Phi^*)^{\hat{y}} \left[ g^{-1/\hat{y}} - \frac{m_0}{\hat{y} t} \right]^{-x \hat{y}} \right)
$$

(A25)

and where $f$ is the same scaling function as encountered before for $\mathfrak{sch}_1$. Finally, the result for the general representations of $\mathfrak{age}_1$, which depend on an arbitrary function $m(v)$ are not particularly inspiring and will not be detailed here. We observe

(i) For $m_0 = 0$, this result is identical to the one found for $\mathfrak{sch}_1$.

(ii) Even for $m_0 \neq 0$, the form of the non-linear potential reduces in the long-time limit $t \to \infty$ to the one found in eq. (A23) for the larger algebra $\mathfrak{sch}_1$.

We can summarize the main results of this appendix as follows.

**Proposition.** Consider the following generators

$$
M_0 = -\mathcal{M} , \quad Y_{-1/2} = -\partial_r , \quad Y_{1/2} = -t \partial_r - \mathcal{M} r , \quad X_{-1} = -\partial_t \\
X_0 = -t \partial_t - \frac{1}{2} r \partial_r - \hat{y} g \partial_y - \frac{x}{2} \\
X_1 = -t^2 \partial_t - tr \partial_r - 2 \hat{y} t g \partial_y - m_0 g^{1+1/\hat{y}} \partial_y - \frac{\mathcal{M} r^2}{2} - xt
$$

(A26)

where $x, \mathcal{M}, m_0$ are parameters. Define the Schrödinger operator $\hat{S} := 2 M_0 X_{-1} - Y_{-1/2}^2$. Then:

(i) the generators $\{X_{0,1}, Y_{\pm 1/2}, M_0\}$ form a representation of the Lie algebra $\mathfrak{age}_1$. If furthermore $m_0 = 0$, then $\{X_{0,\pm 1}, Y_{\pm 1/2}, M_0\}$ is a representation of the Lie algebra $\mathfrak{sch}_1$.

(ii) These representations are dynamical symmetries of the Schrödinger equation $\hat{S} \Phi =$
0, under the auxiliary condition $(1 - 2x - 4\hat{y}g\partial_x)\Phi = 0$.

(iii) For the Schrödinger-algebra $\mathfrak{sch}_1$ and also in the asymptotic limit $t \to \infty$ for the ageing algebra $\mathfrak{age}_1$, the semi-linear Schrödinger equation invariant under these representations has the form

$$\hat{S}\Phi = \Phi (\Phi \Phi^*)^{1/2} f \left(g^x (\Phi \Phi^*)^{\hat{y}}\right) \quad (A27)$$

where $f$ is an arbitrary differentiable function.

This general form includes our potential (58) since the scaling dimension $\hat{y}$ is a remaining free parameter in our considerations.

**Appendix B. The $n$-point function**

In this appendix we use the generators (A26) from appendix A to find the most general form of the $n$-point functions compatible with the symmetries for $n \geq 3$. We shall do this for the case $m_0 = 0$ only, as this will be enough to reproduce the exact results of table 2. The case $n = 2$ needs a special treatment and is presented in appendix C.

We restrict ourselves to the case $d = 1$ for simplicity, but the generalisation to arbitrary dimension will be obvious. First we introduce some notation. We fix an arbitrary index $k$ and define the shifted coordinates

$$\tilde{r}_b := r_b - r_k, \quad \tilde{t}_b := t_b - t_k \quad \text{for} \quad j \neq k \quad \text{and} \quad \tilde{t}_k := t_k \quad (B1)$$

In the sequel, we will adopt the following convention: The index $a$ always runs from 1 to $n$, the index $b$ runs from 1 to $n$ but skips $k$. The prime on a sum means that the index $k$ is left out, viz.

$$\sum_{i=1}^{n'} A_i := \sum_{i=1}^{n} A_i \quad (B2)$$

We denote the $n$-point function by

$$F(\{r_a\}, \{t_a\}, \{g_a\}) := \langle \varphi_1(r_1,t_1) \ldots \varphi_n(r_n,t_n) \rangle \quad (B3)$$

where we assume one coupling constant for each field. This quantity has to satisfy the following four linear partial differential equations

$$\left(\sum_{i=1}^{n} X_k^{(i)} \right) F(\{r_a\}, \{t_a\}, \{t_a\}) = 0 \quad k \in \{0, 1\} \quad (B4)$$

$$\left(\sum_{i=1}^{n} Y_m^{(i)} \right) F(\{r_a\}, \{t_a\}, \{t_a\}) = 0 \quad m \in \left\{-\frac{1}{2}, \frac{1}{2} \right\} \quad (B5)$$
To solve these equations, we use the method of characteristics [37]. We solve (B4) first for spatial translation invariance with the result

$$F(\{r_a\}, \{t_a\}, \{g_a\}) = \tilde{F}(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$$

(B6)

with a new function $\tilde{F}$ with $3n - 1$ arguments. In order to solve for $X_0$ we set

$$x = \frac{1}{2} \sum_{i=1}^{n} x_i$$

(B7)

and make the ansatz

$$\tilde{F}(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = \prod_{i<j} (t_i - t_j)^{-\rho_{ij}} G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$$

(B8)

where the parameters $\rho_{ij}$ and the function $G$ remain to be determined. We also change to the new independent temporal variables $\tilde{t}_a$. Then one finds after a short calculation

$$\left( \sum_{i=1}^{n} \tilde{t}_i \partial_{\tilde{t}_i} + \frac{1}{2} \sum_{i=1}^{n} \tilde{r}_i \partial_{\tilde{r}_i} + \sum_{i=1}^{n} \tilde{y}_i g_i \partial_{g_i} \right) G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0$$

(B9)

together with the condition

$$x = \sum_{i<j} \rho_{ij}.$$  

(B10)

Before proceeding to solve this equation, we turn to the generators $Y_{1/2}$. We find for the function $G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$

$$\left( \sum_{i=1}^{n'} \tilde{t}_i \partial_{\tilde{t}_i} + \sum_{i=1}^{n'} \tilde{r}_i \partial_{\tilde{r}_i} + \sum_{i=1}^{n} \tilde{M}_i \tilde{r}_i + \sum_{i=1}^{n} \tilde{M}_i \tilde{t}_i \right) G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0.$$  

(B11)

Since $G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$ does not depend on $\tilde{r}_k$, we recover the Bargman superselection rule

$$\sum_{i=1}^{n} \tilde{M}_i = 0$$  

(B12)

as expected. For $G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$ we make another ansatz:

$$G(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = \exp \left( -\sum_{i=1}^{n'} \tilde{M}_i \frac{\tilde{r}_i^2}{2} \right) H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$$

(B13)

where the function $H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})$ remains to be determined. With (B12) and (B13), equation (B11) reduces to

$$\left( \sum_{i=1}^{n'} \tilde{t}_i \partial_{\tilde{t}_i} \right) H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0.$$  

(B14)
We thus have to solve the homogenous equations (B14), (B15) and (B20). This yields
\[
\left( \sum_{i=1}^{n} \tilde{t}_i \partial_{\tilde{t}_i} + \frac{1}{2} \sum_{i=1}^{n} \tilde{r}_i \partial_{\tilde{r}_i} + \sum_{i=1}^{n} \check{y}_i g_i \partial_{g_i} \right) H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0 \tag{B15}
\]

The last equation for \(H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})\) we obtain from \(X_1\). Using the ansatz (B8) yields an equation for \(X_i\), see be adding all equations. Also, this system is always solvable for \(n \geq 3\), as for \(n \geq 4\) it is underdetermined and for \(n = 3\) the corresponding determinant does not vanish \(\ddagger\).

Lastly, we often have the case \(x_1 = x_2 =: x\) and \(x_3 = \ldots = x_n =: \tilde{x}\). In this case, we can set
\[
\rho_{12} = x - \frac{n-2}{2} \tilde{x} ; \quad \rho_{2i} = \frac{1}{2} \tilde{x} , \quad \rho_{1i} = \frac{1}{2} \tilde{x} \quad \text{for} \quad i = 3, \ldots, n \tag{B19}
\]

and \(\rho_{ij} = 0\) for all the remaining \(\rho_{ij}\). We still have to rewrite equation (B16) in terms of the variables \(\{\tilde{r}_b\}\) and \(\{\tilde{t}_a\}\). Here we take equations (B9) and (B11) and the ansatz (B13) into account and get for \(H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\})\)
\[
\left( \sum_{i=1}^{n} \tilde{t}_i^{\prime 2} \partial_{\tilde{t}_i} - \tilde{r}_k^{\prime 2} \partial_{\tilde{r}_k} + \sum_{i=1}^{n} \tilde{r}_i \tilde{r}_i \partial_{\tilde{r}_i} + 2 \sum_{i=1}^{n} \check{y}_i \check{t}_i g_i \partial_{g_i} \right) H(\{\tilde{r}_b\}, \{\tilde{t}_a\}, \{g_a\}) = 0. \tag{B20}
\]

We thus have to solve the homogenous equations (B14),(B15) and (B20). This will eliminate three more variables and yields
\[
F(\{t_a\}, \{t_a\}, \{g_a\}) = \prod_{i<j} (t_i - t_j)^{-\rho_{ij}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} \mathcal{M}_i \frac{(r_i - r_k)^2}{t_i - t_k} \right)
\]

\(\ddagger\) This system is not solvable for \(n = 2\) when \(x_1 \neq x_2\). This case is considered in appendix C.
with an arbitrary function \( \tilde{\Psi}_n \), which depends on \( 3n - 4 \) variables. Here the index \( c \) runs from 1 to \( n \) but skips \( k \) and another arbitrarily fixed index \( r \neq k \), and the expressions \( u_c, v_c \) and \( \beta_a \) are given by

\[
\begin{align*}
  u_c &= \frac{t_k ((r_c - r_k)(t_r - t_k) - (r_r - r_k)(t_c - t_k))^2}{(t_c - t_k)(t_r - t_k)^2 t_c} \\
  v_c &= \frac{t_k ((r_c - r_k)(t_r - t_k) - (r_r - r_k)(t_c - t_k))^2}{(t_r - t_k)(t_c - t_k)^2 t_r} \\
  \beta_k &= g_k^{(1/\tilde{g}_k)} \left( \frac{t_r}{(t_r - t_k)t_k} \right), \\
  \beta_b &= g_b^{(1/\tilde{g}_b)} \left( \frac{t_k(t_b - t_k)^2}{(t_r - t_k)t_r} \right)
\end{align*}
\] (B22)

We remind the reader of our convention that the index \( c \) runs from 1 to \( n \) skipping \( r \) and \( k \) and that the index \( b \) runs from 1 to \( n \) skipping only \( k \).

In higher dimensions rotational invariance has to be satisfied as well and then the generalization to arbitrary \( d \) is straightforward.

If we consider instead the algebra \( \mathfrak{g}_1 \) with dimensionless couplings \( g_i \) we merely have to make the replacement

\[
\tilde{\Psi}_n (\{u_c\}, \{v_c\}, \{\beta_a\}) \longrightarrow \Psi_n (\{u_c\}, \{v_c\})
\] (B23)

where \( \Psi_n \) is also an arbitrary functional such that only the dependence on \( \{\beta_a\} \) drops out.

Finally, we explicitly list the three- and four-point functions in the form in which they are needed in the main text. The three-point function with fixed indices \( r = 2 \) and \( k = 3 \) and the special situation (B19) assumed reads

\[
F(\{r_a\}, \{t_a\}, \{g_a\}) = (t_1 - t_2)^{-\frac{1}{2}(x - \tilde{z})} (t_1 - t_3)^{-\frac{1}{2} \tilde{z}} (t_2 - t_3)^{-\frac{1}{2} \tilde{z}}
\]

\[
\times \exp \left( -\frac{1}{2} \sum_{i=1}^{2} \mathcal{M}_i \frac{(r_i - r_3)^2}{t_i - t_3} \right) \tilde{\Psi}_n (\{u_c\}, \{v_c\}, \{\beta_a\})
\] (B24)

with

\[
\begin{align*}
  u_1 &= \frac{t_3 ((r_1 - r_3)(t_2 - t_3) - (r_2 - r_3)(t_1 - t_3))^2}{t_1 (t_1 - t_3)(t_2 - t_3)^2} \\
  v_1 &= \frac{t_3 ((r_1 - r_3)(t_2 - t_3) - (r_2 - r_3)(t_1 - t_3))^2}{t_2 (t_1 - t_3)^2(t_2 - t_3)} \\
  \beta_1 &= g_1^{1/\tilde{g}_1} \frac{t_3(t_1 - t_3)^2}{t_2(t_2 - t_3)}, \\
  \beta_2 &= g_2^{1/\tilde{g}_2} \frac{t_3(t_2 - t_3)}{t_2} \\
  \beta_3 &= g_3^{1/\tilde{g}_3} \frac{t_2}{t_3(t_2 - t_3)}
\end{align*}
\] (B25)

The four point function with \( r = 1 \) and \( k = 2 \) in the special situation (B19) reads

\[
F(\{r_a\}, \{t_a\}, \{g_a\}) = (t_1 - t_2)^{-\frac{1}{2}(x - \tilde{z})} (t_1 - t_3)^{-\frac{1}{2} \tilde{z}} (t_1 - t_4)^{-\frac{1}{2} \tilde{z}} (t_2 - t_3)^{-\frac{1}{2} \tilde{z}}
\]

\[
\times (t_2 - t_4)^{-\frac{1}{2} \tilde{z}} \exp \left( -\frac{1}{2} \sum_{i=1}^{3} \mathcal{M}_i \frac{(r_i - r_4)^2}{t_i - t_4} \right) \tilde{\Psi}_n (\{u_c\}, \{v_c\}, \{\beta_a\})
\]
with

\[ u_3 = \frac{t_2 [(r_3 - r_2)(t_1 - t_2) - (r_1 - r_2)(t_3 - t_4)]^2}{t_3 (t_3 - t_2)(t_1 - t_2)^2} \]
\[ u_4 = \frac{t_2 [(r_4 - r_2)(t_1 - t_2) - (r_1 - r_2)(t_4 - t_2)]^2}{t_4 (t_4 - t_2)(t_1 - t_2)^2} \]
\[ v_3 = \frac{t_2 [(r_3 - r_2)(t_1 - t_2) - (r_1 - r_2)(t_3 - t_2)]^2}{t_1 (t_1 - t_2)(t_3 - t_2)^2} \]
\[ v_4 = \frac{t_2 [(r_4 - r_2)(t_1 - t_2) - (r_1 - r_2)(t_4 - t_2)]^2}{t_1 (t_1 - t_2)(t_4 - t_2)^2} \]
\[ \beta_1 = g_1^{1/\tilde{g}_1} \frac{t_2(t_1 - t_2)}{t_1}, \quad \beta_2 = g_2^{1/\tilde{g}_2} \frac{t_1}{(t_1 - t_2)t_2} \]
\[ \beta_3 = g_3^{1/\tilde{g}_3} \frac{t_2(t_3 - t_2)^2}{t_1(t_1 - t_2)}, \quad \beta_4 = g_4^{1/\tilde{g}_4} \frac{t_2(t_4 - t_2)^2}{(t_1 - t_2)t_1} \]

\[(B26)\]

**Appendix C. The two-point function**

In this appendix we calculate the two-point function, which was not included in the treatment of appendix B. Again, we only treat the case \( m_0 = 0 \). Apart from the generator \( X_1 \), the calculations are similar to those done in appendix B, so we only give the essential steps. First we define

\[ \tau := t_1 - t_2, \quad r := r_1 - r_2 \]

and then we proceed as follows. We solve for \( M_0, Y_{-1/2}, Y_{1/2}, X_0 \) in exactly the same way as before with the result

\[ F(r_1, r_2, t_1, t_2, g_1, g_2) = \langle \varphi_1(r_1, t_1, g_1)\varphi_2(r_2, t_2, g_2) \rangle_0 = \tau^{-x} G(r, \tau, t_2, g_1, g_2) \]

\[(C2)\]

where \( x = \frac{1}{2}(x_1 + x_2) \) and \( G(r, \tau, t_2, g_1, g_2) \) satisfies the equations

\[ \left( \tau \partial_\tau + t_2 \partial_{t_2} + \frac{1}{2} r \partial_r + y_1 g_1 \partial_{g_1} + y_2 g_2 \partial_{g_2} \right) G(r, \tau, t_2, g_1, g_2) = 0 \]
\[ (\tau \partial_\tau + r M_1)G(r, \tau, t_2, g_1, g_2) = 0 \]

\[(C3)(C4)\]

and the Bargman superselection rule

\[ M_1 + M_2 = 0 \]

\[(C5)\]

holds true. Now (C3) is solved by

\[ G(r, \tau, t_2, g_1, g_2) = \tilde{G}(u_1, u_2, v_1, v_2) \]

\[(C6)\]

where we have defined

\[ u_1 := \frac{r^2}{\tau}, \quad u_2 := \frac{r^2}{t_2}, \quad v_1 := \frac{g_1^{1/\tilde{g}_1}}{\tau}, \quad v_2 := \frac{g_2^{1/\tilde{g}_2}}{\tau} \]

\[(C7)\]
and rewriting (C4) in terms of the new variables yields

\[
\left( u_1 \partial_{u_1} + u_2 \partial_{u_2} + \frac{1}{2} u_1 M_1 \right) \tilde{G}(u_1, u_2, v_1, v_2) = 0
\]

which is solved by

\[
\tilde{G}(u_1, u_2, v_1, v_2) = \exp \left( -\frac{1}{2} u_1 M_1 \right) H(w, v_1, v_2), \quad w := \frac{u_2}{u_1}
\]

The function \( H(w, v_1, v_2) \) is found through the generator \( X_1 \). Using again the invariance under \( Y_{1/2} \) and \( X_0 \), we readily obtain in terms of \( v_1, v_2 \) and \( w \)

\[
\left( (w + 1) \partial_w + v_1 \partial_{v_1} - v_2 \partial_{v_2} + \frac{1}{2} (x_1 - x_2) \right) H(w, v_1, v_2) = 0.
\]

The most general solution of this equation is

\[
H = (w + 1)^{-\frac{1}{2} (x_1 - x_2)} \tilde{\Psi}_2 \left( \frac{(w + 1)}{v_1}, v_1 v_2 \right)
\]

where the function \( \tilde{\Psi}_2 \) remains arbitrary. Substituting back the values for \( v_1, v_2 \) and \( w \) our final result is

\[
F(r_1, t_1, r_2, t_2) = \delta_{M_1+M_2,0} (t_1 - t_2)^{-\frac{1}{4} (x_1 + x_2)} \left( \frac{t_1}{t_2} \right)^{-\frac{1}{2} (x_1 - x_2)}
\]

\[
\times \exp \left( -\frac{M_1}{2} \frac{(r_1 - r_2)^2}{t_1 - t_2} \right) \tilde{\Psi}_2 \left( \frac{(t_1)}{t_2}, \frac{(t_1 - t_2)\hat{\gamma}_1}{g_1}, \frac{g_1 g_2}{(t_1 - t_2)\hat{\gamma}_1 + \hat{\gamma}_2} \right)
\]

For applications to semi-linear equations, one now sets \( g := g_1 = g_2 \) with a scaling dimension \( \hat{y} := \hat{\gamma}_1 = \hat{\gamma}_2 \). In the limit \( \hat{y} \to 0 \), the function \( \tilde{\Psi}_2 \) reduces to a \( g \)-dependent normalization constant and we recover the standard result [29].

In many applications, one expects the scaling functions to be universal, up to normalization. On the other hand, the coupling \( g \) should be a non-universal quantity so that a universal scaling function cannot contain \( g \) in its arguments. This leads to \( \tilde{\Psi}_2 = \tilde{\Psi}_2((t_1/t_2)\hat{y}) \) and we point out that such a scaling form would be compatible (one still has \( z = 2 \), however) with what is found from the field-theoretical renormalization group and numerical simulations in non-equilibrium critical dynamics [5, 13]. An extension to different values of \( z \) would as a first step require the generalization of the Bargman superselection rules. We hope to come back elsewhere to this open problem.