The asymptotics of stationary electro-vacuum metrics 
in odd space-time dimensions

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Abstract

We show that stationary, asymptotically flat solutions of the electro-vacuum Einstein equations are analytic at \( i^0 \), for a large family of gauges, in odd space-time dimensions higher than seven. The same is true in space-time dimension five for static vacuum solutions with non-vanishing mass.

1 Introduction

There is currently interest in asymptotically flat solutions of the vacuum Einstein equations in higher dimensions \([5, 9]\). It is thus natural to enquire which part of our body of knowledge of \((3 + 1)\)-dimensional solutions carries over to higher dimensions. In this note we study that question for asymptotic expansions at spatial infinity of stationary or static electro-vacuum metrics. We prove analyticity at \( i^0 \), up to a conformal factor, for a family of natural geometric gauges, in even dimensions \( n \geq 6 \). The same result is established in space-dimension \( n = 4 \) for static vacuum metrics with non-vanishing ADM mass.

2 Static vacuum metrics

We write the space-time metric in the form

\[
ds^2 = -e^{2u}dt^2 + e^{-\frac{2u}{n-2}}\tilde{g}_{ij}dx^idx^j,
\]

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where \( \tilde{g} \) is an asymptotically flat Riemannian metric, with \( \partial_t u = \partial_i \tilde{g}_{ij} = 0 \). The vacuum Einstein equations show that \( u \) is \( \tilde{g} \)-harmonic, with \( \tilde{g} \) satisfying the equation

\[
\tilde{R}_{ij} = \frac{n-1}{n-2} \partial_i u \partial_j u ,
\]

where \( n \) is the space-dimension, and \( \tilde{R}_{ij} \) is the Ricci tensor of \( \tilde{g} \). We assume \( n \geq 3 \) throughout.

It is a standard consequence of those equations that, in harmonic coordinates in the asymptotically flat region, and whatever \( n \geq 3 \), both \( u \) and \( \tilde{g}_{ij} \) have a full asymptotic expansion in terms of powers of \( \ln r \) and inverse powers of \( r \). Solutions without \( \ln r \) terms are of special interest, because they can be used to construct smoothly compactifiable hyperboloidal initial surfaces. In even space-time dimension initial data sets containing such asymptotic regions, when close enough to Minkowskian data, lead to asymptotically simple space-times \([1, 11]\). It has been shown by Beig and Simon that logarithmic terms can always be gotten rid of by a change of coordinates when space-dimension equals three \([4, 14]\).

From what has been said one can infer that the leading order corrections in the metric can be written in a Schwarzschild form, which in “isotropic” coordinates reads

\[
g_m = - \left( \frac{1-m/2|x|^{n-2}}{1+m/2|x|^{n-2}} \right)^2 dt^2 + \left( 1 + \frac{m}{2|x|^{n-2}} \right)^{\frac{4}{n-2}} \sum_{i=1}^{n} dx_i^2 \approx - \left( 1 - \frac{m}{r^{n-2}} \right)^2 dt^2 + \left( 1 + \frac{m}{r^{n-2}} \right)^{\frac{2}{n-2}} \sum_{i=1}^{n} dx_i^2 ,
\]

where \( m \) is of course a constant, and \( \tilde{r} = |x| \) is a radial coordinate in the asymptotically flat region. This gives the asymptotic expansion

\[
u = - \frac{m}{\tilde{r}^{n-2}} + O(\tilde{r}^{-n+1}) ,
\]

Further we have

\[
\tilde{g}_{ij} = \delta_{ij} + O(\tilde{r}^{1-n}) .
\]

Equation (2.3) shows that for \( m \neq 0 \) the function

\[
\omega := (u^2)^{\frac{1}{n-2}}
\]

behaves asymptotically as \( \tilde{r}^{-2} \), and can therefore be used as a conformal factor in the usual one-point compactification of the asymptotic region. Indeed, assuming that \( m \neq 0 \) and setting

\[
g_{ij} := \omega^2 \tilde{g}_{ij} .
\]

one obtains a \( C^{n-2,1} \) metric\(^1\) on the manifold obtained by adding a point (which we denote by \( \nu^0 \)) to the asymptotically Euclidean region.

\(^1\)The differentiability class near \( \nu^0 \) can be established by examining Taylor expansions there.
From the fact that \( u \) is \( \tilde{g} \)-harmonic one finds
\[
\Delta \omega = \mu , \tag{2.7}
\]
where the auxiliary function \( \mu \) is defined as
\[
\mu := \frac{n}{2} \omega^{-1} g^{ij} \partial_i \omega \partial_j \omega . \tag{2.8}
\]

Note that, in spite of the negative power of \( \omega \), this function can be extended by continuity to \( \tilde{g} \), the extended function, still denoted by \( \mu \), being of \( C^{n-2,1} \) differentiability class.

Let \( L_{ij} \) be the Schouten tensor of \( g_{ij} \),
\[
L_{ij} := \frac{1}{n-2} \left( R_{ij} - \frac{R}{2(n-1)} g_{ij} \right) . \tag{2.9}
\]

Using tildes to denote the corresponding objects for the metric \( \tilde{g} \), from (2.1) one obtains
\[
\tilde{L}_{ij} = \frac{1}{4} \omega^{n-4} \left( (n-1) \partial_i \omega \partial_j \omega - \frac{1}{2} g_{ij} g^{kl} \partial_k \omega \partial_l \omega \right) . \tag{2.10}
\]

We see that for \( n \geq 3 \) the tensor \( \tilde{L}_{ij} \) is bounded on the one-point compactification at infinity, and for \( n \geq 4 \) it is as differentiable as \( d\omega \) and the metric allow. This last property is not true anymore for \( n = 3 \), however the following objects are well behaved:
\[
\tilde{L}_{ij} D^j \omega = \frac{2n-3}{4n} \omega^{n-3} \mu D_i \omega , \quad \tilde{L}_{i[j} D_{k]} \omega = -\frac{1}{4n} \omega^{n-3} \mu g_{i[j} D_{k]} \omega . \tag{2.11}
\]

### 3 Conformal rescalings

We recall the well-known transformation law of the Schouten tensor under the conformal rescaling (2.6)
\[
L_{ij} = \tilde{L}_{ij} + \omega^{-1} D_i D_j \omega - \frac{1}{2} \omega^{-2} g_{ij} g^{kl} \partial_k \omega \partial_l \omega ; \tag{3.1}
\]
we emphasize that this holds whether or not \( \omega \) is related to \( \tilde{L} \) as in (2.10). Taking a trace of (3.1) and using (2.7) one finds
\[
R = \frac{(n-1)(n-2)}{2n} \omega^{n-3} \mu . \tag{3.2}
\]

In our subsequent manipulations it is convenient to rewrite (3.1) as an equation for \( D_i D_j \omega \),
\[
D_j D_i \omega = \omega (L_{ij} - \tilde{L}_{ij}) + \frac{1}{n} \mu g_{ij} . \tag{3.3}
\]

We note that the right-hand-side is well-behaved at \( \omega = 0 \) for all \( n \geq 3 \).

Let \( C_{ijk} \) denote the Cotton tensor,
\[
C_{ijk} := D_k L_{ij} - D_j L_{ik} , \tag{3.4}
\]
and let $C_{ijkl}$ be the Weyl tensor of $g$. Note the identity

$$D^i C_{ijkl} = (3 - n) C_{ijkl} .$$  \(3.5\)

Applying $D_k$ to (3.3) and anti-symmetrising over $j$ and $k$ one obtains

$$\omega C_{ijk} + C_{kji\ell} D^\ell \omega = \tilde{L}_{ijk} ,$$  \(3.6\)

where

$$\tilde{L}_{ijk} := 2D_k(\omega \tilde{L}_{j[i}) + 2g_{ij} \tilde{L}_{k]\ell} D^\ell \omega .$$  \(3.7\)

Writing down the second term in (3.7) and using (2.11) and again (3.3), we find that the terms with $\omega^{n-4}$ drop out and there results

$$\tilde{L}_{ijk} = \frac{1}{2} \omega^{n-2} \left( - (n-1) D_j \omega (L_k|l - \tilde{L}_{k}|i) + g_{ij} (L_{k|l} - \tilde{L}_{k|l}) D^\ell \omega \right) .$$  \(3.8\)

Here $\tilde{L}_{ij}$ should be expressed in terms of $\omega$, $d\omega$ and $\mu$ using (2.11). It should be emphasized that the underbraced expression is regular at $\omega = 0$.

Let $B_{ij}$ denote the Bach tensor,

$$B_{ij} := D^k C_{ijk} - L^{k\ell} C_{kji\ell} .$$  \(3.9\)

Applying $D_k$ to (3.6) and using (3.5) and (3.3) one obtains

$$B_{ij} - (n - 4)\omega^{-1} C_{ijkl} D^k \omega = \omega^{-1} D^k \tilde{L}_{ijk} - C_{kji\ell} \tilde{L}^{k\ell} .$$  \(3.10\)

Note that the factor $\omega^{-1}$ in front of the divergence $D^k \tilde{L}_{ijk}$ is compensated by $\omega^{n-2}$ in (3.7), so that for $n \geq 4$ the right-hand-side is a well-behaved function of the metric, of $\omega$, and of their derivatives at zeros of $\omega$. Alternatively we can, using (3.6), rewrite (3.10) as

$$B_{ij} + (n - 4)\omega^{-2} C_{kji\ell} D^k \omega D^\ell \omega = \omega^{3-n} D^k (\omega^{n-4} \tilde{L}_{ijk}) - C_{kji\ell} \tilde{L}^{k\ell} .$$  \(3.11\)

Note that the right-hand-side of (3.11) is regular also for $n = 3$. Recall, now, the identity

$$B_{ij} = \Delta L_{ij} - D_i D_j (\text{tr} L) + \mathcal{F}_{ij}$$
$$= \frac{1}{n-2} \Delta R_{ij} - \frac{1}{2(n-1)} \left( \frac{1}{n-2} \Delta R g_{ij} + D_i D_j R \right) + \mathcal{F}_{ij} ,$$  \(3.12\)

where $\mathcal{F}_{ij}$ depends upon the metric and its derivatives up to order two. We eliminate the Ricci scalar terms using (3.2). The terms involving derivatives of $R$ will introduce derivatives of $\mu$, which can be handled as follows. Differentiating (2.8) and using (3.3) one obtains

$$D_i \mu = - n(L_{ij} - \tilde{L}_{ij}) D^i \omega$$
$$= - n L_{ij} D^i \omega + \frac{2n-3}{4} \omega^{n-3} \mu D_i \omega ,$$  \(3.13\)

which allows us to eliminate each derivative of $\mu$ in terms of $\mu$, $\omega$ and $d\omega$. 

4
3.1 Space-dimensions three and four

In dimension three the term involving $\omega^{-2}C_{kji\ell}L^{k\ell}$ on the left-hand-side of (3.11) goes away because the Weyl tensor vanishes. In dimension four its coefficient vanishes. In those dimensions one therefore ends up with an equation of the form

$$\Delta R_{ij} = F_{ij}(n, \omega, d\omega, g, \partial g, \partial^2 g).$$

with a tensor field $F_{ij}$ which is well behaved at $\omega = 0$. Here we have used the expression of $\mu$ as a function of the metric, $\partial g$, $\partial \omega$ and $\partial^2 \omega$ which follows from (2.7).

We can calculate the laplacian of $\mu$ by taking a divergence of (3.13) and eliminating again the second derivatives of $\omega$ in terms of $\mu$, and the first derivatives of $\mu$, as before. This leads to a fourth-order equation for $\omega$ of the form

$$\Delta^2 \omega = F(n, \omega, d\omega, \partial^2 \omega, g, \partial g, \text{Ric}),$$

with $F$ — well behaved at $\omega = 0$, where Ric stands for the Ricci tensor. Note that one should use the Bianchi identities to eliminate the term involving the divergence of $L_{ij}$ which arises in the process:

$$D^j L_{ij} = \frac{1}{2(n-1)} D_i R.$$

In harmonic coordinates, Equations (3.14)-(3.15) can be viewed as a system of equations of fourth order for the metric $g$ and the function $\omega$, with diagonal principal part $\Delta^2$. The system is elliptic so that usual bootstrap arguments show smoothness of all fields. In fact the solutions are real-analytic by [13], as we wished to show.

3.2 Higher even dimensions

A natural generalisation of the Bach tensor in even dimensions $n \geq 6$ is the obstruction tensor $\mathcal{O}_{ij}$ of Fefferman and Graham [10, 12]. It is of the form

$$\mathcal{O}_{ij} = \Delta^{n-4} [\Delta L_{ij} - D_i D_j (\text{tr} L)] + \mathcal{F}^n_{ij},$$

where $\mathcal{F}^n_{ij}$ is a tensor constructed out of the metric and its derivatives up to order $n - 2$. This leads us to expect that further differentiations of the equations above leads to a regular expression for $\Delta^{n-2} B_{ij}$ in terms of $\omega$ and its derivatives up to order $n - 3$. However, we have not been able to conclude using this approach. Instead, we proceed as in [6]:

In coordinates $x^I$ which are harmonic with respect to the metric $\tilde{g}$, (2.1) and the harmonicity condition for $u$ lead to a set of equations for $u$ and

$$f := (\tilde{g}_{ij} - \delta_{ij})$$

of the form

$$\tilde{g}^{ij} \partial_i \partial_j f = F(f)(\partial f)^2 + (\partial u)^2, \quad \tilde{g}^{ij} \partial_i \partial_j u = 0.$$
Setting 
\[ \Omega = \frac{1}{r^2}, \quad \tilde{f} = \Omega^{-n-2} f, \quad \tilde{u} = \Omega^{-n-2} u, \quad y^i = \frac{x^i}{r^2}, \]
one obtains a set of regular elliptic equations in the coordinates \( y^i \) after a conformal rescaling \( \delta_{ij} \rightarrow \Omega^2 \delta_{ij} \) of the flat metric, provided that \( n \geq 6 \). The reader is referred to [6] for a detailed calculation in a Lorentzian setting, which carries over with minor modifications (due to the quadratic rather than linear zero of \( \Omega \)) to the current situation; note that \( n \) in the calculations there should be replaced by \( n - 1 \) for the calculations at hand. We further note that the leading order behavior of \( \tilde{f} \) is governed by the mass, which can be made arbitrarily small by a constant rescaling of the metric and of the original harmonic coordinates \( x^i \); this freedom can be made use of to ensure ellipticity of the resulting equations. Finally we emphasise that this result, contrary to the one for \( n \) equal three or four, does not require the non-vanishing of mass.

4 Stationary vacuum solutions

We consider Lorentzian metrics \( n+1 \ g \) in odd space-time-dimension \( n + 1 \geq 7 \), with Killing vector \( X = \partial/\partial t \). In adapted coordinates those metrics can be written as

\[ n+1 \ g = -V^2(dt + \theta_i dx^i)^2 + g_{ij} dx^i dx^j, \quad (4.1) \]
\[ \partial_t V = \partial_t \theta = \partial_k g = 0. \quad (4.2) \]

The vacuum Einstein equations (with vanishing cosmological constant) read (see, e.g., [8])

\[ \left\{ \begin{array}{l}
V \nabla^i \nabla V = \frac{1}{2} |\gamma|^2_g, \\
\text{Ric}(g) - V^{-1} \text{Hess}_g V = \frac{1}{2V^2} \lambda \circ \lambda, \\
\text{div}(V \lambda) = 0,
\end{array} \right. \quad (4.3) \]

where

\[ \lambda_{ij} = -V^2 (\partial_i \theta_j - \partial_j \theta_i), \quad (\lambda \circ \lambda)_{ij} = \lambda_i^k \lambda_{kj}. \]

We consider metrics satisfying, for some \( \alpha > 0 \),

\[ g_{ij} - \delta_{ij} = O(r^{-\alpha}) , \quad \partial_k g_{ij} = O(r^{-\alpha-1}) , \quad V = O(r^{-\alpha}) , \quad \partial_k V = O(r^{-\alpha-1}). \quad (4.4) \]

As is well known [2], one can then introduce new coordinates, compatible with the above fall-off requirements, which are harmonic for \( g \).

Next, a redefinition \( t \rightarrow t + \psi \), introduces a gauge transformation

\[ \theta \rightarrow \theta + d\psi, \]

and one can exploit this freedom to impose restrictions on \( \theta \). We will assume a condition of the form

\[ g^{ij} \partial_i \theta_j = Q(g, V; \partial g, \partial V, \theta), \quad (4.5) \]

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where $Q$ is a smooth function of the variables listed near $(\delta, 1; 0, 0, 0)$, with a zero of order two or higher with respect to $q$:

$$Q(p; 0) = \partial_q Q(p; 0) = 0.$$  

Examples include the harmonic gauge, $\Box_{n+1} g t = 0$, which reads

$$\partial_i (\sqrt{\det g} g^{ij}\theta_j) = 0,$$  

as well as the maximal gauge,

$$\partial_i \left( \frac{V^3 \sqrt{\det g} g^{ij}}{\sqrt{1 - V^2 g^{kl} \theta_k \theta_l}} \theta_j \right) = 0.$$  

Equation (4.6) can always be achieved by solving a linear equation for $\psi$, cf., e.g., [2, 7] for the relevant isomorphism theorems. On the other hand, (4.7) can always be solved outside of some large ball [3]. More generally, when non-linear in $\theta$, equation (4.5) can typically be solved outside of some large ball using the implicit function theorem in weighted Hölder or weighted Sobolev spaces.

In harmonic coordinates, and in a gauge (4.5), the system (4.3) is elliptic and, similarly to the static case, standard asymptotic considerations show that $g_{ij}$ is Schwarzschild in the leading order, and that there exist constants $\alpha_{ij}$ such that

$$\theta_i = \alpha_{ij} x^j + O(r^{-n}).$$

To prove analyticity at $r^0$ one proceeds as in Section 3.2: thus, one first rewrites the second of equations (4.3) as an equation for

$$\tilde{g}_{ij} := e^{2u/n} g_{ij} \equiv V^{n/2} g_{ij},$$

which gets rid of the Hessian of $V$ there. It should then be clear that, in coordinates which are harmonic for $\tilde{g}$, the first two equations in (4.3) have the right structure for the argument of Section 3.2. It remains to check the third one. For this we note that, in $\tilde{g}$–harmonic coordinates so that $\partial_i (\sqrt{\det \tilde{g}} \tilde{g}^{ij}) = 0$,

$$\text{div}(V \lambda)_k = \frac{1}{\sqrt{\det \tilde{g}}} \partial_i \left( \sqrt{\det g} V^3 g^{ij} (\partial_j \theta_k - \partial_k \theta_j) \right)$$

$$= V^{n/2} \partial_i \left( \sqrt{\det \tilde{g}} V^2 \tilde{g}^{ij} (\partial_j \theta_k - \partial_k \theta_j) \right)$$

$$= V^{n/2} \tilde{g}^{ij} \partial_i \left( V^2 (\partial_j \theta_k - \partial_k \theta_j) \right)$$

$$= V^{n/2} (\tilde{g}^{ij} \partial_i \partial_j \theta_k + 2V \tilde{g}^{ij} \partial_i V (\partial_j \theta_k - \partial_k \theta_j)$$

$$- \tilde{g}^{ij} \partial_i \partial_k \theta_j) \right).$$

If $Q$ in (4.5) is zero, then the vanishing of $\text{div}(V \lambda)$ immediately gives an equation of the right form for $\theta$. Otherwise, $\partial Q$ leads to nonlinear terms of the form $\partial^2 g \theta$, etc., which are again of the right form, see the calculations in [6]. Note that such terms do not affect the ellipticity of the equations because of their off-diagonal character.
5 Einstein-Maxwell equations

The above considerations immediately generalise to the stationary Einstein-
Maxwell equations, with a Killing vector which approaches a time-translation
in the asymptotically flat region. Indeed, the calculations of Section 4 carry
over to this setting, as follows:

Stationary Maxwell fields can be described by a time-independent scalar
field $\varphi = A_0$ and a vector potential $A = A_i dx^i$, again time-independent. Here
one needs to assume that, in addition to (4.4), one has

\[ A_\mu = O(r^{-\alpha}) , \quad \partial_k A_\mu = O(r^{-\alpha-1}) . \]

Maxwell fields lead to supplementary source terms in the right-hand-sides of
(4.3) which are quadratic in the first derivatives of $\varphi$ and $A$, hence of the right
form for the argument so far. Next, if we write the Maxwell equations as

\[
\frac{1}{\sqrt{n+1} g} \partial_\mu \left( \sqrt{n+1} g^{\mu\rho} A^{n+1} \{\nu A_{\sigma}\} \right) = 0 ,
\]

and impose the Lorenz gauge,

\[
\frac{1}{\sqrt{n+1} g} \partial_\mu \left( \sqrt{n+1} g^{\mu\nu} A_{\nu} \right) = 0 ,
\]

the equations $\partial_t A_\mu = 0$ allow one to rewrite the above as

\[
g^{ij} \partial_i \partial_j a = H(f, V, \theta; \partial f, \partial V, \partial \theta; \partial a) ,
\]

where $a = (\varphi, A_i)$, with a function $H$ which is bilinear in the second and third
groups of arguments. This is again of the right form, which finishes the proof
of analyticity of $\tilde{f}, \tilde{\varphi}, \tilde{A}$ and $\tilde{\theta}$ at $i_0$ for even $n \geq 6$, where the original fields are
related to the tilde-ones via a rescaling by $\Omega^{n+2}$, e.g. $\varphi = \Omega^{n+2} \tilde{\varphi}$, and so on.

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