SET THEORETIC SOLUTIONS OF THE YANG-BAXTER
EQUATION, GRAPHS AND COMPUTATIONS

TATIANA GATEVA-IVANOVA AND SHAHN MAJID

Abstract. We extend our recent work on set-theoretic solutions of the Yang-Baxter or braid relations with new results about their automorphism groups, strong twisted unions of solutions and multipermutation solutions. We introduce and study graphs of solutions and use our graphical methods for the computation of solutions of finite order and their automorphisms. Results include a detailed study of solutions of multipermutation level 2.

1. Introduction

It is well-known that certain matrix solutions of the braid or Yang-Baxter equations lead to braided categories, knot invariants, quantum groups and other important constructions, see [M02] for an introduction. However, these equations are also very interesting at the level of set maps \( r : X \times X \to X \times X \) where \( X \) is a set and \( r \) is a bijection, a line of study proposed in [Dri, Wei]. Solutions extend linearly to very special linear solutions but also lead to a great deal of combinatorics and to algebras with very nice homological properties including those relating to the existence of noncommutative Groebner bases. We start with the definitions, using conventions from recent works [GI, GIM].

Definition 1.1. Let \( X \) be a nonempty set and let \( r : X \times X \to X \times X \) be a bijective map. We use notation \( (X, r) \) and refer to it as a quadratic set. The image of \( (x, y) \) under \( r \) is presented as

\[
(1.1) \quad r(x, y) = (x^y, x^y).
\]

The formula (1.1) defines a “left action” \( \mathcal{L} : X \times X \to X \), and a “right action” \( \mathcal{R} : X \times X \to X \), on \( X \) as:

\[
(1.2) \quad \mathcal{L}_x(y) = x^y, \quad \mathcal{R}_y(x) = x^y,
\]

for all \( x, y \in X \). The map \( r \) is nondegenerate, if the maps \( \mathcal{R}_x \) and \( \mathcal{L}_x \) are bijective for each \( x \in X \). In this paper we shall always be interested in the case where \( r \) is nondegenerate, as will be indicated. As a notational tool, we shall often identify the sets \( X \times X \) and \( X^2 \), the set of all monomials of length two in the free semigroup \( \langle X \rangle \).

Date: May 11, 2007.
1991 Mathematics Subject Classification. Primary 81R50, 16W50, 16S36.
Key words and phrases. Yang-Baxter, Semigroups, Quantum Groups, Graphs.

The first author was partially supported by Grant SAB2005-0094 of the Ministry of Education and Culture of Spain, the Royal Society, UK, and by Grant MI 1503/2005 of the Bulgarian National Science Fund of the Ministry of Education and Science. Collaboration was during a visit by both authors to the Isaac Newton Institute, Cambridge, where the work originates.
Definition 1.2.  
1. $r$ is square-free if $r(x, x) = (x, x)$ for all $x \in X$.
2. $r$ is a set-theoretic solution of the Yang-Baxter equation or, shortly a solution (YBE) if the braid relation

$$r_{12} r_{23} r_{12} = r_{23} r_{12} r_{23}$$

holds in $X \times X \times X$, where the two bijective maps $r_{ii}^{i+1} : X^3 \rightarrow X^3$, $1 \leq i \leq 2$ are defined as $r_{12}^{12} = r \times \text{Id}_X$, and $r_{23}^{23} = \text{Id}_X \times r$. In this case some authors also call $(X, r)$ a braided set. If in addition $r$ is involutive $(X, r)$ is called a symmetric set.

To each quadratic set $(X, r)$ we associate canonical algebraic objects generated by $X$ and with quadratic defining relations $\mathcal{R} = \mathcal{R}(r)$ defined by

\[ xy = zt \in \mathcal{R}(r), \quad \text{whenever} \quad r(x, y) = (z, t). \]

Definition 1.3. Let $(X, r)$ be a quadratic set.
1. The semigroup

$$S = S(X, r) = \langle X; \mathcal{R}(r) \rangle,$$

with a set of generators $X$ and a set of defining relations $\mathcal{R}(r)$, is called the semigroup associated with $(X, r)$.
2. The group $G = G(X, r)$ associated with $(X, r)$ is defined as

$$G = G(X, r) = g_r(X; \mathcal{R}(r)).$$

3. For arbitrary fixed field $k$, the $k$-algebra associated with $(X, r)$ is defined as

\[ A = A(k, X, r) = k \langle X \rangle / (\mathcal{R}(r)). \]

If $(X, r)$ is a solution, then $S(X, r)$, resp. $G(X, r)$, resp. $A(k, X, r)$ is called the Yang-Baxter semigroup, resp. the Yang-Baxter group, resp. the Yang-Baxter algebra associated to $(X, r)$.

Example 1.4. For arbitrary nonempty set $X$, the trivial solution $(X, r)$ is defined as $r(x, y) = (y, x)$, for all $x, y \in X$. It is clear that $(X, r)$ is the trivial solution iff $xy = yx$, and $x^y = x$, for all $x, y \in X$, or equivalently $L_x = \text{id}_X = R_x$ for all $x \in X$. In this case $S(X, r)$ is the free abelian monoid, $G(X, r)$ is the free abelian group, $A(k, X, r)$ the algebra of commutative polynomials in $X$.

The algebras $A(X, r)$ provided new classes of Noetherian rings [GI94, GI96-1], Gorestein (Artin-Schelter regular) rings [GI96-2, GI00, GI04] and so forth. Artin-Schelter regular rings were introduced in [AS] and are of particular interest. The algebras $A(X, r)$ are similar in spirit to the quadratic algebras associated to linear solutions, particularly studied in [M], but have their own remarkable properties in the set theoretic case. The semigroups $S(X, r)$ were studied particularly in [GIM] with a systematic theory of ‘exponentiation’ from the set to the semigroup by means of the ‘actions’ $L_x, R_x$ (which in the process become a matched pair of semigroup actions) somewhat in analogy with the Lie theoretic exponentiation in [M90]. Also in [GIM] are first results about construction of extensions of solutions that we shall take further. There are many other works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [ESS, GB, GJO, JO, LYZ, R, T]. Of particular interest in the present paper, we define:
**Definition 1.5.** When \((X, r)\) is nondegenerate, \(L : G(X, r) \to \text{Sym}(X)\) is a group homomorphism defined via the left action. We denote by \(G = G(X, r)\) the subgroup \(L(G(X, r))\) of \(\text{Sym}(X)\).

For example, \(G(X, r) = \{\text{id}_X\}\) in the case of the trivial solution.

**Remark 1.6.** As is well known, see for example [GIM], a quadratic set \((X, r)\) is a braided set (i.e. \(r\) obeys the YBE) iff the following conditions hold
\[
\begin{align*}
\text{l1} & : \quad x(yz) = x(y)xz, \\
\text{r1} & : \quad (x^y)^z = (x^y^z)^y, \\
\text{l3} & : \quad (x^y)^{(x^y)}(z) = (x^{y^z})^{y^z},
\end{align*}
\]
for all \(x, y, z \in X\). Clearly, conditions l1, respectively, r1 imply that the group \(G(X, r)\) acts on the left, respectively on the right, on the set \(X\). This is needed for the definition of \(G(X, r)\) above to make sense.

We also have some technical conditions and special cases of interest, particularly certain ‘cyclicity’ conditions. Systematic treatments were provided in [GI, GIM] and we recall some essentials needed for the paper in the remainder of this introduction.

**Definition 1.7.** [GIM] Given \((X, r)\) we extend the actions \(x \cdot \cdot\) and \(\cdot\cdot x\) on \(X\) to left and right actions on \(X \times X\) as follows. For \(x, y, z \in X\) we define:
\[
\begin{align*}
x(y, z) & := (x^y, x^y^z), \\
(x, y^z) & := (x^y^z, y^z).
\end{align*}
\]
We say that \(r\) is respectively left and right invariant if
\[
\begin{align*}
\text{l2} & : \quad r(x(y, z)) = x(r(y, z)), \\
\text{r2} & : \quad r((x, y)^z) = (r(x, y))^z
\end{align*}
\]
hold for all \(x, y, z \in Z\).

**Definition 1.8.** [GIM], A quadratic set \((X, r)\) is called cyclic if the following conditions are satisfied
\[
\begin{align*}
\text{cl1} & : \quad y^x = y^x \quad \text{for all } x, y \in X; \\
\text{cr1} & : \quad x^{y^x} = x^y, \quad \text{for all } x, y \in X; \\
\text{cl2} & : \quad y^x = y^x \quad \text{for all } x, y \in X; \\
\text{cr2} & : \quad x^{y^x} = x^y \quad \text{for all } x, y \in X.
\end{align*}
\]
We refer to these conditions as cyclic conditions.

Theorem 1.13 given below shows that every square-free solution \((X, r)\) is cyclic, furthermore it satisfies condition lri and is a left and a right cycle set, see the definitions below.

**Definition 1.9.** [GIM] Let \((X, r)\) be a quadratic set. We define the condition
\[
\text{lri} : \quad (x^y)^x = y = (y^x)^x \quad \text{for all } x, y \in X.
\]
In other words lri holds if and only if \((X, r)\) is nondegenerate and \(R_x = L_x^{-1}\) and \(L_x = R_x^{-1}\).

**Remark 1.10.** By contrast with this involutiveness does not entail nondegeneracy. For example the identity solution \((X, r_X)\), with \(r_X = \text{id}_X \times X\) is obviously involutive, but for each \(X\) of order \(\geq 2\) it is not nondegenerate.

**Definition 1.11.** [R, GIM] Let \((X, r)\) be nondegenerate. \((X, r)\) is called a left cycle set if
\[
\text{csl} : \quad (x^y)^t = (t^y)^t \quad \text{for all } x, y, t \in X.
\]
The definition of a right cycle set is analogous, see [GIM].
Proposition 1.12. [GIM, Proposition 2.24] Let \((X, r)\) be a quadratic set. Then any two of the following conditions imply the remaining third condition.

1. \((X, r)\) is involutive.
2. \((X, r)\) is nondegenerate and cyclic, see Definition 1.8.
3. \(Iri\) holds.

Theorem 1.13. [GIM, Theorem 2.34] Suppose \((X, r)\) is nondegenerate, involutive and square-free quadratic set. Then the following conditions are equivalent:

1. \((X, r)\) is a set-theoretic solution of the Yang-Baxter equation.
2. \((X, r)\) satisfies \(l1\).
3. \((X, r)\) satisfies \(l2\).
4. \((X, r)\) satisfies \(r1\).
5. \((X, r)\) satisfies \(r2\).
6. \((X, r)\) satisfies \(l3\).
7. \((X, r)\) satisfies \(csl\).

In this case \((X, r)\) is cyclic and satisfies \(Iri\).

Corollary 1.14. Every nondegenerate involutive square-free solution \((X, r)\) is uniquely determined by the left action \(L : X \times X \to X\), more precisely,

\[ r(x, y) = (L_x(y), L_y^{-1}(x)). \]

Furthermore it is cyclic.

2. Homomorphisms of solutions, automorphisms

Definition 2.1. Let \((X, r_X)\) and \((Y, r_Y)\) be arbitrary solutions (braided sets). A map \(\varphi : X \to Y\) is a homomorphism of solutions, if it satisfies the equality

\[ (\varphi \times \varphi) \circ r_X = r_Y \circ (\varphi \times \varphi). \]

A bijective homomorphism of solutions is called (as usual) an isomorphism of solution. An isomorphism of the solution \((X, r)\) onto itself is an \(r\)-automorphism.

We denote by \(Hom((X, r_X), (Y, r_Y))\) the set of all homomorphisms of solutions \(\varphi : X \to Y\). The group of \(r\)-automorphisms of \((X, r)\) will be denoted by \(Aut(X, r)\). Clearly, \(Aut(X, r)\) is a subgroup of \(Sym(X)\).

Remark 2.2. Let \((X, r_X)\), \((Y, r_Y)\) be finite symmetric sets with \(Iri\). Every homomorphism of solutions \(\varphi : (X, r_X) \to (Y, r_Y)\) induces canonically a homomorphism of their graphs (see Definition 5.1):

\[ \varphi_\Gamma : \Gamma(X, r_X) \to \Gamma(Y, r_Y). \]

Furthermore there is a one-to one correspondence between \(Aut(X, r)\) and \(Aut(\Gamma(X, r))\), the group of automorphisms of the multigraph \(\Gamma(X, r)\).

The next lemma is straightforward from the definition.

Lemma 2.3. Let \((X, r_X)\) and \((Y, r_Y)\) be solutions.

1. A map \(\varphi : X \to Y\) is a homomorphism of solutions iff:

\[ \varphi \circ L_x = L_{\varphi(x)} \circ \varphi \quad \text{and} \quad \varphi \circ R_x = R_{\varphi(x)} \circ \varphi \quad \text{for all} \ x \in X. \]
(2) If both \((X, r_X)\) and \((Y, r_Y)\) satisfy \textbf{iri}, then \(\varphi\) is a homomorphism of solutions \(\text{iff}
\varphi \circ \mathcal{L}_x = \mathcal{L}_{\varphi(x)} \circ \varphi, \text{ for all } x \in X.\)

(3) If \((X, r)\) obeys \textbf{iri}, then \(\tau \in \text{Sym}(X)\) is an automorphism of \((X, r_X)\) \(\text{iff}\)
\begin{equation}
\tau \circ \mathcal{L}_x \circ \tau^{-1} = \mathcal{L}_{\tau(x)}, \text{ for all } x \in X.
\end{equation}

For example, if \((X, r)\) is the trivial solution then, clearly, \(\text{Aut}(X, r) = \text{Sym}(X).\) This is because each \(\mathcal{L}_x = \text{id}_X.\) More generally, (2.1) clearly implies:

**Corollary 2.4.** The group \(\text{Aut}(X, r)\) is a subgroup of \(\text{Nor}_{\text{Sym}(X)} \mathcal{G}(X, r),\) the normalizer of \(\mathcal{G}(X, r)\) in \(\text{Sym}(X).\)

We now turn to two basic examples which will be used throughout the paper. The first is a solution \((X, r)\) of order 6, with \(\mathcal{G}(X, r) \subsetneq \text{Aut}(X, r)\) for which \(\text{Aut}(X, r) \subsetneq \text{Nor}_{\text{Sym}(X)}(\mathcal{G}).\) We shall show in general in Section 4 that \(\mathcal{G}(X, r)\) is a subgroup of \(\text{Aut}(X, r)\) \(\text{iff}\) \((X, r)\) is a multipermutation solution of level 2, see Theorem 5.24.

**Example 2.5.** Let \((X, r)\) be the nondegenerate involutive square-free solution with
\[X = \{x_1, x_2, x_3, x_4, b, c\}, \quad \mathcal{L}_b = (x_1 x_2)(x_3 x_4), \quad \mathcal{L}_c = (x_1 x_3)(x_2 x_4)\]
and the remaining \(\mathcal{L}_{x_i} = \text{id}_X.\) Here and elsewhere we use Corollary 1.14. Then \(\mathcal{G}(X, r) = \langle \mathcal{L}_b, \mathcal{L}_c \rangle,\) so it is isomorphic to the Klein’s group \(\mathbb{Z}_2 \times \mathbb{Z}_2.\) Direct computations show that the set of automorphisms consists of the following eight elements:
\[\text{id}_X, \quad \tau_1 = (bc)(x_2 x_3), \quad \tau_2 = (bc)(x_1 x_2 x_4 x_3), \quad \tau_3 = (bc)(x_1 x_3 x_4 x_2), \quad \tau_4 = (bc)(x_1 x_4), \quad \mathcal{L}_b, \quad \mathcal{L}_c, \quad \mathcal{L}_b \circ \mathcal{L}_c.\]

Furthermore, one has
\[
\tau_1^2 = 1, \quad \tau_2^4 = 1, \quad \tau_1 \tau_2 \tau_1^{-1} = \tau_3, \quad \tau_1 \circ \tau_2 = \mathcal{L}_c, \quad \tau_2 \circ \tau_1 = \mathcal{L}_b, \quad \tau_1 \circ \tau_2^2 = \tau_4.
\]

It is easy to see that
\[\text{Aut}(X, r) = \langle \tau_1, \tau_2 \mid \tau_1^2 = 1, \tau_2^4 = 1, \tau_1 \tau_2 \tau_1^{-1} = \tau_2 \rangle \cong D_4.\]

Next we show that \(\text{Aut}(X, r) \subsetneq \text{Nor}_{\text{Sym}(X)}(\mathcal{G}).\) Consider \(\sigma = (x_1 x_3)(x_2 x_4)(bc) \in \text{Sym}(X).\) One has:
\begin{equation}
\sigma \circ \mathcal{L}_b \circ \sigma^{-1} = \mathcal{L}_b, \quad \sigma \circ \mathcal{L}_c \circ \sigma^{-1} = \mathcal{L}_c,
\end{equation}
so \(\sigma \in \text{Nor}_{\text{Sym}(X)}(\mathcal{G}).\) On the other hand \(\sigma\) does not satisfy the necessary condition (2.1) for being an automorphism, so \(\text{Aut}(X, r)\) is a proper subgroup of \(\text{Nor}_{\text{Sym}(X)}(\mathcal{G}).\) The graph \(\Gamma(X, r)\) and its automorphism group is given on figure 1.

The second example will contain the above solution \((X, r)\) an \(r\)-invariant subset. (In fact \((X, r)\) itself contains \(X_0 = \{x_1, x_2, x_3, x_4\}\) as an \(r\)-invariant subset.)

**Definition 2.6.** Let \((Z, r)\) be a quadratic set, \(\emptyset \neq X \subseteq Z.\) \(X\) is \(r\)-\textit{invariant} if \(r(X \times X) \subseteq X \times X.\) In this case we consider the restriction \(r_X = r|_{X \times X}.\) Clearly, if \((Z, r)\) obeys YBE then \((X, r_X)\) is also a solution, and inherits all “good” properties of \((Z, r)\) like being non degenerate, involutive, square-free, satisfying \textbf{iri}, the cyclic conditions, etc.
Lemma 2.7. Let \((Z, r)\) be a nondegenerate solution, with \(I^r\) and \(\alpha \in Z\). Suppose \(X\) is an \(r\)-invariant subset of \(Z\) and let \(r_X\) be the restriction of \(r\) on \(X \times X\). Then

1) \(L_\alpha|_X \in \text{Aut}(X, r_X)\) iff
\[ \alpha^y x = \alpha x \quad \text{for all } x, y \in X. \]

In particular, \(L_\alpha \in \text{Aut}(Z, r)\) iff the displayed condition holds for all \(x, y \in Z\).

2) Furthermore, suppose that \(r\) is involutive, then \(L_\alpha|_X \in \text{Aut}(X, r_X)\) iff
\[ ^y \alpha^x = \alpha^x \quad \text{for all } x, y \in X. \]

Proof. Let \(\alpha \in Z\). By (2.1) \(L_\alpha|_X\) is an automorphism of \((X, r)\) iff
\[ L_\alpha|_X \circ L_y = L_{\alpha y} \circ L_\alpha|_X, \quad \text{for all } y \in X \]
or equivalently
\[ (2.3) \quad \alpha(y x) = ^y(\alpha x), \quad \text{for all } x, y \in X \]

By condition 11 on \((Z, r)\) one has
\[ \alpha(y x) = ^y(\alpha x), \quad \text{for all } x, y \in X, \]
which together with (2.3) implies
\[ (2.4) \quad ^y(\alpha x) = ^y(\alpha x), \quad \text{for all } x, y \in X \]

By the non degeneracy of \((Z, r)\) (2.4) holds iff \(^y \alpha^x = \alpha^x\). This proves part (1).
Assume now that \( r \) is involutive. Then \((Z, r)\) is a symmetric set so its retraction \((\{Z\}, r|_Z)\) is well defined, see Definition 3.1. By Lemma 3.4 \((\{Z\}, r|_Z)\) inherits \textbf{ri}. We write (2.3) in the notation of retractions and obtain the implications
\[
(2.3) \iff [\alpha]^y = [\alpha]^y = [\alpha] \iff [\alpha] = [y][\alpha]
\]
\[
[\alpha] = [y][\alpha] \iff ^D \alpha x = ^y \alpha x \quad \text{for all } x \in X.
\]
These equations together with part (1) imply part (2) of the lemma. \(\square\)

For instance, we have seen that \(L_b, L_c\) in our example above are automorphisms. For our next example they are joined by \(L_a = L_b \circ L_c\).

**Example 2.8.** Let \((Z, r_Z)\) be the nondegenerate involutive square-free solution given by
\[
Z = \{x_1, x_2, x_3, x_4, a, b, c\}, \quad L_a = (x_1x_4)(x_2x_3), \quad L_b = (x_1x_2)(x_3x_4), \quad L_c = (x_1x_3)(x_2x_4).
\]
\((Z, r_Z)\) is an extension of the preceding solution \((X, r)\) by the trivial solution on the one element set \(\{a\}\). Furthermore, \(L_a = L_b \circ L_c\), so \(G(Z, r_Z) = G(X, r) \approx \mathbb{Z}_2 \times \mathbb{Z}_2\).

More sophisticated arguments in Section 5 will show that the group \(\text{Aut}(Z, r_Z)\) is isomorphic to \(S_4\), the symmetric group on 4 elements. Furthermore, each automorphism \(\tau \in \text{Aut}(X, r)\) can be extended uniquely to an automorphism in \(\text{Aut}(Z, r_Z)\), by \(\tau(a) = a\). These are the automorphisms denoted \(\tau_i, L_a, L_b, L_c\) shown in Figure 2. The remaining automorphisms and the graph of the solution are also shown.

We conclude the section with some straightforward generalities.

**Remark 2.9.** Let \((X, r_X), (Y, r_Y)\) be solutions. By definition every homomorphism of solutions \(\varphi : (X, r_X) \to (Y, r_Y)\) agrees with the defining relations \(\mathcal{R}(X, r_X)\) and \(\mathcal{R}(Y, r_Y)\) of the related algebraic objects, so it can be extended to
\[
\begin{align*}
(1) & \quad \text{a semigroup homomorphism } \varphi_S : S(X, r_X) \to S(Y, r_Y) \text{ of their Yang-Baxter semigroups; } \\
(2) & \quad \text{a group homomorphism } \varphi_G : G(X, r_X) \to G(Y, r_Y) \text{ of their Yang-Baxter groups;} \\
(3) & \quad \text{an algebra homomorphism } \varphi_A : \mathcal{A}(k, X, r_X) \to \mathcal{A}(k, Y, r_Y) \text{ of their Yang-Baxter algebras.}
\end{align*}
\]

If furthermore, \(X\) is embedded in \(G(X, r)\) (respectively in \(S(X, r)\)), each \(r\)-automorphism \(\tau\) of the solution \((X, r)\) can be extendeded to a group automorphism \(\tau_G : G(X, r) \to G(X, r)\), (respectively, to a semigroup automorphism \(\tau_S : S(X, r) \to S(X, r)\)). In this case, we have an embedding \(\text{Aut}(X, r) \hookrightarrow \text{Aut}(G(X, r))\), (respectively, an embedding \(\text{Aut}(X, r) \hookrightarrow \text{Aut}(S(X, r))\))

3. **Multipermutation solutions**

In this section we shall consider only nondegenerate symmetric sets \((X, r)\).

**Definition 3.1.** [ESS], 3.2. Let \((X, r)\) be a nondegenerate symmetric set. An equivalence relation \(\sim \) is defined on \(X\) as
\[
x \sim y \iff Lx = Ly.
\]
In this case we also have \(R_x = R_y\), see [ESS].
(a) \[ Z = \text{X}_\{a\} = \{x_1, x_2, x_3, x_4, a, b, c\} \]

(b) \[ \text{Aut}(Z, \tau_2) \]

\[ \tau = \]

\[ \pi = \]

\[ \eta = \]

\[ \rho = \]

\[ \sigma = \]

\[ L_a, L_b, L_c = \]

\[ \text{Figure 2.} \ \ \text{(a) Graph of the solution of Example 2.8 and (b) its automorphism group.} \]

We denote by \([x]\) the equivalence class of \(x \in X\), \([X] = X/\sim\) is the set of equivalence classes.
It is shown in [ESS] that the solution \( r \) induces a canonical map

\[
 r_{\{X\}} : [X] \times [X] \to [X] \times [X],
\]

which makes \((\{X\}, r_{\{X\}})\) a nondegenerate symmetric set, called the retraction of \((X, r)\), and denoted \( Ret(X, r) \).

For our purposes we need concrete expressions of the left and the right actions on \([X]\) canonically induced by the left and right actions on \(X\).

**Definition 3.2.** Let \((X, r)\) be a nondegenerate symmetric set. Then the left and the right actions of \(X\) onto itself induce naturally left and right actions on the retraction \([X]\), via

\[
 (\alpha) x := \alpha x, \quad [\alpha x] := [\alpha^x], \quad \text{for all} \, \alpha, x \in X.
\]

Note that in the following there is no need to assume \( lri \), since \( L_x = L_y \iff R_x = R_y \).

**Lemma 3.3.** Suppose \((X, r)\) is a nondegenerate symmetric set. Then the left and the right actions (3.1) are well defined.

**Proof.** We need to show that \([\alpha] x\) does not depend on the representatives of \([\alpha]\) and \([x]\). It will be enough to show that

\[
 L_y = L_x \implies [\alpha y] = [\alpha x], \quad \text{for all} \, \alpha \in X.
\]

So fix \(\alpha, x \in X\), and assume \(y \in [x]\). Let \(z \in X\). We have to verify:

\[
 (\alpha y) z = (\alpha x) z.
\]

By hypothesis \((X, r)\) is nondegenerate, so there exists a \(t \in X\), with

\[
 z = \alpha^z t = \alpha^y t
\]

(The right hand side equality comes from \( L_y = L_x \), which implies \( \alpha^z = \alpha^y \)). Then we have

\[
 (\alpha y) z = \alpha y (\alpha y t) =^1 \alpha (y t) =^2 [z] = [y] \alpha (z t) = \alpha x (\alpha x t) = (3.3) (\alpha x) z.
\]

\(\square\)

**Lemma 3.4.** Suppose \((X, r)\) is a nondegenerate symmetric set. Then

1. The left and the right actions (3.1) define (as usual) a canonical map

\[
 r_{\{X\}} : [X] \times [X] \to [X] \times [X],
\]

which makes \((\{X\}, r_{\{X\}})\) a nondegenerate symmetric set.

2. \((X, r)\) cyclic \(\implies\) \((\{X\}, r_{\{X\}})\) cyclic.

3. \((X, r)\) is lri \(\implies\) \((\{X\}, r_{\{X\}})\) is lri.

4. \((X, r)\) square-free \(\implies\) \((\{X\}, r_{\{X\}})\) square-free.

**Proof.** It is easy to see that the left and the right actions (3.1) on \([X]\) inherit conditions \(l1, rl1, lr3\). Therefore \(r_{\{X\}}\) obeys YBE. The implication in (3) follows easily from (3.1). We leave conditions (2), (4) (and so forth) to the reader. \(\square\)
Definition 3.5. [ESS]. The solution \([X, [r]]\) is called retraction of \((X, r)\) and is also denoted \(\text{Ret}(X, r)\). For all integers \(m \geq 1\), \(\text{Ret}^m(X, r)\) is defined recursively as \(\text{Ret}^m(X, r) = \text{Ret}(\text{Ret}^{m-1}(X, r))\). A nondegenerate symmetric set \((X, r)\) is a multipermutation solution of level \(m\) if \(m\) is the minimal number such that \(\text{Ret}^m(X, r)\) is finite of order 1, this will be denoted by \(mpl(X, r) = m\). By definition \((X, r)\) is a multipermutation solution of level 0 iff \(X\) is a one element set.

Example 3.6. An involutive permutation solution \((X, r)\) is defined [Dri] (attributed to Lyubashenko) as \(r(x, y) = (\sigma(y), \sigma^{-1}(x))\), where \(\sigma\) is a fixed permutation in \(\text{Sym}(X)\). In this case \((X, r)\) is a nondegenerate symmetric set with \(\text{mpl}(X, r)\). Clearly, \(L_x = \sigma\), for all \(x \in X\), thus \(\text{Ret}(X, r)\) is a one element set and \(mpl(X, r) = 1\). The converse also holds [ESS], so \(mpl(X, r) = 1\) iff \((X, r)\) is a permutation solution.

In particular, the trivial solution \((X, r)\) with \(X\) of order \(\geq 2\) has \(mpl(X, r) = 1\).

Remark 3.7. Let \((X, r)\) be a nondegenerate symmetric set. Then there is a surjective homomorphism of solutions \(\mu: (X, r) \rightarrow ([X], [r]); \mu(x) = [x]\), Each finite symmetric set \((X, r)\) with \(\text{mpl}\) has a well defined oriented graph \(\Gamma(X, r_X)\), see [GIM] and 5.1. In this case \(\mu\) induces a homomorphism of graphs \(\mu_\Gamma: \Gamma(X, r_X) \rightarrow \Gamma([X], r_{[X]})\). The graph \(\Gamma([X], r_{[X]})\) is a retraction of \(\Gamma(X, r_X)\).

The following is straightforward from Definition 3.5

Lemma 3.8. Suppose \((X, r)\) is a multipermutation solution. The following implications hold.

\[mpl(X, r) = m \implies mpl(\text{Ret}^k(X, r)) = m - k, \quad \text{for all } 1 \leq k \leq m - 1.\]

Conversely, if \(k \geq 0\) is an integer, then

\[mpl(\text{Ret}^k(X, r)) = s \implies mpl(X, r) = s + k.\]

Lemma 3.9. Let \((X, r)\) be a nondegenerate square-free symmetric set. Then \([x] = [y], x, y \in X \implies r(x, y) = (y, x)\).

Proof. Indeed, \([x] = [y]\) implies \(^y x = ^x x = x\), from which by \(\text{mpl}\), one has \(^x y = x\) and by an analogous argument with \(x, y\) swapped one has \(^y x = y\). □

Corollary 3.10. Suppose \((X, r)\) is a nondegenerate square-free symmetric set (of order \(\geq 2\)). Then the following conditions are equivalent.

1. \(mpl(X, r) = 1\).
2. \((X, r)\) is the trivial solution.
3. \(^x y = y\), for all \(x, y \in X\).
4. \(S(X, r)\) is the free abelian monoid generated by \(X\).
5. \(G(X, r)\) is the free abelian group generated by \(X\).
6. \(G(X, r) = \{id_X\}\).
The $k$th retract orbit of an element was introduced first in the case of square-free solutions, see [GI].

**Definition 3.11.** [GI] Let $(X, r)$ be a nondegenerate symmetric set. We denote by $[x]_k$ the image of $x$ in $\text{Ret}^k(X, r)$. The set

$$\Omega(x, k) := \{\xi \in X \mid [\xi]_k = [x]_k\}$$

is called the $k$th retract orbit of $x$.

Suppose $X$ is an $r$-invariant subset of the solution $(Z, r)$, and $x \in X$. Then $[x]_k, X$ will denote the $k$-th retract of $x$ in $X$.

**Remark 3.12.** cf. [GI, Lemma 8.9] Let $(X, r)$ be a square-free symmetric set. For every positive integer $k \leq \text{mpl}(X, r)$ the $k$th retract orbit $\Omega(x, k)$ is $r$-invariant. Furthermore if we denote by $r_{x,k}$ the corresponding restriction of $r$, then $(\Omega(x, k), r_{x,k})$ is a multipermutation solution and

$$\text{mpl}(\Omega(x, k), r_{x,k}) \leq k. \quad (3.4)$$

In [GI] (3.4) is actually written as an equality, but this should be corrected as the following example shows.

**Example 3.13.** Let $X = \{x_1, \ldots, x_k, y\}$, and $r$ be defined via the actions $L_y = (x_1 \cdots x_k), L_{x_k} = id_X$. Then $\Omega(x_1, 1) = \{x_1, \ldots, x_k\}$, $\Omega(y, 1) = \{y\}$. Clearly, $[X] = \{[x_1], [y]\}$ is the trivial solution, so $\text{mpl}X = 2$. Note that $\text{mpl}\Omega(y, 1) = 0$, $\text{mpl}\Omega(x_1, 1) = 1$.

4. Strong twisted unions of solutions

In this section we study special extensions of solutions called *strong twisted unions*. We recall first some basic facts and definitions.

The notion of a *union* of solutions and one-sided extensions were introduced in [ESS], but only for nondegenerate involutive solutions $(X, r_X), (Y, r_Y)$. In [GIM] are introduced and studied more general extensions $(Z, r)$ of arbitrary solutions $(X, r_X), (Y, r_Y)$, and given necessary and sufficient conditions (in terms of left and right actions) so that a regular extension $(Z, r)$ satisfies YBE.

**Definition 4.1.** [GIM] Let $(X, r_X)$ and $(Y, r_Y)$ be disjoint quadratic sets (i.e. with bijective maps $r_X : X \times X \rightarrow X \times X$, $r_Y : Y \times Y \rightarrow Y \times Y$). Let $(Z, r)$ be a set with a bijection $r : Z \times Z \rightarrow Z \times Z$. We say that $(Z, r)$ is a (general) extension of $(X, r_X), (Y, r_Y)$, if $Z = X \cup Y$ as sets, and $r$ extends the maps $r_X$ and $r_Y$, i.e. $r|_{X \times X} = r_X$, and $r|_{Y \times Y} = r_Y$. Clearly in this case $X, Y$ are $r$-invariant subsets of $Z$. $(Z, r)$ is a YB-extension of $(X, r_X), (Y, r_Y)$ if $r$ obeys YBE.

**Remark 4.2.** In the assumption of the above definition, suppose $(Z, r)$ is a non-degenerate extension of $(X, r_X), (Y, r_Y)$, (without any further restrictions on the solutions). Then the equalities $r(x, y) = (x y, x y^2)$, $r(y, x) = (y x, y^2 x)$, and the non-degeneracy of $r$, $r_X$, $r_Y$ imply that

$$y x, x y^2 \in X, \quad x y, y^2 x \in Y, \quad \text{for all } x \in X, y \in Y.$$ 

Therefore, $r$ induces bijective maps

$$\rho : Y \times X \rightarrow X \times Y, \quad \sigma : X \times Y \rightarrow Y \times X, \quad (4.1)$$
and left and right “actions”

\[(4.2) \quad \bullet : Y \times X \rightarrow X, \quad \bullet : Y \times X \rightarrow Y, \quad \text{projected from } \rho \]

\[(4.3) \quad \triangleright : X \times Y \rightarrow Y, \quad \triangleleft : X \times Y \rightarrow X, \quad \text{projected from } \sigma.\]

Clearly, the 4-tuple of maps \((r_X, r_Y, \rho, \sigma)\) uniquely determine the extension \(r\). The map \(r\) is also uniquely determined by \(r_X, r_Y\), and the maps (4.2), (4.3).

In the present paper we restrict our attention to particular extensions called \textit{strong twisted unions}, also introduced in [GIM]. However, in the present paper we prefer to avoid the most general form of this and focus on the case where the extension is nondegenerate as in the remark above, and involutive.

\textbf{Definition 4.3.} [GIM] In the notation of Remark 4.2 a nondegenerate involutive extension \((Z, r)\) is a \textit{strong twisted union} of the quadratic sets \((X, r_X)\) and \((Y, r_Y)\) if

1. The assignment \(\alpha \rightarrow \circ \bullet\) extends to a left action of the associated group \(G(Y, r_Y)\) (and the associated semigroup \(S(Y, r_Y)\)) on \(X\), and the assignment \(x \rightarrow \bullet^x\) extends to a right action of the associated group of \(G(X, r_X)\) (and the associated semigroup \(S(X, r_X)\)) on \(Y\);
2. The pair of ground actions satisfy

\[\text{STU: } \quad \alpha^{\circ x} = \alpha x; \quad \alpha^{\bullet x} = \alpha^x, \quad \text{for all } \quad x, y \in X, \alpha, \beta \in Y\]

We shall use notation \((Z, r) = (X, r_X) \sharp (Y, r_Y)\) (or shortly, \((Z, r) = X_0 Y\)) for a strong twisted union. A strong twisted union \((Z, r)\) of \((X, r_X)\) and \((Y, r_Y)\) is nontrivial if at least one of the actions in (1) is nontrivial. In the case when both actions (1) are trivial we write \((Z, r) = X_0 Y\). In this case one has \(r(x, \alpha) = (\alpha, x)\) for all \(x \in X, \alpha \in Y\).

The following example is extracted from [ESS, Definition 3.3, Proposition 3.9],

\textbf{Example 4.4.} Let \((X, r_X), (Y, r_Y)\) be nondegenerate symmetric sets and \(\sigma \in Aut(X, r), \rho \in Aut(Y, r)\). Define the nondegenerate involutive extension \((Z, r)\) via \(r_X, r_Y\) and the formulæ

\[r(\alpha, x) = (\sigma(x), \rho^{-1}(\alpha)), \quad r(x, \alpha) = (\rho(\alpha), \sigma^{-1}(x)).\]

This is, moreover, a symmetric set (obeys the YBE) and is called a \textit{twisted union} in [ESS]. We denote it \((Z, r) = X_0 Y\) as a special case of a strong twisted union. Note that

\[L_{x|Z} = L_{x|X} \circ \rho, \quad L_{\alpha|Z} = L_{\alpha|Y} \circ \sigma, \quad \text{for all } x \in X, \alpha \in Y\]

from which it is clear that \((Z, r)\) is nondegenerate as \((X, r_X)\) and \((Y, r_Y)\) are.

Clearly a trivial extension \((Z, r)\) of \((X, r_X), (Y, r_Y)\) is a particular case of twisted union. Another easy example is:

\textbf{Example 4.5.} Let \((X, r)\) be a nondegenerate symmetric set and \(Y = \{a\}\) a one element set with trivial solution. A strong twisted union (in fact any regular extension) of these that obeys the YBE is necessarily a twisted union \(Z = X_0 Y\) given by \(L_a \in Aut(X, r)\) and \(id_Y\). This will be clear from Proposition 4.10 below. Our previous Example 2.8 is of this form.
The following lemma is straightforward.

**Lemma 4.6.** Suppose \((Z,r) = X_0 \bowtie Y\), is a solution and \(mpl(X, r_X) < \infty, mpl(Y, r_Y) < \infty\). Then \(mpl(Z, r) = \max\{mpl(X, r_X), mpl(Y, r_Y)\}\).

In [ESS, Definition 3.3] the notion of a *generalized twisted union* \((Z, r)\) of the solutions \((X, r_X)\) and \((Y, r_Y)\), is introduced in the class of symmetric sets.

**Definition 4.7.** A symmetric set \((Z, r)\) is a generalized twisted union of the disjoint symmetric sets \((X, r_X)\) and \((Y, r_Y)\) if it is an extension, and for every \(x \in X, \alpha \in Y\) the ground action \(\alpha^* \bullet : Y \times X \rightarrow X\) does not depend on \(x\), and the ground action \(\bullet^x : Y \times X \rightarrow Y\) does not depend on \(\alpha\).

**Remark 4.8.** Note that a strong twisted union \((Z, r)\) of \((X, r_X)\) and \((Y, r_Y)\) does not necessarily obey YBE, in contrast with twisted unions and generalized twisted unions. It follows straightforwardly from Definition 4.7 that a strong twisted union \((Z, r)\) which is symmetric set is a generalized twisted union of \((X, r_X)\) and \((Y, r_Y)\).

Furthermore, it is shown in [GI, Proposition 8.3] that an YB-extension \((Z, r)\) of two involutive square-free solutions \((X, r_X)\), \((Y, r_Y)\) is a generalized twisted union iff it is a strong twisted union. Proposition 4.9 now generalizes this result for arbitrary symmetric sets with lri, without necessarily assuming that the solutions are square-free.

**Proposition 4.9.** Suppose the symmetric set \((Z, r)\) has lri and is an extension of the solutions \((X, r_X)\), \((Y, r_Y)\). Then \((Z, r)\) is a generalized twisted union iff it is a strong twisted union.

**Proof.** By hypothesis \((Z, r)\) is an involutive solution with lri, thus Proposition 1.12 implies that \((Z, r)\) also satisfies the cyclic conditions, see Definition 1.8. Assume \((Z, r)\) is a generalized twisted union. Then by Definition 4.7 for every \(x \in X, \alpha \in Y\) the ground action \(\alpha^* \bullet : Y \times X \rightarrow X\) does not depend on \(x\), and the ground action \(\bullet^x : Y \times X \rightarrow Y\) does not depend on \(\alpha\). It follows then that for all \(x, y \in X, \alpha, \beta \in Y\) there are equalities:

\[
\alpha^x y = \alpha^y = \text{cycl} \alpha^y \quad \text{and} \quad \beta^x = \beta^x = \text{cycl} \beta^x.
\]

We have shown that conditions stu are satisfied, it follows then that \((Z, r) = (X, r_X) \bowtie (Y, r_Y)\). The converse implication is straightforward, see Remark 4.8. \(\square\)

Let \(\text{Ext}^2(X, Y)\) denote the set of strong twisted unions which obey the YBE.

**Proposition 4.10.**

1. A strong twisted union of solutions \((Z, r) = (X, r_X) \bowtie (Y, r_Y)\) obeys YBE iff
   (a) The assignment \(\alpha \rightarrow \alpha^* \bullet\) extends to a a group homomorphism
   \[
   \varphi : G(Y, r_Y) \rightarrow \text{Aut}(X, r);
   \]
   and
   (b) The assignment \(x \rightarrow \bullet^x\) extends to a a group homomorphism
   \[
   \psi : G(X, r_X) \rightarrow \text{Aut}(Y, r).
   \]
   In this case the pair of group homomorphisms \((\varphi, \psi)\) is uniquely determined by the pair of the ground actions, or equivalently by \(r\).
Furthermore, there is a one-to-one correspondence between the sets $\text{Ext}^2(X, Y)$ and $\text{Hom}(G(Y, r_Y), \text{Aut}(X, r_X)) \times \text{Hom}(G(X, r_X), \text{Aut}(Y, r_Y))$ using (1).

Proof. We use [GIM, Theorem 4.9] which breaks down the condition for any regular extension $(Z, r)$ (of which a strong twisted union is an example) to obey the YBE into explicit conditions $\text{ml1}, \text{ml2}, \text{mr1}, \text{mr2}$. Particularly, as equalities in $X \times X$

$$\text{ml2}: \quad \alpha r(x, y) = r(\alpha(x, y)) \quad \text{for all } x, y \in X, \alpha \in Y$$

(where the condition $\text{mr2}$ similar but for the right action and the roles of $X, Y$ swapped.) We can interpret this condition by computing both sides as

$$\alpha r(x, y) = \alpha(x y, \alpha^y(x y)) = \alpha(x y, \alpha(y)) = L_\alpha \times L_\alpha \circ r(x, y)$$

and

$$r(\alpha(x, y)) = r(\alpha x, \alpha^y y) = r(\alpha x, \alpha y) = r(L_\alpha \times L_\alpha)(x, y).$$

using our assumption $\text{stu}$. Thus the condition has the meaning

$$L_\alpha \times L_\alpha \circ r = r \circ (L_\alpha \times L_\alpha)$$

that is $L_\alpha|X \in \text{Aut}(X, r)$, for every $\alpha \in Y$ as in part (a). Similarly for $\text{mr2}$ as in part (b). By definition every strong twisted union obeys $\text{ml1}, \text{mr1}$ so the above are the only conditions for $(Z, r)$ obeys YBE. Note that this proof works for general strong twisted unions as defined in [GIM] not only the nondegenerate involutive case in Definition 4.3, as long as the same definition is used for $\text{Ext}^2$.

Lemma 4.11. Let $(Z, r)$ be a strong twisted union of $(X, r_X)$ and $(Y, r_Y)$ which is a symmetric set with $\text{li}$. Then the retraction $([Z], r_{[Z]})$ is a strong twisted union of the retractions $([X], r_{[X]})$ and $([Y], r_{[Y]})$.

Proof. By hypothesis $Z$ is a disjoint union of $X, Y$ thus $[Z]$ is a disjoint union of $[X], [Y]$. It follows from Lemma 3.4 and conditions $\text{stu}$ on $Z$ that

$$[\alpha]^{(\phi)} [x] = (3.1) \quad [\alpha^y] = \text{stu} \quad [\alpha] = (3.1) \quad [\alpha] [x].$$

This gives the left hand side part of $\text{stu}$, which together with $\text{li}$ implies the right hand side of $\text{stu}$. In view of Proposition 4.9 this could be said equally well in terms of generalised twisted unions; the present version can also be applied to general twisted unions obeying the YBE with $\text{li}$ provided the retracts make sense.

In the following example the strong twisted union $Z = X \sqcup Y$ satisfies $\text{mpl}Z = \max(\text{mpl}X, \text{mpl}Y)$.

Example 4.12. Let $Z = X \sqcup Y$, where $(X, r_X), (Y, r_Y)$, are the nondegenerate involutive square-free solutions defined by follows. $X = \{x, y, z\}, Y = \{\alpha, \beta, \gamma\}$, with

$$L_{X|X} = \text{id}_X, \quad L_{Y|X} = \text{id}_X, \quad L_{\alpha|Y} = (\beta \gamma), \quad L_{\beta|Y} = L_{\gamma|Y} = \text{id}_Y.$$

It is easy to see that $\text{mpl}X = \text{mpl}Y = 2$. Define

$$L_{X|Y} = (\beta \gamma), \quad L_{Y|Y} = L_{Z|Y} = \text{id}_Y, \quad L_{\alpha|X} = (yz), \quad L_{\beta|X} = L_{\gamma|X} = \text{id}_X.$$
and extend these to $Z$ (necessarily for any regular extension) by \( L_z = L_{z,X}L_{z,Y} \) where the restricted parts are considered to act trivially on the rest. So we have actions on $Z$

\[
L_x = L_\alpha = (yz)(\beta\gamma), \quad L_y = L_z = L_\beta = L_\gamma = id_x,
\]

which define a nondegenerate involutive square-free solution \((Z, r)\). It is a strong twisted union of \((X, r_X)\), and \((Y, r_Y)\). Clearly, \([x] = [\alpha], [y] = [z] = [\beta] = [\gamma]\), thus \([Z] = \{[x], [y]\}\) is a two element set, and \(([Z], [r])\) is the trivial solution. This gives \(mpl = 2 = mpl_X = mpl_Y\).

However, \((Z, r)\) can also be looked at as a strong twisted union of the trivial solutions \(X_0 = \{x, \alpha\}, Y_0 = \{y, z, \beta, \gamma\}\) with

\[
L_{x|Y_0} = L_\alpha|Y_0 = (yz)(\beta\gamma); \quad L_{y|X_0} = L_\beta|X_0 = L_\gamma|X_0 = id_X.
\]

This way we have again \(mpl = X_0\|Y_0\), but

\[
mp_{pl}(Z, r) = mp_{pl}(X_0) + 1 = mp_{pl}(Y_0) + 1.
\]

**Remark 4.13.** Theorem 5.24 in the next section shows that in the case when \(mp_{pl}(Z, r) = 2\), \(Z\) can always be presented as a strong twisted union of a finite number of solutions \(X_i, 1 \leq i \leq s\) where \(mp_{pl}(X_i) \leq 1\), and there exists an \(i\), such that \(mp_{pl}(X_i) \geq 1\). The case \(mp_{pl}(Z, r) \geq 3\) is more complicated. We show in Proposition 5.27 that every solution \((Z, r)\) with \(mp_{pl}(Z, r) = 3\) splits into \(r\)-invariant components \(X_i, 1 \leq i \leq s\), where \(X_i = V(\Gamma_i)\), \(mp_{pl}X_i \leq 2\), and for each pair \(i, j, 1 \leq i \leq s\), the subset \(X_{ij} = X_i \cup X_j\) is \(r\)-invariant and has presentation as \(X_{ij} = X_i \| X_j\). However, in order to understand the nature of solutions with higher multipermutation level and for their classification we should understand under what conditions \(\|\) associates or what replaces this. We believe that it is always possible to present \((Z, r)\) as a strong twisted union of components of strictly smaller multipermutation level.

We now give two more examples of strong twisted unions of solutions. In the first we have a strong twisted union \(Z = X_2Y\), with \(mp_{pl}X = 2, mp_{pl}Y = 1\) and \(mp_{pl}Z = 3\). \(Z\) is also split as a strong twisted union of three \(r\)-invariant components, each of multipermutation level 1, and the “associative law” holds. The second example, see Example 4.15, gives a solution \((Z, r)\) as an extension (but not necessarily a strong twisted union) of two \(r\)-invariant subsets. It illustrates Remark 4.13 with respect to \(\|\) not being nonassociative in general.

**Example 4.14.** Let \((X, r_X)\) be the nondegenerate involutive square-free solution defined by \(X = \{a, b, c, x_1, x_2, y_1, y_2, z_1, z_2\}\) and left action:

\[
(4.4) \quad L_{a|x} = L_{b|x} = L_{c|x} = (x_1 x_2)(y_1 y_2)(z_1 z_2),
\]

and \(L_{x_i|x} = L_{y_i|x} = L_{z_i|x} = id_x\) for \(i = 1, 2\).

Let \((Y, r_Y)\) be the trivial solution on the set \(Y = \{\alpha, \beta\}\). We extend the left actions on \(Z = X \cup Y\) as

\[
L_{\alpha} = L_{\beta} = (abc)(x_1 y_1 z_1 x_2 y_2 z_2), \quad L_{a|Y} = L_{b|Y} = L_{c|Y} = L_{y_1|Y} = L_{z_1|Y} = id_Y, i = 1, 2.
\]

This defines a left action of \(Z\) which we verify defines \((Z, r)\) as a nondegenerate involutive square-free solution. Clearly, \(mp_{pl}Y = 1\), and it is easy to see that \(mp_{pl}(X, r_X) = 2\). We leave the reader to verify that \(mp_{pl}(Z, r) = 3\).
The graph $\Gamma(X, r_X)$ has three nontrivial components and three one vertex components, as shown in Figure 3 part (b). Consider the presentation of $X = X_1 \cup X_2$, where $X_1 = \{x_1, x_2, y_1, y_2, z_1, z_2\}$, $X_2 = \{a, b, c\}$. Both are $r$-invariant sets and the restrictions $r_1 = r|_{X_1 \times X_1}$, $r_2 = r|_{X_2 \times X_2}$, are the trivial solutions as shown along with $Y$ in part (a) of the figure. Moreover, the actions given by (4.4) make $X$ a strong twisted union $X = X_1 \natural X_2$. This way we have

$Z = (X_1 \natural X_2) \natural Y = X_1 \natural (X_2 \natural Y) = X_2 \natural (X_1 \natural Y)$

mpl$X_1 = mplX_2 = mplY = 1 = mpl(Z) = 3$.

The graph $\Gamma(Z, r)$ is shown in part (c) of the figure.

**Example 4.15.** Let $X_3 = \{a\}$ be the one element solution $(r_3 = id_{X_3 \times X_3})$, and let $(X_1, r_1), (X_2, r_2)$ be the trivial solutions on the sets

$X_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \quad X_2 = \{a, b, c\}$

as depicted in Figure 4 (a). Consider the strong twisted union $(X, r_0) = X_1 \natural X_2$ with $r_0$ defined by the left actions

$L_b = (x_1 x_2)(x_3 x_4)(x_5 x_6)(x_7 x_8), \quad L_c = (x_1 x_5)(x_2 x_6)(x_3 x_7)(x_4 x_8)$

and the condition $r_{\text{i}}$. As usual, in this way we find a nondegenerate involutive square-free solution, as depicted in part (b). It is easy to see that $\text{Ret}(X, r_0)$ is the trivial solution on the set $[X] = \{[x_1], [b], [c]\}$, therefore $mpl(X, r_0) = 2$.

Consider now the set $Z = X \cup X_3$. Let

$L_a = (bc)(x_1 x_4)(x_2 x_3)(x_5 x_8)(x_6 x_7)$.

We leave the reader to verify that $L_a \in Aut(X, r_0)$. Next we define the extension $(Z, r) = X_2 \natural X_3$, with $r$ defined via $L_a$, the condition $r_{\text{i}}$ that we are extending $r_0, r_3$. Clearly, $(Z, r)$ is nondegenerate involutive square-free solution and $mpl(Z, r) = 3$.

This way we have a presentation which we denote $Z = (X_1 \natural X_2) \natural X_3 = (X_2 \natural X_1) \natural X_3$ as in part (c).
Consider now the set \( Y = X_1 \cup X_3 \), which is an \( r \)-invariant subset of \((Z,r)\). Let \( r_Y = r_{|Y \times Y} \), then \((Y, r_Y) = X_1 \sharp X_3\), \((r_Y\) can be defined also via the action \( \mathcal{L}_{\alpha|X_1} = (x_1 x_4)(x_2 x_3)(x_5 x_8)(x_6 x_7)\)). One has \( mpl(Y, r_Y) = 2 \). Clearly \((Z, r)\) as an extension of \((Y, r_Y)\) and \((X_2, r_2)\). However, this extension is not a strong twisted union, since neither \( \mathcal{L}_b \) nor \( \mathcal{L}_c \) is in \( Aut(Y, r_Y) \). Or equivalently, a direct verification shows that the pair \( Y, X_3 \) does not satisfy condition \( stu \), since

\[
 b^x x_1 = c^x x_1 = x_5 \neq b^x x_1 = x_2.
\]

Thus, while one also has \( Z = X_1 \sharp (X_3 \sharp X_2) \) by similar computations, the parentheses do not exactly associate with \( \sharp \); we have shown that \((Z, r)\) is a YB-extension of the nondegenerate square-free involutive solutions \( Y = X_1 \sharp X_3 \) and \( X_2 \) as desired but this extension is not a strong twisted union.

**Theorem 4.16.** Let \((X, r_X)\) and \((Y, r_Y)\), be square-free multipermutation solutions, with \( mpl(X) = m, mpl(Y) = k \). Suppose \((Z, r) = X_1 \sharp Y \) is a nondegenerate symmetric set. Then

\[
 mpl(Z, r) \leq max\{m, k\} + 1.
\]

**Proof.** Without loss of generality we can assume \( k \leq m \). We shall prove (4.5) by induction on \( m \). For the base for the induction, \( m = 1 \), we have to show that \( mpl(Z, r) \leq 2 \), which by Lemma 4.5 is equivalent to \( mpl(Ret(Z, r)) = 1 \). By hypothesis \( stu \) holds on \( Z \), so

\[
 x^\alpha x = x^\alpha \beta = x^\beta \beta, \text{ for every } x, \alpha, \beta \in Y,
\]

or equivalently,

\[
 L_{\alpha y} | X = L_{\alpha | X} = L_{\alpha y} | X; \ L_{x^\alpha y} | Y = L_{x^\alpha y} | Y = L_{x^\beta y} | Y, \text{ for all } x, y \in X, \alpha, \beta \in Y.
\]
Clearly, \( m_{\text{pl}} \geq L \) (4.7) Ret (4.9)

Now from (4.6), (4.7), and (4.8) we deduce the equalities

\[ L_x = L_x \circ L_x \circ id_X = L_x \circ L_x \circ L_x = L_x, \quad \forall x \in X. \]

Analogously,

\[ L_{\alpha|x} = L_{\alpha|x}, \quad \forall \alpha \in Y. \]

Now from (4.6), (4.7), and (4.8) we deduce the equalities

\[ L_{\alpha|Z} = L_{\alpha|Z} = L_{\alpha|Z}; \quad L_{x|x} = L_{x|x} = L_{x|x}, \quad \text{for all } x, y \in X, \alpha, \beta \in Y. \]

This, together with Corollary 3.10 imply that either i) \( \text{Ret}(Z, r) \) is a trivial solution of order \( \geq 2 \), hence \( m_{\text{pl}}(Z, r) = 2 \); or ii) \( \text{Ret}(Z, r) \) is one element solution so \( m_{\text{pl}}(Z, r) = 1 \). We have verified (4.5) for \( m = 1 \). Assume now (4.5) holds for all \( m \leq m_0 \). Suppose \( m_{\text{pl}}(X) = m_0 + 1 \), thus by Lemma 4.5, \( m_{\text{pl}}([X, r_{|X}]) = m_0 \). Clearly, \( m_{\text{pl}}([Y, r_{|Y}]) = m_{\text{pl}}([Y, r_{|Y}]) - 1 \leq m_0 \). By Lemma 4.11 the retract \( ([Z, r_{|Z}] \times [X]) \) is a strong twisted union \( ([Z, r_{|Z}] \times [X]) \), hence by the inductive assumption one has \( m_{\text{pl}}([Z, r_{|Z}]) \leq m_0 + 1 \). This and the equality \( m_{\text{pl}}(Z, r) = m_{\text{pl}}([Z, r_{|Z}]) + 1 \) implies \( m_{\text{pl}}(Z, r) \leq m_0 + 2 \). It follows then that (4.5) holds for all \( m, m \geq 1 \), which proves the theorem.

The following lemma is straightforward.

**Lemma 4.17.** Let \((X_1, r_1), (X_2, r_2)\) be disjoint solutions with \( m_{\text{pl}}(X_1, r_1) = m_1, m_{\text{pl}}(X_2, r_2) = m_2 \), let \( m = \max\{m_1, m_2\} \). Suppose \((Z, r) = X_1 \times X_2\) is a symmetric set. Then \( m_{\text{pl}} Z = m + 1 \) if and only if for some \( i, 1 \leq i \leq 2 \) with \( m_i = m \) there are \( x \in X_i \), and \( \alpha \in Z \backslash X_i \) such that the orbit \( O_{m-1}(x) \) does not contain \( \alpha x \).

Lemma 4.17 has a clear interpretation in terms of the graphs \( \Gamma(X_i, r_i), i = 1, 2, \Gamma(Z, r) \), see Definitions 5.1, 5.3. In the case discussed by the lemma, \( L_{\alpha|X} \) acts as an automorphism of \((X_i, r_i)\) which maps the connected component \( \Gamma_x \) of \( \Gamma(X_i, r_i) \) onto a different isomorphic component \( \Gamma_{\alpha x} \) of \( \Gamma(X_i, r_i) \). Clearly this is possible only in the case when \( \Gamma(X_i, r_i) \) has at least two connected components which are isomorphic as graphs. The last is a necessary but not a sufficient condition.

### 5. Graphs of symmetric sets with \( \text{lri} \)

Each finite involutive solution \((X, r)\) with \( \text{lri} \) can be represented geometrically by its graph of the left action \( \Gamma(X, r) \). It is an oriented labeled multi-graph (although we refer to it as a graph). It was introduced in [GI00] for square-free solutions, see also [GIM]. Here we recall the definition.

**Definition 5.1.** [GIM] Let \((X, r)\) be a finite symmetric set with \( \text{lri} \), we define the (complete) graph \( \Gamma = \Gamma(X, r) \) as follows. It is an oriented graph, which reflects the left action of \( G(X, r) \) on \( X \). The set of vertices of \( \Gamma \) is exactly \( X \). There is a labeled arrow \( x \xrightarrow{\alpha} y \), if \( x, y, \alpha \in X \) and \( \alpha x = y \). An edge \( x \xrightarrow{\alpha} y \), with \( x \neq y \) is called a nontrivial edge. We will often consider the simplified graph in which to avoid clutter we typically omit self-loops unless needed for clarity or contrast. Also for the same reason, we use the line type to indicate when the same type of element acts, rather than labeling every arrow. Clearly, \( x \xleftarrow{\alpha} y \) indicates that \( \alpha x = y \) and \( \alpha y = x \). (One can make such graphs for arbitrary solutions but then it should be indicated which action is considered).
Note that two solutions are isomorphic if and only if their (complete) oriented graphs are isomorphic. Various properties of a solution \((X, r)\) are reflected in the properties of its graph \(\Gamma(X, r)\), see for example the remark below, Proposition 5.6, Theorem 5.22.

**Remark 5.2.** Let \((Z, r)\) be a symmetric set with \(\text{lri}\), \(\Gamma = \Gamma(Z, r)\).

1. \((Z, r)\) is a square-free solution iff \(\Gamma\) does not contain a nontrivial edge \(x \to y, x \neq y\).
2. In this case, \((Z, r)\) is a trivial solution, or equivalently, \(\text{mpl}(Z, r) = 1\) iff \(\Gamma\) does not contain nontrivial edges \(x \to y, x \neq y\).

By Proposition 1.12 our assumptions that \((X, r)\) is a symmetric set with \(\text{lri}\) imply the cyclic conditions (without necessarily assuming \((X, r)\) square-free). We will need the assumption \((X, r)\) square-free any time when we claim \(x \to x\), for every \(x \in X\).

Examples of graphs of square-free solutions were already given in Section 2, see Examples 2.5, and 2.8, as well as in Section 4.

We will find now various properties of \(\Gamma = \Gamma(X, r)\).

**Notation 5.3.** Suppose \(\Gamma_0\) is a subgraph of \(\Gamma\). Denote by \(V(\Gamma_0)\) the set of all vertices of \(\Gamma_0\), \(E(\Gamma_0)\) denotes the set of all labels of (nontrivial) edges that occur in \(\Gamma_0\), i.e.

\[
E(\Gamma_0) = \{a \in X \mid \exists \text{ an edge } x \to y \subset \Gamma_0, x \neq y\}.
\]

Clearly, each \(x \in X\) determines uniquely a connected component of \(\Gamma\) which contains \(x\) as a vertex, we shall denote it by \(\Gamma_x\).

Let \(a, x \in X\). Suppose \(a x \neq x\). Then the orbit of \(x\) under the left action of the cyclic group \(\langle a \rangle\) (or equivalently under the action of \(\mathcal{L}_a\)) on \(X\) is a cycle \((x_1 x_2 \cdots x_m)\) of length \(m \geq 2\) in the symmetric group \(\text{Sym}(X)\), where for symmetry we set \(x_1 = x\). One has \(a x_i = x_{i+1}, 1 \leq i \leq m-1, a x_m = x_1\). This cycle participates in the presentation of the permutation \(\mathcal{L}_a \in \text{Sym}(X)\) as a product of disjoint cycles. Clearly, \(x_2 \cdots x_m \in \Gamma_x\).

**Notation 5.4.** For \(x, a\) as above we use notation \(\mathcal{L}_a^x = (x_1 \cdots x_m) = \mathcal{L}_a^{x_i}, 1 \leq i \leq m\).

**Conventions 5.5.** Till the end of the section we shall consider only finite square-free nondegenerate symmetric sets \((X, r)\). We recall from Theorem 1.13 that such solutions satisfy both \(\text{lri}\) and the cyclic conditions. In particular, one has

\[
(5.1) \quad \mathcal{L}_{a^e}^x = \mathcal{L}_a^x = \mathcal{L}_{(a^e)}^x.
\]

Moreover, it is known that for each such square-free solution \((X, r)\) with \(1 < |X| < \infty\), its YB group \(G(X, r)\) acts nontransitively on \(X\), see [GI, R], and therefore the graph \(\Gamma(X, r)\) has at least two connected components.

**Proposition 5.6.** Let \((X, r)\) be a finite nondegenerate involutive square-free solution, \(G = G(X, r)\), be its YB group, and \(\Gamma = \Gamma(X, r)\) be its graph. Suppose \(\Gamma_1, \Gamma_2, \cdots, \Gamma_s\) is the set of all connected components, with sets of vertices, respectively, \(X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s\).

1. Each set \(X_i, 1 \leq i \leq s\), is precisely an orbit of the left action of the group 
   \(G\) on \(X\).
(2) $X_i$ is $r$-invariant, so $(X_i, r_i)$, with $r_i = r_i|_{X_i \times X_i}$, is a nondegenerate involutive square-free solution. Its graph $\Gamma(X_i, r_i)$ is the subgraph of $\Gamma$ obtained by erasing the edges labeled with elements $a \in \mathcal{E}(\Gamma_1) \setminus X_i$.

(3) Furthermore, if $mpl(X, r) = m$, then $mpl(X_i, r_i) \leq m - 1$, for each $i, 1 \leq i \leq s$. More precisely, in this case, $x \in X_i$ implies $X_i \subseteq O(x, m - 1)$.

Proof. Sketch of the proof. Condition (1) follows straightforwardly from the definition of the graph $\Gamma(X, r)$. The definition of $\Gamma(X, r)$ also implies that the set of vertices $X_i$ of each connected component is invariant under the left action. By $lri$ it is also invariant with respect to the right action, and therefore it is $r$-invariant. We shall prove (3). By 3.8 one has $mpl(\text{Ret}^{m-1}(X, r)) = 1$. $\text{Ret}^{m-1}(X, r)$ inherits the property of being square-free, so by Lemma 3.10, $\text{Ret}^{m-1}(X, r)$ is a trivial solution.

In terms of the left action this is equivalent to

$$y x \in O(x, m - 1) \text{ for all } x, y \in X.$$

It follows then that the orbit of $x$, $X_i = O_G(x)$ is contained in the $(m - 1)$st retract orbit $O(x, m - 1)$, and Remark 3.12, implies $mpl(X_i) \leq m - 1$. □

Definition 5.7. Let $a \in X$, and let $\Gamma_1$ be a connected component of $\Gamma$. We shall use notation $\mathcal{L}_a|\Gamma_1$ for the restriction of the action $\mathcal{L}_a$ on the set of vertices $V(\Gamma_1)$.

We say that the (left) actions of $a$ and $b$ commute on $\Gamma_1$ if the following equalities hold:

$$\mathcal{L}_{(a)\Gamma_1} = \mathcal{L}_{a|\Gamma_1}, \quad \mathcal{L}_{(b)\Gamma_1} = \mathcal{L}_{b|\Gamma_1}. \quad (5.2)$$

In this case we also say that each two edges of $\Gamma_1$ labeled by $a$ and $b$ commute.

The graph $\Gamma(X, r)$ and the combinatorial properties of the solution $(X, r)$ can be used as a “mini-computer” to deduce missing relation of $(X, r)$. For example, see the proofs of Lemmas 5.8, 5.11, 5.13 and Example 5.9.

The following straightforward lemma describes the “cells” from which we build $\Gamma(X, r)$ in the general case of $(X, r)$ with $lri$.

Lemma 5.8. Let $(X, r)$ be a symmetric set with $lri$.

(1) For each $x_1, a, b \in X, a \neq b$, there exists a uniquely determined subgraph $\Gamma_0$ of $\Gamma$:

$$\xymatrix{ x_1 \ar[r]^a \ar[d]^b & x_2 \\ x_3 \ar[r]^{b_a} & x_4 } \quad (5.3)$$

where the vertices $x_1, x_2, x_3, x_4 \in X$ are not necessarily pairwise distinct.

(2) Given $a, b$, each vertex $x_i$, of $\Gamma_0$ in (5.3) determines the remaining three vertices $x_j, 1 \leq j \leq 4, j \neq i$.

(3) In the notation of (1) let $\Gamma_{x_1}$ be the connected component which contains $x_1$ as a vertex. Suppose that the left actions of $a$ and $b$ commute on $\Gamma_{x_1}$,
see Definition 5.7. Then the subgraph $\Gamma_0$ has the shape
\begin{equation}
\begin{array}{c}
\begin{array}{c}
x_1 \quad \rightarrow \quad x_2 \\
\downarrow \quad \downarrow \\
x_3 \quad \rightarrow \quad x_4
\end{array}
\end{array}
\end{equation}
and the following implications hold:
\begin{equation}
\begin{array}{c}
x_1 \neq x_2 \iff x_3 \neq x_4 \\
x_1 \neq x_3 \iff x_2 \neq x_4.
\end{array}
\end{equation}

Proof. (1) is clear. Note that given $\xi, a \in X$ there exists unique $x \in X$, such that $^ax = \xi$. Indeed, by $\text{lri}$, $x = \xi^a$. One uses this and $\text{lri}$ again to deduce (2). Now we prove the first implication of (3). It follows from the diagram (5.4) that one has $x_3 = b x_1, x_4 = a x_1 = b x_2$, so by $\text{lri}$ $x_3 b = x_1, x_4 b = x_2$. Clearly this implies $x_1 = x_2 \iff x_3 = x_4$.

Analogous argument proves the second implication. \hfill \square

Note that the diagram (5.3) is just a formal graphic expression of condition $\text{II}$, so it always presents a subgraph of $\Gamma(X, r)$, although in the case when some vertices coincide it is deformed to a segment or to a single vertex with loops. It is convenient (and mathematically correct) for computational purposes to use sometimes diagrams of the shape of squares (as in (5.3)) even when some vertices coincide. See for example Corollary 5.12.

**Example 5.9.** Let $(Z, r)$ be a nondegenerate involutive square-free solution with at least 7 elements. Suppose where $a, b, c, x_1, x_2, x_3 \in Z$, and
\begin{equation}
\begin{array}{c}
r(a, b) = (c, a), \\
r(a, c) = (b, a), \\
r(a, x_1) = (x_2, a) \\
r(a, x_2) = (x_1, a) \\
r(a, x_3) = (x_3, a) \\
r(b, c) = (c, b) \\
r(b, x_1) = (x_3, b)
\end{array}
\end{equation}
Then $r(c, x_2) = (x_3, c)$ and there exists some $x_4 \in X$, distinct from $a, b, c, x_1, x_2, x_3$, such that
\begin{equation}
\begin{array}{c}
r(c, x_1) = (x_4, c) \\
r(b, x_2) = (x_4, b) \\
r(a, x_4) = (x_4, a)
\end{array}
\end{equation}
This is indicated in terms of the graph in Figure 5. Furthermore, if $(Z, r)$ has exactly 7 elements then (5.6) determines uniquely the solution $(Z, r)$. It is not
difficult to see that in addition to (5.6), (5.7), \((Z,r)\) satisfies \(r(x_i, x_j) = (x_j, x_i)\),
for all \(1 \leq i, j \leq 4\). For the left actions we have:
\[ L_{x_i} = id_{Z}, 1 \leq i \leq 4, \quad L_{b} = (x_1x_3)(x_2x_4), \quad L_{c} = (x_1x_4)(x_2x_3), \quad L_{a} = (bc)(x_1x_2). \]

So the retracts of \((Z, r)\) are:
\[ \text{Ret}^1(Z, r) = \{[x_1], [a], [b], [c]\}, \quad \text{L}_a = ([b][c]), \quad \text{L}_b = \text{L}_{[c]} = \text{L}_{[x_1]} = id_{[Z]} \]
\[ \text{Ret}^2(Z, r) = \{[a]^{(2)}, [b]^{(2)}\}, \quad \text{the trivial solution} \quad \text{Ret}^3(Z, r) = \{[a]^{(3)}\}. \]

It follows then that \(mpl(Z, r) = 3\).

**Example 5.10.** The solution from the previous example can be constructed as a strong twisted union of disjoint solutions of lower multipermutation levels. Consider the trivial solutions \((X_1, r_1), (X_2, r_2)\), where
\[ X_1 = \{x_1, x_2, x_3, x_4\}, \quad X_2 = \{b, c\}. \]

Let \((X, r_X) = X_1 \oplus X_2\) be the strong twisted union defined via the actions
\[ L_b = (x_1x_3)(x_2x_4), \quad L_c = (x_1x_4)(x_2x_3), \quad L_a = id_{X}. \]

We leave the reader to verify that \((X, r_X)\) is a solution. The graph \(\Gamma(X, r_X)\) has three connected components: two one vertex components, \(b\) and \(c\), and \(\Gamma_1\), with \(\psi(\Gamma_1) = X_1, \psi(\Gamma_1) = X_2\). Moreover, \(\Gamma_1\) is a ‘graph of first type’, see Definition 5.17 below. One has \(mpl(X, r_X) = 2\).

Let \((Y, r_Y)\) be the one element solution, with \(Y = \{a\}\). We will build an extension \((Z, r) = X_1 \oplus Y\). Since \(Y\) is one element solution, the left action on \(Y\) on \(X\) has to be via an automorphism. Consider the permutation \(\tau = (bc)(x_1x_2) \in Sym(X)\). It is easy to see that this is an automorphism of \((X, r_X)\). Now one defines strong twisted union \((Z, r) = X_1 \oplus Y\) via the action \(L_a = (bc)(x_1x_2)\). We obtain again the solution of Example 5.10. Now one has \(mpl(Z, r) = 3 = \max\{mpl(X), mpl(Y)\} + 1\).

**Lemma 5.11.** Let \(x, a, b \in X, L^x_a = (x_1x_2 \cdots x_k)\), where \(x_1 = x, k \geq 2\), and let \(b^x = x_{21} \neq x\). Suppose that the left actions of \(a\) and \(b\) commute on \(\Gamma_x\). Then

1. \(L^x_a^{(b^x)} = (x_1x_2x_2 \cdots x_{2k})\) is a cycle of length exactly \(k\).
2. If \(b^x = x_{m+1}\) for some \(m, 1 \leq m \leq k-1\), then \(L^{x} = L^{x}_a\), and \(L^{x}_b = (L^{x}_a)^m\).
3. If part (2) does not apply then \(L^{x}_a\) and \(L^{x}_a\) are two disjoint cycles.

**Proof.** We apply Lemma 5.8 to the commuting edges
\[ x_1 \xrightarrow{a} x_2 \]
\[ \downarrow b \]
\[ x_21 \]
and obtain the “cell” subgraph of \(\Gamma\):
(5.8)
\[ x_1 \xrightarrow{a} x_2 \]
\[ \downarrow b \]
\[ x_21 \xrightarrow{a} x_{22}. \]
Now we use this method to recursively build the diagram

(5.9) \[ \begin{array}{cccccccc}
  x_1 & \xrightarrow{a} & x_2 & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_k & \xrightarrow{a} & x_1 \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  x_{21} & \xrightarrow{a} & x_{22} & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{2k} & \xrightarrow{a} & x_{21} \xrightarrow{a} \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots
\end{array} \]

It is clear from the first two rows of the diagram that the cycle $\mathcal{L}_{a}^{x_{21}}$ (represented by the second row) has length at most $k$. Assume that its length $q$ is strictly less than $k$ then we obtain from (5.9)

(5.10) \[ \begin{array}{cccccccc}
  x_1 & \xrightarrow{a} & x_2 & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_q & \xrightarrow{a} & x_{q+1} \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  x_{21} & \xrightarrow{a} & x_{22} & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{2q} & \xrightarrow{a} & x_{21} \xrightarrow{a} \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots
\end{array} \]

which implies $b x_1 = x_{21}$ and $b x_{q+1} = x_{21}$. The last is impossible in view of the nondegeneracy of $(X, r)$ and $x_1 \neq x_{q+1}$. We have shown that $\mathcal{L}_{a}^{b x}$ is a cycle of length exactly $k$.

Assume now $b x = x_{21} = x_{m+1}$, for some $m \geq 1$. Then we deduce from (5.10) the following diagram:

\[ \begin{array}{cccccccc}
  x_1 & \xrightarrow{a} & x_2 & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{m+1} & \xrightarrow{a} & x_k & \xrightarrow{a} & x_1 \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  x_{m+1} & \xrightarrow{a} & x_{m+2} & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{2m+1} & \xrightarrow{a} & x_{m+k} & \xrightarrow{a} & x_{m+1} \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  x_{2m+1} & \xrightarrow{a} & x_{2m+2} & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{3m+1} & \xrightarrow{a} & x_{2m+k} & \xrightarrow{a} & x_{2m+1} \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  x_{sm+1} & \xrightarrow{a} & x_{sm+2} & \xrightarrow{a} & \cdots & \xrightarrow{a} & x_{sm+k} & \xrightarrow{a} & x_{sm+1} \xrightarrow{a} \cdots \\
  b & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots & \downarrow & b & \downarrow & \cdots \\
  \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots
\end{array} \]

Looking at the first column of the diagram we see that $\mathcal{L}_{a}^{b} = (\mathcal{L}_{a}^{x})^{m}$. It follows from (5.1) that $\mathcal{L}_{a}^{b x} = \mathcal{L}_{a}^{x_{m+1}} = \mathcal{L}_{a}^{x}$. This proves (1). The rest follows from (1).
Corollary 5.12. Let $x, a, b \in X$ and suppose that the left actions of $a$ and $b$ commute on $\Gamma_x$.

1. If $a^x = b^x$ then $L^x_a = L^x_b$.
2. If $b^{(a^x)} = x$ then $L^x_b = (L^x_a)^{-1}$.

Proof. There is nothing to prove if $a^x = x$. So suppose $a^x = x \neq b^x$. Let $L^x_a = (x_1 x_2 \cdots x_k), k \geq 2$, where, as usual, we set $x_1 = x$. Suppose $a^x = b^x$. Then

$$
\begin{array}{c}
  x_1 \\
  \downarrow^a \\
  x_2 \\
  \downarrow^b \\
  x_1
\end{array}
$$

is a subgraph of $\Gamma$. The hypothesis of Lemma 5.11 part (2) is satisfied (here $m = 1$). Therefore $L^x_b = L^x_a$. This proves part (1). Assume now $b^{(a^x)} = x$. This implies that

$$
\begin{array}{c}
  x_1 \\
  \downarrow^a \\
  x_2 \\
  \downarrow^b \\
  x_1
\end{array}
$$

is a subgraph of $\Gamma$. It follows then by Lemma 5.8 that $\Gamma$ contains also the following subgraph:

$$
\begin{array}{c}
  x_1 \\
  \downarrow^a \\
  x_2 \\
  \downarrow^b \\
  \downarrow^b \\
  x_1
\end{array}
$$

Apply Lemma 5.11 part (2) again to obtain $L^x_b = (L^x_a)^k = (L^x_a)^{-1}$. (In the notation of the Lemma, this time $m = k - 1$).

Clearly, in the case when $L^x_a = (x_1 x_2)$ the diagram in the proof of part (2) of the corollary becomes simply

$$
\begin{array}{c}
  x_1 \\
  \downarrow^a \\
  x_2 \\
  \downarrow^b \\
  x_2
\end{array}
$$

which in the real graph $\Gamma$ deforms to $x_1 \overset{a,b}{\rightarrow} x_2$.

Lemma 5.13. Let $a, b, x \in X$, let $L^x_a = (x_1 x_2 \cdots x_k)$, $L^x_b = (x_1 y_2 \cdots y_k)$, where $x = x_1$ and $s \leq k$. Suppose that the left actions of $a$ and $b$ commute on $\Gamma_x$ and suppose that the cycles $L^x_a$ and $L^x_b$ have at least two common elements, i.e. $x_i = y_j$ for some $i, j \neq 1$. Then $L^x_b = (L^x_a)^m$, where $1 \leq m \leq k - 1$.

Proof. We claim that $b^x \in \{x_2 \cdots x_k\}$. Assume the contrary, then we have $L^x_b = (x_1 y_2 \cdots y_{p+1} \cdots)$, where $p \geq 1$ and $y_{21}, \cdots, y_{p+1}$ are distinct from $x_2, \cdots, x_k$. 


We want to deduce the vertices of the following subgraph of $\Gamma$:

\[x_1 \to a \to x_2 \to a \to \ldots \to a \to x_k \to a \to x_1 \to a \to \ldots\]

\[y_1 \to b \to y_2 \to b \to \ldots \to b \to y_k \to b \to y_1 \to b \to \ldots\]

\[\ldots \]

\[x_{m+1} \to a \to x_{m+2} \to a \to \ldots \to a \to x_{m+k} \to a \to x_{m+1} \to a \to \ldots\]

So starting with $i = 2$, we apply Lemma 5.11 to successively deduce the vertices of $i$-th row, $i = 2, 3, \ldots$. This way we obtain

\[x_1 \to a \to x_2 \to a \to \ldots \to a \to x_k \to a \to x_1 \to a \to \ldots\]

Note first that each row of (5.12) represents a cycle of length exactly $k$. Compare $L_a^x$ with $L_b^x$ (represented by the first column of (5.12)). The hypothesis of Lemma 5.11 is satisfied but this time we have that $L_a^x = (L_b^x)^{qp}$, where $qp \equiv 2 (mod k)$. Thus the lengths of the two cycles $k$ and $s$ satisfy $k \leq s$. An equality is impossible, since by our assumption the entry $y_{21}$ of $L_b^x$ does not occur in $L_a^x$. This yields $k < s$, a contradiction with the hypothesis. It follows then that $b_x = x_{m+1}$ for some $m \geq 1$ as we claimed. Now Lemma 5.11 again implies $L_b^x = (L_a^x)^m$. \qed
Definition 5.14. Let \((X, r)\) be a nondegenerate involutive square-free solution of the YBE, and \(x \in X\). The set \(\text{Star}(x)\) is defined as
\[
\text{Star}(x) = \{L_x^a \mid a \in X\}.
\]

Definition 5.15. By \(\Gamma(x)\) we denote the subgraph of (the complete graph) \(\Gamma(X, r)\) with a set of vertices consisting of all \(x_j\) that occur in the cycles of \(\text{Star}(x)\), and all edges inherited from \(\Gamma(X, r)\).

On the set of stars we introduce an equivalence relation \(\approx\) defined as \(\text{Star}(x) \approx \text{Star}(y)\) iff the complete graphs \(\Gamma(x)\) and \(\Gamma(y)\) are isomorphic (as labeled graphs).

As before, when we draw the graph \(\Gamma(x)\) we shall often present it as a simplified graph in which if \(L_x^a = L_x^b\), for \(a \neq b \in X\), we only draw it once. In various cases it is convenient to use even more schematic graph in which for each \(L_x^a = (x_1 x_2 \cdots x_k) \in \text{Star}(x)\) with \(x_1 = x\) we draw
\[
\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet
\]
where each variable occurring in \(L_x^a\) is represented by a \(\bullet\) (the last corresponds to \(x_k\)).

Example 5.16. Consider the nondegenerate square-free involutive solution \((X, r)\) defined on the set
\[
X = \{x_1, x_2, x_3, a, b, c\},
\]
via the left actions
\[
L_{x_1} = L_{x_2} = L_{x_3} = \text{id}_X, \quad L_b = (x_1 x_2 x_3), \quad L_c = (x_1 x_3 x_2), \quad L_a = (bc)(x_2 x_3),
\]
The graph \(\Gamma(X, r)\) is presented in figure 6 (a). The graphs of the Stars of all elements of \(X\) are represented in part (b).

We introduce now a particular type of subgraphs \(\Gamma_1\) we call them \textit{graphs of first type}.

Definition 5.17. In our usual assumption and notation, let \(\Gamma_1\) be a connected component of \(\Gamma = \Gamma(X, r), X_1 = V(\Gamma_1)\) be its set of vertices. \(\Gamma_1\) is a \textit{graph of first type} if it contains a nontrivial edge and for each \(a, b \in E(\Gamma_1)\) the left actions of \(a\) and \(b\) commute on \(\Gamma_1\). In other words, if
\[
L_{(a)\mid X_1} = L_{a\mid X_1}, \quad \forall a, b \in E(\Gamma_1).
\]
Next we define
\[(5.13) \quad G_1 = G(\Gamma_1) := gr(\mathcal{L}_a|_{X_1}; a \in E(\Gamma_1)).\]

**Lemma 5.18.** Let \( \Gamma_1 \) be a nontrivial connected component of \( \Gamma(X, r) \) with a set of labels \( E_1 = E(\Gamma_1) \). Then \( \Gamma_1 \) is a graph of first type iff
\[\mathcal{L}_{(a^x)}|_{\Gamma_1} = \mathcal{L}_a|_{\Gamma_1} \quad \forall a, x \in E_0.\]

**Proof.** The lemma follows from Definition 5.17, the cyclic conditions and \textbf{iir}. \( \square \)

Note that since \((X, r)\) is a solution, condition \textbf{iir} holds and can be written as
\[\mathcal{L}_a \circ \mathcal{L}_b = \mathcal{L}_{a \circ b} \quad \forall a, b \in X.\]

It follows then from Definition 5.17 that the group \( G(\Gamma_1) \) is an abelian subgroup of \( Sym(X_1) \). We recall the following fact about finite abelian groups.

**Fact 5.19.** **Basis Theorem** [Sc, Ch. 2] Let \( G \) be a finitely generated abelian group. If \( s \) is the least integer such that \( G \) is generated by \( s \) elements, then \( G \) is the direct sum (in abelian notation) of \( s \) cyclic subgroups.

**Remark 5.20.** Note that, in general, the presentation of a finite abelian group \( G \) as a direct product of cyclic subgroups is not an invariant of the group, it becomes invariant if one considers only direct products of cyclic subgroups of prime power order. (For example one has \( G = \mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3 \), here \( s(G) = 1 \).

However, the least number \( s = s(G) \) of a generating set of a finite group is well determined, and we shall use it to introduce the notion of \textit{basis of labels} for \( \Gamma_1 \).

It follows by Fact 5.19, that the group \( G_1 = G(\Gamma_1) \) is the direct product of cyclic subgroups:
\[(5.14) \quad G_1 = (\mathcal{L}_{a_1}|_{X_1}) \times (\mathcal{L}_{a_2}|_{X_1}) \times \cdots \times (\mathcal{L}_{a_s}|_{X_1}),\]
where \( s = s(\Gamma_1) \) is the least integer such that \( G_1 \) is generated by \( s \) elements. Note that \( s \) is also an invariant of \( \Gamma_1 \).

**Definition 5.21.** A set of labels \( \mathbb{B} = \{a_1, \ldots, a_s\} \subseteq E(\Gamma_1) \) is a \textit{basis of labels} for \( \Gamma_1 \) if \((5.14)\) holds and \( s \) is the least number of a generating set of \( G_1 \).

For computations it is convenient to use the arrows in \( \Gamma_1 \) labeled by \( a_i, 1 \leq i \leq s \) as “axes” of an \( s \)-dimensional discrete ”coordinate system”. Let \( k_i = k(a_i) \) as in Theorem 5.22 part (4). We can chose an arbitrary vertex \( x \) of \( \Gamma_1 \) as an “origin”. It has coordinates \( x = (0 \cdots 0) \). Then each vertex \( y \in X_1 \) has \( s \) “coordinates” which we define as follows.
\[\mathcal{L}_{a_1}(00 \cdots 0) = (10 \cdots 0), \quad \mathcal{L}_{a_2}(00 \cdots 0) = (01 \cdots 0), \quad \cdots, \quad \mathcal{L}_{a_s}(00 \cdots 0) = (00 \cdots 1).\]

Clearly, every vertex \( y \) of \( \Gamma_1 \) can be obtained by applying a finite sequence of left actions:
\[(5.15) \quad y = (\mathcal{L}_{a_1})^{m_1} \circ (\mathcal{L}_{a_2})^{m_2} \circ \cdots \circ (\mathcal{L}_{a_s})^{m_s}(0 \cdots 0) = (m_1m_2 \cdots m_s),\]
where \( 0 \leq m_i \leq k_i \).

We will now give a detailed description of the shape of a graph of first type.
Theorem 5.22. Let \((X, r)\) be a finite nondegenerate square-free symmetric set and suppose the connected component \(\Gamma_1\) of \(\Gamma(X, r)\) is a graph of first type. Let \(X_1 = \mathcal{V}(\Gamma_1)\), and \(G_1 = \mathcal{G}(\Gamma_1)\), as in (5.13). Then the following conditions hold.

1. If the nontrivial edge \(x \xrightarrow{a} y, x \neq y\) occurs in \(\Gamma_1\), then for every vertex \(z\) in \(\mathcal{V}(\Gamma_1)\) there is an edge \(z \xrightarrow{a^{-1}} t, z \neq t\) in \(\mathcal{V}(\Gamma_1)\).
2. \(X_1\) is \(r\)-invariant, and \((X_1, r_1)\) with \(r_1 = r|_{X_1 \times X_1}\) is a trivial solution.
3. Each \(a \in X \) acts on \(X_1\) as an automorphism, and \(G_1 = \mathcal{G}(\Gamma_1)\) is a normal subgroup of \(\text{Aut}(X_1, r_1)\).
4. For every \(a \in \mathcal{E}(\Gamma_1)\), there exists an integer \(k = k(a, \Gamma_1)\), which is an invariant of \(a\) and \(\Gamma_1\) such that for every vertex \(x \in X_1\) the cycle \(\mathcal{L}_a^k\) is of length \(k\). Moreover
   \[
   \mathcal{L}_{a|}X_1 = (x_1 \cdots x_k)(y_1 \cdots y_k) \cdots (z_1 \cdots z_k),
   \]
   where each vertex of \(\Gamma_1\) occurs exactly once and the product contains all disjoint cycles with edges labeled \(a\).
5. All vertices \(x_1, \cdots, x_N \in X_1\) have equivalent stars, \(\text{Star}(x_1) \approx \text{Star}(x_i), 1 \leq i \leq N\).
6. Let \(\mathcal{B} = \{a_1, \cdots, a_s\}\) be a basis of labels for \(\Gamma_1\), and let \(k_i = k(a_i, \Gamma_1), 1 \leq i \leq s\), as in 4. Then \(\Gamma_1\) is a "multi-dimensional cube" of dimension \(s\), and the order of \(X_1\) (i.e. the size of the cube) is exactly the product \(N = k_1k_2 \cdots k_s\).

Proof. We start with part (1). By hypothesis \(\Gamma_1\) is a connected component of \(\Gamma\), and all its edges commute. Then it follows from Lemma 5.8 that for any "neighbour" \(z\) of \(x\), \((z = y\) is also possible) with

\[
\begin{align*}
  x & \xrightarrow{a} y \\
  b & \downarrow \\
  z & 
\end{align*}
\]

\(\Gamma_1\) contains the subgraph:

\[
\begin{align*}
  x & \xrightarrow{a} y \\
  b & \downarrow \\
  z & \xrightarrow{a^{-1}} t.
\end{align*}
\]

Furthermore, by Lemma 5.8 part (3), \(x \neq y\) implies \(z \neq t\), which verifies (1).

(2). \(X_1\) is the set of vertices of a connected component of \(\Gamma(X, r)\), so by Lemma 5.6 it is \(r\)-invariant. Let \(r_1 = r|_{X_1 \times X_1}\), we will show that \((X, r_1)\) is a trivial solution, that is \(r(x, y) = (y, x)\), for all \(x, y \in X_1\). In view of \(\text{irr}\) this is equivalent to \(y x = x\), for all \(x, y \in X_1\). Assume the contrary, there exist \(x, y \in X_1\), such that \(y x \neq x\). Then \(\Gamma_1\) contains the nontrivial edge \(x \xrightarrow{a} \xi\). This by (1) implies that every vertex \(z \in X_1\) has an edge \(z \xrightarrow{a^{-1}} t, t \neq z\). In particular \(y \xrightarrow{a^{-1}} \xi, \xi \neq y\), which is impossible, since \((X, r)\) is a square-free solution, thus \(y x = x, \forall x, y \in X\). This implies (2).

The lemma below proves part (3) and will be needed also for the proof of other statements in the sequel. It remains to prove the last three items of the theorem. Suppose \(x \in X_1, a, b \in \mathcal{E}(\Gamma_1)\), then by Lemma 5.11 the cycles \(\mathcal{L}_a^x\) and \(\mathcal{L}_a^x\) are of the
The following equalities hold:
\[ a^y x = a^x = y^a x \quad \forall x, y \in X_1, \text{ and } \forall a \in X. \]

(2) Each \( a \in X \) acts on \( X_1 \) as an automorphism, and \( G(\Gamma_1) \) is a normal subgroup of \( \text{Aut}(X_1, r_1) \).

Proof. Suppose \( x, y \in X_1, a \in X \). We use part (2) of the theorem and \( \Pi \) on \( (X, r) \) to obtain
\[ a^x = (2) a^y x = \Pi a^y x = (2) a^y x. \]
This implies \( a^y x = a^x \), for every \( x, y \in X_1 \), and \( a \in X \), which verifies the left hand side of the equality of part (1). From there, using \( \Pi \) and the non-degeneracy of \( r \) one easily deduces the righthand side of the required equality.

It follows from part (1) and Lemma 2.7 that
\[ \mathcal{L}_{a|\Gamma_1} \in \text{Aut}(X_1, r_1), \forall a \in X \]
thus \( G(\Gamma_1) \) is a subgroup of \( \text{Aut}(X_1, r_1) \). Now Lemma 2.3, and the equality (2.1) imply that \( G(\Gamma_1) \) is a normal subgroup of \( \text{Aut}(X_1, r_1) \). The lemma has been proved, which also completes the proof of the theorem.

Theorem 5.24. Let \( (X, r) \) be a finite nondegenerate square-free symmetric set of order \( \geq 2 \), \( G = G(X, r) \), \( G = G(X, r) \), \( \Gamma = \Gamma(X, r) \), \( \text{Aut}(X, r) \) in the usual notation. Let \( \Gamma_1, \Gamma_2, \cdots, \Gamma_s \) be all connected components, and \( X_i = V(\Gamma_i), 1 \leq i \leq s \), respectively, be their sets of vertices. The following conditions are equivalent.

(1) \( (X, r) \) is multipermutation solution with \( \text{mpl}(X, r) = 2 \).
(2) \( G(X, r) \) is an abelian group of order \( \geq 2 \).
(3) \( \{id_X\} \neq G(X, r) \) is a subgroup of the automorphism group \( \text{Aut}(X, r) \).
(4) The set of nontrivial components is nonempty. Suppose these are \( \Gamma_1, \Gamma_2, \cdots, \Gamma_p \), with \( 1 \leq p \leq s \). Every nontrivial connected component \( \Gamma_1 \) is a graph of first type. Furthermore, in this case, for each pair \( i, j, 1 \leq i, j \leq s \) and each \( a, b \in X_j \) one has \( \mathcal{L}_{a|X_i} = \mathcal{L}_{b|X_i} \).
(5) \( (X, r) \) can be split into disjoint \( r \)-invariant subsets \( X_i, 1 \leq i \leq s \), where each \( (X_i, r|X_i) \) is a trivial solution and \( X = X_1, X_2, \cdots, X_s \), in the sense that we can put parentheses and "apply" \( \natural \) in any order. In particular, for any pair \( i, j, r \) induces \( X_i \natural X_j \) and
\[ X = X_i \natural \bigcup_{1 \leq j \leq s, j \neq i} X_j \]

Proof. We shall show that
\[(1) \iff (2) \iff (3) \quad \text{and} \quad (2) \iff (4) \iff (5) \iff (2).\]

Under the hypothesis of the theorem we start with the following easy lemma.
Lemma 5.25. The following conditions are equivalent.

(i) $mpl(X, r) = 2$

(ii) $mpl(Ret(X, r)) = 1$,

(iii) $Ret(X, r) = ([X], r_{[X]})$ is the trivial solution.

(iv) $[x] a = [x^a] = [a]$, for all $a, x \in X$

(v) There is an equality:

\[
\text{(5.18)} \quad L^*_a = L_a \quad \forall x, a \in X
\]

(vi) There are equalities

\[
\text{(5.19)} \quad L^*_a = L_a = L_{a x}, \quad \forall x, a \in X.
\]

Proof. It follows from Definition 3.5 and Lemma 3.8 that (i) $\iff$ (ii). Lemma 3.10 gives the equivalence (ii) $\iff$ (iii). For (iii) $\iff$ (vi), the condition that the retract $([X], r_{[X]})$ is the trivial solution, implies that each pair $x, a \in X$ satisfies

\[
[x] a = [x^a] = [a] = [a^x] = [a^x]
\]

These equalities, “translated” in terms of the left actions give (5.19). (iii) $\iff$ (iv) follows from Definition 3.2 and Lemma 3.10 again. By Definition 3.1 the equality (5.18) is just expression of $[x^a] = [a]$ in terms of the left action, thus (iv) $\iff$ (v). $\square$

We continue with the proof of the theorem. (1) $\implies$ (2). Suppose $mpl X = 2$. Recall that $G(X, r)$ is the subgroup of $Sym(X)$ generated by the permutations $L_a, a \in X$. By Lemma 5.25 the equalities (5.19) are in force, so we yield:

\[
L_a \circ L_b = L_a \circ L_{a b} = (5.19) L_{a b} \circ L_a, \quad \forall a, b \in X
\]

Thus $G(X, r)$ is an abelian group. The following implications are clear

\[
(2) \implies (5.18) \implies \text{by Lemma 5.25} \quad (1)
\]

The implications (5.18) $\iff$ (3) follow from Lemma 2.7, and together with Lemma 5.25 yield (1) $\iff$ (3).

(2) $\iff$ (4). Suppose (2) holds. Then $G(X, r) \neq id_X$, so there is a nontrivial connected component of $\Gamma$. In addition by (5.18) each such a component $\Gamma_i$ is a graph of first type.

Lemma 5.26. (4) $\implies$ (5.18).

Proof. Suppose (4) holds and assume the contrary, there exist a pair $a, x \in X$, such that $L^*_a \neq L_a$.

Hence there exist a $t$, for which

\[
(5.20) \quad ^x a t \neq ^a t.
\]

Clearly, then the connected component $\Gamma_t$ which contains $t$ is nontrivial. Denote by $X_0$ the set of vertices of $\Gamma_t$, the set of labels we denote by $E_0 = E(\Gamma_t)$. By our assumption $\Gamma_t$ is a graph of first type, and therefore all its edges commute, so at
least one of \(a\) and \(x\) is not in \(E_0\). Suppose \(a\) is not in \(E_0\). Then \(x^a t \neq a t = t\), implies that

\[(5.21) \quad x^a \in E_0.\]

Two cases are possible. \textbf{a}) \(x \in E_0\). Then, since \(\Gamma_1\) is a graph of first type, \(L_{x^a}\) and \(L_x \) commute on \(\Gamma_1\). Now the equalities

\[(5.22) \quad L_{(x^a)x}|\Gamma_1) = L_{(x^a)x}|\Gamma_1) = L_{x^a|\Gamma_1}, \]

give a contradiction with (5.20). \textbf{b}) \(x\) is not in \(E_0\), so \(x t = t\), and \(x^a t = a t\). By \textbf{11} one has \(x^a t = (x^a)(x^a) t\), so (5.20) imply \((x^a) t \neq t\). Thus \(x^a \in E_0\), and by (5.21) Lemma 5.18 one has

\[(5.23) \quad L_{(x^a)|\Gamma_1} = L_{x^a|\Gamma_1}.\]

For the right hand side of (5.23) we used the following equalities which come from the cyclic condition \textbf{cl1}, see Definition 1.8, and \textbf{iri}:

\[(x^a)^x = \text{by cl1} = (x^a)^x = \text{by iri} a.\]

This way (5.23) gives a contradiction with (5.20). It follows then that \(a \in E_0\). Next (5.20) implies that \(x\) is not in \(E_0\), so it commutes with every element of \(X_0\) and in particular with \(a t\). Hence one has

\[a t = x(a t) = (x^a)(x^a) t,\]

and therefore \(x^a \in E_0\). It follows then that

\[L_{a|\Gamma_1} = L_{(x^a)x|\Gamma_1} = \text{by cl1} L_{x^a|\Gamma_1}.\]

(Remind that due to \textbf{cl1} one has \(x^a a = x^a\), which was used in the right hand side of the above equality). It follows than that (5.20) is impossible. We have shown that for each \(a, x \in X\) one has \(L_{x^a} = L_a\). This proves the lemma. \(\square\)

It follows from Lemmas 5.25 and 5.26 that (4) \(\implies\) (2)

Next we show (4) \(\implies\) (5). So, assume now (4) and consider the sets \(X_i, 1 \leq i \leq s\).

By Theorem 5.22 \((X_i, r_i)\) is a trivial solution of order \(\geq 2\) for all \(i, 1 \leq i \leq p\) and (in case that \(p < s\)) it is one element solution for \(i, p + 1 \leq i \leq s\). by Lemma 5.26 the equalities (5.18) hold, and it is easy to see that (5.18) implies (5).

Conversely, suppose (5) holds. Let \(a \in X\). Then there exist unique \(i, 1 \leq i \leq s\) such that \(a \in X_i\). It follows then from (5.17) that \(L_a\) acts on \(X \setminus X_i\) as an automorphism. Note that since \(X_i\) is the trivial solution, or one element solution thus in both cases one has \(L_a|X_i = id_{X_i}\), which clearly implies \(L_a = L_a|X \setminus X_i\). It follows then that \(L_a \in Aut(X, r)\). We have shown (5) \(\implies\) (2). The theorem has been proved. \(\square\)

**Proposition 5.27.** Let \((X, r)\) be a nondegenerate square-free symmetric set. Let \(\Gamma_1, \Gamma_2, \ldots, \Gamma_s\) be all connected components, and \(X_i = \mathcal{V}(\Gamma_i), 1 \leq i \leq s,\) respectively, be their sets of vertices. Suppose \(mpl(X, r) = 3\). Then for each \(i, 1 \leq i \leq s,\) \((X_i, r_i)\) with \(r_i = r_{X_i \setminus X_i}\) is a multipermutation solutions of level \(mpl(X_i) \leq 2\). Furthermore, for each pair \(i \neq j, 1 \leq i, j \leq s,\) one has \(X_i \subseteq O(x, 2)\), and therefore \(mpl(X_i, r_i) \leq 2\).

**Proof.** Proposition 5.6 implies that for each \(i, 1 \leq i \leq s\) and each \(x \in X_i\) one has \(X_i \subseteq O(x, 2)\), and therefore \(mpl(X_i, r_i) \leq 2\).
Chose now an arbitrary pair \( i \neq j, 1 \leq i, j \leq s \). We claim that \( X_i \gamma X_j \). It will be enough to show

\[
\alpha^\gamma x = x^\alpha \quad \forall x, y \in X_i, \alpha, \beta \in X_j.
\]

Let \( x, y \in X_i, \alpha \in X_j \). Then \( \alpha^\gamma \sim \alpha^x \), so

\[
\alpha^\gamma x = \alpha^x x = e11 \alpha x,
\]

which proves the left hand side equality of (5.24). Similar argument verifies the remaining equality in (5.24). \( \square \)

We will illustrate our theory on Examples 2.5 and 2.8. To begin with, it follows straightforwardly from the definition of \((X, r)\) in these examples that for \( Ret(X, r) = ([X], r|_{[X]}) \) one has \([X] = \{[x_1], [b], [c]\}, and r|_{[X]} \) the trivial solution. Therefore \( mpl(X) = 2 \). Moreover, \( Z = X^\sharp(a) \), where \( L_a = L_b \circ L_c \in Aut(X, r) \). We leave the reader to verify \( mpl(Z, r_Z) = 3 \).

Clearly the graph \( \Gamma(Z, r_Z) \) is obtained by adding to \( \Gamma(X, r) \) a new vertex and two new edges labeled by \( a \).

Next we shall describe the group \( Aut(Z, r_Z) \). Each automorphism \( \varphi \in Aut(Z, r_Z) \) is a product \( \varphi = \varphi' \varphi'' \) where \( \varphi' \in Sym(a, b, c), \varphi'' \in Sym(x_1, x_2, x_3, x_4) \). We claim that \( \varphi \) is uniquely determined by the data \((\varphi', \varphi''(x_1))\). Indeed, knowing the image \( \varphi''(x_1) \), one applies the equalities

\[
\varphi \circ L_a = L_{\varphi(a)} \circ \varphi, \quad \varphi \circ L_b = L_{\varphi(b)} \circ \varphi, \quad \varphi \circ L_c = L_{\varphi(c)} \circ \varphi.
\]

to find \( \varphi(x_i), 2 \leq i \leq 4 \). We can then tabulate the automorphisms with row corresponding to \( \varphi' \in S_3 \) and column corresponding to the value of \( \varphi''(x_1) \), which in all except the last row (where we use a different notation) this value supplies the index used. For simplicity we write \( 1, 2, \cdots \) instead of \( x_1, x_2, \cdots \). Then the full list of automorphisms is:

\[
\begin{align*}
\tau_1 &= (bc)(23), & \tau_2 &= (bc)(1243), & \tau_3 &= (bc)(1342), & \tau_4 &= (bc)(14) \\
\pi_1 &= (abc)(234), & \pi_2 &= (abc)(124), & \pi_3 &= (abc)(132), & \pi_4 &= (abc)(143) \\
\eta_1 &= (acb)(243), & \eta_2 &= (acb)(123), & \eta_3 &= (acb)(134), & \eta_4 &= (acb)(142) \\
\rho_1 &= (ab)(24), & \rho_2 &= (ab)(1234), & \rho_3 &= (ab)(13), & \rho_4 &= (ab)(1432) \\
\sigma_1 &= (ac)(34), & \sigma_2 &= (ac)(12), & \sigma_3 &= (ac)(1234), & \sigma_4 &= (ac)(1423) \\
id_2 &= (), & \mathcal{L}_b &= (12)(34), & \mathcal{L}_c &= (13)(24), & \mathcal{L}_a &= (14)(23).
\end{align*}
\]

For example, \( \pi_2 \) corresponds to the data \( (\pi_2)(abc) = (abc), \pi_2(x_1) = x_2 \). We will do the first step of the “search” for the missing information to see how this was obtained.

\[
\pi_2 \circ \mathcal{L}_b(x_1) = \mathcal{L}_{\pi_2}(b) \circ \pi_2(x_1), \quad \pi_2(x_2) = \mathcal{L}_c(x_2) = x_4
\]

so \( \pi_2 \) acts as \( x_1 \mapsto x_2 \mapsto x_4 \). Similar computation with \( \pi_2 \circ \mathcal{L}_a(x_4) \) shows that \( \pi_2(x_4) = x_1 \). Hence \( \pi_2 = (abc)(x_1x_2x_4) \). Note that the \( \eta \) row consists of the inverses of the \( \pi \) row. In the same way one obtains the rest of the table. This list was illustrated in Figure 2.

We claim that \( Aut(Z, r_Z) \) is isomorphic to the symmetric group \( S_4 \). Indeed, we know that \( S_4 \) is generated by \( \{(1234), (12)\} \). Direct computation shows that the assignment

\[
(1234) \mapsto (ab)(1234) = \rho_2, \quad (12) \mapsto (ac)(12) = \sigma_2
\]
extends to an isomorphism of groups $S_4 \longrightarrow Aut(X, r)$.

The computation of $Aut(X, r)$ as a set is done analogously, and the relations show directly that there is a group isomorphism $Aut(X, r) \approx D_4$. Clearly, $Aut(X, r)$ is a subgroup of $Aut(Z, rz)$ as expected.

References


TGI: INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, SOFIA 1113, BULGARIA, S.M: QUEEN MARY, UNIVERSITY OF LONDON, SCHOOL OF MATHEMATICS, MILE END RD, LONDON E1 4NS, UK

E-mail address: tatianagateva@yahoo.com, tatyana@aubg.bg, s.majid@qmul.ac.uk