Abstract. Let \( l \) be a prime number. In the present paper, we prove that the isomorphism class of an \( l \)-monodromically full hyperbolic curve of genus zero over a finitely generated extension of the field of rational numbers is completely determined by the kernel of the natural pro-\( l \) outer Galois representation associated to the hyperbolic curve. This result can be regarded as a genus zero analogue of a result due to S. Mochizuki which asserts that the isomorphism class of an elliptic curve which does not admit complex multiplication over a number field is completely determined by the kernels of the natural Galois representations on the various finite quotients of its Tate module.

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INTRODUCTION

Throughout the present paper, let \( k \) be a field of characteristic zero, \( \overline{k} \) an algebraic closure of \( k \), and \( G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k) \). In the present paper, we prove that if \( l \) is a prime number, then the isomorphism class of an \( l \)-monodromically full hyperbolic curve of genus zero over a finitely generated extension of the field of rational numbers is completely determined by the kernel of the associated pro-\( l \) outer Galois representation.

In [14], §1, S. Mochizuki proved the following theorem (cf. [14], Theorem 1.1):

Let \((E_1, o_1) \in E_1(k))\), \((E_2, o_2) \in E_2(k))\) be elliptic curves over \( k \) which do not admit complex multiplication over \( k \). Suppose that \( k \) is a number field — i.e., a finite extension of the field of rational numbers. Then the following conditions are equivalent:

(i) \((E_1, o_1)\) is isomorphic to \((E_2, o_2)\) over \( k \).

(ii) For \( i = 1, 2 \), write \( T(E_i, o_i) \) for the full Tate module of \((E_i, o_i)\) and for the natural Galois representation on \( T(E_i, o_i) \otimes \mathbb{Z} (\mathbb{Z}/n\mathbb{Z}) \). Then \( \text{Ker}(\rho_{(E_1, o_1)/k}^{(n)}) = \text{Ker}(\rho_{(E_2, o_2)/k}^{(n)}) \) for any positive integer \( n \).

In the present paper, we prove a genus zero analogue of the above result of Mochizuki. The main theorem of the present paper is as follows (cf. Theorem 6.1):

**Theorem A** (Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero). Let \( l \) be a prime number; \( k \) a finitely generated field of characteristic zero, i.e., a finitely generated extension of the field of rational numbers; \( X_1 = (C_1, D_1 \subseteq C_1) \), \( X_2 = (C_2, D_2 \subseteq C_2) \) hyperbolic curves (cf. Definition 1.1, (ii)) of genus zero over \( k \) which are \( l \)-monodromically full (cf. Definition 2.2, (i)). Suppose that the following condition \((\dagger)^{\prime}\) is satisfied:

\((\dagger)^{\prime}\) : There exists a finite Galois extension \( k' \subseteq \overline{k} \) of \( k \) of extension degree prime to \( l \) such that \( X_1 \otimes_k k' \) and \( X_2 \otimes_k k' \) are split (cf. Definition 1.5, (i)).

(For example, if one of the following conditions is satisfied, then the above condition \((\dagger)^{\prime}\) is satisfied:

- \( X_1 \) and \( X_2 \) are split.
- If we write \( r_i \) for the number of the cusps of \( X_i \) — i.e., if \( X_i \) is of type \((0, r_i)\) — then \( l \) is prime to \( r_1! \) and \( r_2! \) — or, equivalently, \( r_1, r_2 < l \).)
Then the following conditions are equivalent:

(i) \( X_1 \) is isomorphic to \( X_2 \) over \( k \).
(ii) For \( i = 1, 2 \), write

\[
\rho_{X_i/k}^{(l)} : G_k \rightarrow \text{Out} \left( \pi_1((C_i \setminus D_i) \otimes_k \mathbb{F})^{(l)} \right)
\]

for the natural pro-\( l \) outer Galois representation associated to \( X_i \). Then \( \text{Ker}(\rho_{X_1/k}^{(l)}) = \text{Ker}(\rho_{X_2/k}^{(l)}) \).

The term “\( l \)-monodromically full” is a term introduced in the present paper, but the corresponding notion was studied by M. Matsumoto and A. Tamagawa in [11]. It is known (cf. [11], Theorem 1.2, as well as Corollary 2.6 of the present paper) that many hyperbolic curves are \( l \)-monodromically full. This property of being \( l \)-monodromically full may be regarded as an analogue for hyperbolic curves of the property of not admitting complex multiplication for elliptic curves. In fact, if a hyperbolic curve \( X \) of type \((g, r)\) over a finitely generated extension \( k \) of the field of rational numbers is \( l \)-monodromically full, then the following hold:

- \( X \) has no special symmetry (i.e., roughly speaking, the automorphism group of \( X \) over \( \overline{k} \) is isomorphic to the automorphism group of a general hyperbolic curve of type \((g, r)\) over \( \overline{k} \) — cf. Definition 3.3, Proposition 3.4).
- \( X \) is of \{\( l \}\}-AIJ-type (i.e., roughly speaking, the \( l \)-adic Tate module of the Jacobian variety of the compactification of \( X \) is, as a Galois module, absolutely irreducible — cf. Definition 3.5, Proposition 3.6).
- \( X \) does not have a JCM-component (i.e., roughly speaking, there is no subabelian variety with complex multiplication over \( \overline{k} \) of the Jacobian variety of the compactification of \( X \) — cf. Definition 3.7, Proposition 3.8).

In the present paper, as an example, we consider hyperbolic curves of type \((0, 4)\) and obtain results concerning sufficient conditions for such a hyperbolic curve to be monodromically full (cf. Theorem 7.8, Corollaries 7.10, 7.11, 8.2). These results, together with Theorem A, imply the following result (cf. Corollaries 7.12, 8.3):

**Theorem B** (Galois-theoretic characterization of isomorphism classes of certain hyperbolic curves of type \((0, 4)\)). Let \( k \) be a finitely generated field of characteristic zero, i.e., a finitely generated extension of the field of rational numbers; \( X_1 = (C_1, D_1 \subseteq C_1) \), \( X_2 = (C_2, D_2 \subseteq C_2) \) hyperbolic curves (cf. Definition 1.1, (ii)) of type \((0, 4)\) over \( k \). Suppose that one of the following conditions is satisfied:
The field $k$ is a number field, i.e., a finite extension of the field of rational numbers, and, moreover, if we write $\mathcal{O}_k$ for the ring of integers of $\overline{k}$, then $m_{X_1} \cap \mathcal{O}_k^+ = m_{X_2} \cap \mathcal{O}_k^+ = \emptyset$ (cf. Definition 7.9).

The hyperbolic curves $X_1$ and $X_2$ are not NF-isotrivial (cf. Definition 8.1).

Then the following conditions are equivalent:

(i) $X_1$ is isomorphic to $X_2$ over $k$.

(ii) There exists an infinite set $\Sigma$ of prime numbers such that, for any $l \in \Sigma$, if we write

$$\rho^{(l)}_{X_i/k} : G_k \longrightarrow \text{Out}\left(\pi_1((C_1 \setminus D_l) \otimes_k \overline{k})^{(l)}\right)$$

for the natural pro-$l$ outer Galois representation associated to $X_i$, then $\text{Ker}(\rho^{(l)}_{X_1/k}) = \text{Ker}(\rho^{(l)}_{X_2/k})$.

On the other hand, one may also take the point of view that Theorems A and B serve to highlight the difference between the profinite and pro-$l$ outer Galois representations associated to a hyperbolic curve.

In [11], Matsumoto and Tamagawa compared the profinite and pro-$l$ outer Galois representations associated to hyperbolic curves. One result obtained in [11] which shows the difference between the profinite and pro-$l$ outer Galois representations is the following:

The image of the profinite outer Galois representation associated to any hyperbolic curve of type $(g, r)$ over a number field $k$ has trivial intersection with the image of the outer profinite geometric universal monodromy representation of $\pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k})$ (cf. [11], Theorem 1.1 and [8], Corollary 6.4). On the other hand, there exist many hyperbolic curves of type $(g, r)$ over number fields $k$ for which the image of the associated pro-$l$ outer Galois representation contains the image of the outer pro-$l$ geometric universal monodromy representation of $\pi_1(\mathcal{M}_{g,r} \otimes_k \overline{k})$ (cf. [11], Theorem 1.2).

By Theorems A, B (cf. also Theorem C below), one obtains another result which highlights the difference between the profinite and pro-$l$ outer Galois representations:

The kernel of the profinite outer Galois representation associated to any hyperbolic curve over a number field is always trivial, namely, the kernel does not depend on the given hyperbolic curve (cf. [8], Theorem C). On the other hand, the kernel of the pro-$l$ outer Galois representation associated to a hyperbolic curve over a number field depends strongly on the given hyperbolic curve (cf. Theorems A, B, also Theorem C below).
Finally, in the Appendix, we prove the following finiteness result, which is related to the main result of the present paper (cf. Corollary A.4):

**Theorem C (Finiteness of the set of isomorphism classes of certain hyperbolic curves).** Let \( l \) be a prime number, \( k \) a number field, i.e., a finite extension of the field of rational numbers, \((g,r)\) a pair of nonnegative integers such that \(2g - 2 + r > 0\), and \( N \subseteq G_k\) a normal closed subgroup of \( G_k \). Then there are only finitely many isomorphism classes over \( k \) of hyperbolic curves \( X \) of type \((g,r)\) over \( k \) for which the kernel of the natural pro-\( l \) outer Galois representation associated to \( X \) coincides with \( N \).

This result follows immediately from various well-known finiteness theorems in number theory and arithmetic geometry, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves. It seems to the author that this result is likely to be well-known. Since, however, this result could not be found in the literature, the author decided to give a proof of it in the Appendix of the present paper.

The present paper is organized as follows: In \( \S 1 \), we review some generalities concerning outer monodromy representations arising from hyperbolic curves. In \( \S 2 \), we define the notion of a \( \Sigma \)-monodromically full hyperbolic curve, as well as the related notion of a \( \Sigma \)-monodromically full point. In \( \S 3 \), we consider the relationship between monodromic fullness and certain properties of hyperbolic curves. In \( \S 4 \), we consider the moduli stacks of hyperbolic curves of genus zero. In \( \S 5 \), we prove a Grothendieck conjecture-type lemma for certain images of the universal monodromy. In \( \S 6 \), we derive Theorem A from the results obtained in \( \S 4 \) and \( \S 5 \). In \( \S 7 \) and \( \S 8 \), we consider the monodromic fullness of hyperbolic curves of type \((0,4)\). In particular, we obtain results concerning sufficient conditions for such a hyperbolic curve to be monodromically full and prove Theorem B. In the Appendix, we derive Theorem C as a consequence of various well-known finiteness theorems in number theory and arithmetic geometry, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves.

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0. Notations and Conventions

Numbers: A finite extension (respectively, finitely generated extension) of the field of rational numbers will be referred to as a number field (respectively, finitely generated field of characteristic zero). If \( p \) is a prime number, then a field which may be embedded as a subfield of a finitely generated extension of the field of fractions of the ring of Witt vectors with coefficients in an algebraic closure of the finite field of \( p \) elements will be referred to as a generalized sub-\( p \)-adic field (cf. [14], Definition 4.11).

Topological groups: Let \( G \) be a topological group and \( \mathbf{P} \) a property for a topological group (e.g., “abelian” or “pro-\( l \)” for some prime number \( l \)). Then we shall say that \( G \) is almost \( \mathbf{P} \) if there exists an open subgroup of \( G \) that is \( \mathbf{P} \).

If \( G \) is a topological group, then we shall write \( G^{ab} \) for the abelianization of \( G \), i.e., the quotient of \( G \) by the closure of the commutator subgroup of \( G \).

If \( G \) is a topological group, and \( H \subseteq G \) is a closed subgroup of \( G \), then we shall write \( Z_G(H) \) for the centralizer of \( H \) in \( G \), i.e.,
\[
Z_G(H) \overset{\text{def}}{=} \{ g \in G \mid ghg^{-1} = h \text{ for any } h \in H \} \subseteq G,
\]
\[
Z_G^{loc}(H) \overset{\text{def}}{=} \lim_{H' \subseteq H} Z_G(H') \subseteq G
\]
— where \( H' \subseteq H \) ranges over the open subgroups of \( H \) — \( Z(G) \overset{\text{def}}{=} Z_G(G) \) for the center of \( G \), and \( Z_G^{loc}(G) \overset{\text{def}}{=} Z_G^{loc}(G) \) for the local center of \( G \). It is immediate from the various definitions involved that \( Z_G(H) \subseteq Z_G^{loc}(H) \) and that if \( H_1, H_2 \subseteq G \) are closed subgroups of \( G \) such that \( H_1 \subseteq H_2 \) (respectively, \( H_1 \subseteq H_2; H_1 \cap H_2 \) is open in \( H_1 \) and \( H_2 \)), then \( Z_G(H_2) \subseteq Z_G(H_1) \) (respectively, \( Z_G^{loc}(H_2) \subseteq Z_G^{loc}(H_1); Z_G^{loc}(H_1) = Z_G^{loc}(H_2) \)).

We shall say that a topological group \( G \) is center-free (respectively, slim) if \( Z(G) = \{1\} \) (respectively, \( Z_G^{loc}(G) = \{1\} \)). Note that it follows from [15], Remark 0.1.3, that a profinite group \( G \) is slim if and only if every open subgroup of \( G \) has trivial center.

If \( G \) is a profinite group, then we shall denote the group of automorphisms of \( G \) by \( \text{Aut}(G) \) and the group of inner automorphisms of \( G \) by \( \text{Inn}(G) \subseteq \text{Aut}(G) \). Conjugation by elements of \( G \) determines a surjection \( G \to \text{Inn}(G) \). Thus, we have a homomorphism \( G \to \text{Aut}(G) \) whose image is \( \text{Inn}(G) \subseteq \text{Aut}(G) \). We shall denote by \( \text{Out}(G) \) the quotient of \( \text{Aut}(G) \) by the normal subgroup \( \text{Inn}(G) \subseteq \text{Aut}(G) \) and refer to an element of \( \text{Out}(G) \) as an automorphism of \( G \). In particular, if \( G \) is center-free, then the natural homomorphism \( G \to \text{Inn}(G) \) is an
isomorphism; thus, we have an exact sequence of groups
\[ 1 \longrightarrow G \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1. \]

If, moreover, \( G \) is topologically finitely generated, then one verifies easily that the topology of \( G \) admits a basis of characteristic open subgroups, which thus induces a profinite topology on the groups \( \text{Aut}(G) \) and \( \text{Out}(G) \) with respect to which the above exact sequence determines an exact sequence of profinite groups. If \( J \) is a profinite group, and \( \rho: J \rightarrow \text{Out}(G) \) is a continuous homomorphism, then we shall denote by \( G \rtimes J \) the profinite group obtained by pulling back the above exact sequence of profinite groups via \( \rho \). Thus, we have a natural exact sequence of profinite groups
\[ 1 \longrightarrow G \longrightarrow G \rtimes J \longrightarrow J \longrightarrow 1. \]

1. Outer monodromy representations

Throughout the present paper, let \( k \) be a field of characteristic zero and \( \overline{k} \) an algebraic closure of \( k \). If \( k' \subseteq \overline{k} \) is a(n) (possibly infinite) algebraic extension of \( k \), then we shall write \( G_{k'} \overset{\text{def}}{=} \text{Gal}(\overline{k}/k') \).

In the present \( \S \), we review some generalities concerning outer monodromy representations arising from hyperbolic curves. In the present \( \S \), let \((g, r)\) be a pair of nonnegative integers such that \( 2g - 2 + r > 0 \) and \( \Sigma \) a nonempty set of prime numbers.

Definition 1.1. Let \( S \) be a scheme.

(i) Let \( C \) be a scheme over \( S \) and \( s_i: S \rightarrow C \) a section of the structure morphism of \( C \) — where \( i = 1, \ldots, r \). Then we shall say that \((C, (s_1, \ldots, s_r))\) is an \( r \)-pointed smooth curve of genus \( g \) over \( S \) whose marked points are equipped with an ordering if \( C \) is smooth and proper over \( S \), any geometric fiber of \( C \rightarrow S \) is a (necessarily smooth and proper) connected curve of genus \( g \), and the image of \( s_i \) does not intersect the image of \( s_j \) if \( i \neq j \).

(ii) Let \( C \) be a scheme over \( S \) and \( D \subseteq C \) a closed subscheme of \( C \). Then we shall say that \((C, D \subseteq C)\) is a hyperbolic curve of type \((g, r)\) over \( S \) if \( C \) is smooth and proper over \( S \), any geometric fiber of \( C \rightarrow S \) is a (necessarily smooth and proper) connected curve of genus \( g \), and the composite \( D \hookrightarrow C \rightarrow S \) is a finite étale covering over \( S \) of degree \( r \).

Definition 1.2.

(i) We shall denote by \( \mathcal{M}_{g,r} \rightarrow \text{Spec } k \) the moduli stack (cf. [5], [10]) of \( r \)-pointed smooth curves of genus \( g \) over \( k \) whose marked points are equipped with orderings (cf. Definition 1.1, (i)) and by \((\mathcal{C}_{g,r} \rightarrow \mathcal{M}_{g,r}, (s_1^M, \ldots, s_r^M))\) the universal curve over \( \mathcal{M}_{g,r} \).
(ii) We shall denote by $\mathcal{M}_{g,[r]} \rightarrow \text{Spec } k$ the moduli stack of hyperbolic curves of type $(g, r)$ over $k$ (cf. Definition 1.1, (ii)) and by $(C_{g,[r]} \rightarrow \mathcal{M}_{g,[r]}, D^M_{g,[r]} \subseteq C_{g,[r]})$ the universal curve over $\mathcal{M}_{g,[r]}$.

It follows from the various definitions involved that we have a commutative diagram

$$
\begin{array}{cccc}
C_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M) & \xrightarrow{\subseteq} & C_{g,r} & \longrightarrow & \mathcal{M}_{g,r} \\
\downarrow & & \downarrow & & \downarrow \\
C_{g,[r]} \setminus D^M_{g,[r]} & \xrightarrow{\subseteq} & C_{g,[r]} & \longrightarrow & \mathcal{M}_{g,[r]}
\end{array}
$$

such that the two squares in this diagram are cartesian; moreover, as is well-known, in this commutative diagram, the right-hand vertical arrow $\mathcal{M}_{g,r} \rightarrow \mathcal{M}_{g,[r]}$ is a finite étale Galois covering whose Galois group is isomorphic to the symmetric group on $r$ letters $\mathfrak{S}_r$. In particular, we obtain a commutative diagram

$$
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & N_{g,r} & \longrightarrow & \pi_1(C_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M)) & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & 1 \\
\| & & \| & & \| & & \| \\
1 & \longrightarrow & N_{g,r} & \longrightarrow & \pi_1(C_{g,[r]} \setminus D^M_{g,[r]}) & \longrightarrow & \pi_1(\mathcal{M}_{g,[r]}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & 1 & \longrightarrow & \mathfrak{S}_r & \longrightarrow & \mathfrak{S}_r & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & 1 & 1
\end{array}
$$

— where $N_{g,r}$ is the kernel of the surjection $\pi_1(C_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M)) \twoheadrightarrow \pi_1(\mathcal{M}_{g,r})$, and the vertical and horizontal sequences are exact. (See [20] for the fundamental groups of stacks.)

**Definition 1.3.**

(i) We shall write

$$
\Delta^\Sigma_{g,r}
$$

for the maximal pro-$\Sigma$ quotient of the kernel $N_{g,r}$ of the surjection $\pi_1(C_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M)) \twoheadrightarrow \pi_1(\mathcal{M}_{g,r})$ (cf. Remark 1.3.1 below).

(ii) We shall write

$$
\rho^\Sigma_{g,r} \ (\text{respectively, } \rho^\Sigma_{g,[r]} ; \rho^\Sigma_{g,r}^{-\text{geom}} ; \rho^\Sigma_{g,[r]}^{-\text{geom}})
$$
for the natural homomorphism determined by the above commutative diagram

\[ \pi_1(\mathcal{M}_{g,r}) \rightarrow \text{Out}(\Delta_{g,r}^\Sigma) \]

(respectively, \( \pi_1(\mathcal{M}_{g,[r]}) \rightarrow \text{Out}(\Delta_{g,r}^\Sigma) \);
\[ \pi_1(\mathcal{M}_{g,r} \otimes_k \bar{k}) \rightarrow \text{Out}(\Delta_{g,r}^\Sigma) ; \]
\[ \pi_1(\mathcal{M}_{g,[r]} \otimes_k \bar{k}) \rightarrow \text{Out}(\Delta_{g,r}^\Sigma) . \]

(iii) Let \( S \) be a scheme that is connected and of finite type over \( k \), and \( X = (C, D \subseteq C) \) a hyperbolic curve of type \( (g, r) \) over \( S \). Then the classifying morphism \( S \rightarrow \mathcal{M}_{g,[r]} \) of \( X \) determines up to \( \pi_1(\mathcal{M}_{g,[r]} \otimes_k \bar{k}) \)-inner automorphism — a section \( s_{X/S} \) of the natural exact sequence

\[ 1 \rightarrow \pi_1(\mathcal{M}_{g,[r]} \otimes_k \bar{k}) \rightarrow \pi_1(\mathcal{M}_{g,[r]}) \times_{G_k} \pi_1(S) \rightarrow \pi_1(S) \rightarrow 1 . \]

Thus, by considering the composite of \( s_{X/S} \) and \( \rho_{g,[r]}^\Sigma \), we obtain a homomorphism

\[ \rho_{X/S}^\Sigma : \pi_1(S) \rightarrow \text{Out}(\Delta_{g,r}^\Sigma) \]

which is determined up to \( \text{Im}(\rho_{g,[r]}^\Sigma) \)-inner automorphism.

Remark 1.3.1. It follows immediately from, for example, [11], Lemma 2.1, that \( \Delta_{g,r}^\Sigma \) is naturally isomorphic to the maximal pro-\( \Sigma \) quotient of the fundamental group of the geometric fiber of the universal curve \( \mathcal{C}_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M) \rightarrow \mathcal{M}_{g,r} \) at a geometric point of \( \mathcal{M}_{g,r} \). In particular, it follows immediately from, for example, [17], Corollary 1.3.4, that \( \Delta_{g,r}^\Sigma \) is slim (cf. the discussion entitled “Topological groups” in §0); moreover, there exists a natural bijection between the following two sets:

- The set of the cusps of the geometric fiber of the universal curve \( \mathcal{C}_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M) \rightarrow \mathcal{M}_{g,r} \) at a geometric point of \( \mathcal{M}_{g,r} \).
- The set of the conjugacy classes of the cuspidal inertia subgroups of \( \Delta_{g,r}^\Sigma \) associated to the cusps of the geometric fiber of the universal curve \( \mathcal{C}_{g,r} \setminus \bigcup_{i=1}^r \text{Im}(s_i^M) \rightarrow \mathcal{M}_{g,r} \) at a geometric point of \( \mathcal{M}_{g,r} \).

Lemma 1.4 (Kernels of the universal outer monodromy representations).

(i) The action of \( \pi_1(\mathcal{M}_{g,[r]}) \) on the set of the conjugacy classes of the cuspidal inertia subgroups of \( \Delta_{g,r}^\Sigma \) induced by \( \rho_{g,[r]}^\Sigma \) factors through the quotient \( \pi_1(\mathcal{M}_{g,[r]}) \rightarrow \pi_1(\mathcal{M}_{g,[r]})/\pi_1(\mathcal{M}_{g,r}) \simeq \mathfrak{S}_r \), and the resulting action of \( \mathfrak{S}_r \) on the set of the conjugacy classes of the cuspidal inertia subgroups of \( \Delta_{g,r}^\Sigma \) is faithful.

(ii) The kernel of \( \rho_{g,[r]}^\Sigma \) is contained in \( \pi_1(\mathcal{M}_{g,r}) \) and coincides with the kernel of \( \rho_{g,r}^\Sigma \).
Proof. Assertion (i) follows immediately from the various definitions involved, together with Remark 1.3.1. Assertion (ii) follows immediately from assertion (i), together with Remark 1.3.1. □

Definition 1.5. Let $S$ be a scheme and $X = (C, D \subseteq C)$ a hyperbolic curve of type $(g, r)$ over $S$.

(i) We shall say that the hyperbolic curve $X$ is split if the finite étale covering obtained as the composite $D \hookrightarrow C \to S$ (cf. Definition 1.1, (ii)) is trivial, i.e., $D$ is isomorphic to the disjoint union of $r$ copies of $S$ over $S$.

(ii) Let $X_0 = (C_0, D_0 \subseteq C_0)$ be a hyperbolic curve over $S$. Then we shall say that $X_0$ is a hyperbolic partial compactification of $X$ if there exists an open immersion $C \setminus D \hookrightarrow C_0 \setminus D_0$ over $S$.

(iii) Suppose that $g \geq 2$. Then it is immediate that the pair $(C, \emptyset \subseteq C)$ is a hyperbolic partial compactification of the hyperbolic curve $X$. We shall write $X^{\text{cpt}} = (C, D \subseteq C)^{\text{cpt}} \overset{\text{def}}{=} (C, \emptyset \subseteq C)$ and refer to as the compactification of $X$.

Remark 1.5.1. Let $S$ be a scheme that is connected and of finite type over $k$, and $X$ a hyperbolic curve of type $(g, r)$ over $S$.

(i) It follows immediately from Lemma 1.4, (i), that the hyperbolic curve $X$ is split if and only if the image $\text{Im}(\rho_{X/S}^\Sigma)$ is contained in the image $\text{Im}(\rho_{g,r}^\Sigma)$. In particular, if $g \geq 2$, then we obtain natural surjections

$$\pi_1(S) \to \text{Im}(\rho_{X/S}^\Sigma) \to \text{Im}(\rho_{X_0/S}^\Sigma).$$

(ii) Let $X_0$ be a hyperbolic partial compactification of $X$. Then it follows immediately from the various definitions involved that the homomorphism $\rho_{X_0/S}^\Sigma$ factors through the homomorphism $\rho_{X/S}^\Sigma$; thus, we obtain natural surjections

$$\pi_1(S) \to \text{Im}(\rho_{X/S}^\Sigma) \to \text{Im}(\rho_{X_0/S}^\Sigma).$$

Lemma 1.6 (Universal pro-$\ell$ outer monodromy representations). Suppose that $\Sigma$ is of cardinality one. Then the following hold:

(i) The natural surjection $\pi_1(\mathcal{M}_{g,r}) \to G_k = \pi_1(\mathcal{M}_{0,3})$ induces a surjection $\text{Ker}(\rho_{g,r}^\Sigma) \to \text{Ker}(\rho_{0,3}^\Sigma)$. In particular, we obtain a commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & \pi_1(\mathcal{M}_{g,r}) & \longrightarrow & G_k & \longrightarrow & 1 \\
& & \rho_{g,r}^{\text{geom}} \downarrow & & \rho_g^{\Sigma} \downarrow & & \bar{\rho}_{0,3}^{\Sigma} & \\
1 & \longrightarrow & \text{Im}(\rho_{g,r}^{\text{geom}}) & \longrightarrow & \text{Im}(\rho_{g,r}^{\Sigma}) & \longrightarrow & \text{Im}(\rho_{0,3}^{\Sigma}) & \longrightarrow & 1 \\
\end{array}
$$

— where the horizontal sequences are exact.
(ii) The natural surjection $\pi_1(M_{g,[r]}) \to G_k = \pi_1(M_{0,3})$ induces a surjection $\text{Ker}(\rho_{g,[r]}^\Sigma) \to \text{Ker}(\rho_{0,3}^\Sigma)$. In particular, we obtain a commutative diagram

$$
1 \longrightarrow \pi_1(M_{g,[r]} \otimes_k \overline{k}) \longrightarrow \pi_1(M_{g,[r]}) \longrightarrow G_k \longrightarrow 1
$$

\[ \rho_{g,[r]}^\Sigma \downarrow \quad \rho_{g,[r]}^\Sigma \downarrow \quad \rho_{0,3}^\Sigma \]

$$
1 \longrightarrow \text{Im}(\rho_{g,[r]}^\Sigma) \longrightarrow \text{Im}(\rho_{g,[r]}^\Sigma) \longrightarrow \text{Im}(\rho_{0,3}^\Sigma) \longrightarrow 1
$$

where the horizontal sequences are exact.

(iii) The commutative diagram

$$
\mathcal{M}_{g,r} \otimes_k \overline{k} \longrightarrow M_{g,[r]} \otimes_k \overline{k}
$$

$$
\mathcal{M}_{g,r} \longrightarrow M_{g,[r]}
$$

induces a commutative diagram

$$
1 \longrightarrow \text{Im}(\rho_{g,r}^\Sigma) \longrightarrow \text{Im}(\rho_{g,[r]}^\Sigma) \longrightarrow \mathcal{G}_r \longrightarrow 1
$$

$$
1 \longrightarrow \text{Im}(\rho_{g,r}^\Sigma) \longrightarrow \text{Im}(\rho_{g,[r]}^\Sigma) \longrightarrow \mathcal{G}_r \longrightarrow 1
$$

where the horizontal sequences are exact, and the vertical arrows are injective.

Proof. Assertion (i) is a consequence of a result concerning Oda’s problem: If $r \neq 0$, then the desired surjectivity was proven in [9], Corollary 4.2.2; on the other hand, if $r = 0$, then the desired surjectivity follows from [9], Theorem 3B, together with [8], Theorem C, or a result obtained in [24].

Assertion (ii) follows immediately from assertion (i), together with Lemma 1.4, (ii). Assertion (iii) follows immediately from Lemma 1.4, (ii).

In the rest of the present §, we consider the almost slimness (cf. the discussion entitled “Topological groups” in §0) of the images of outer monodromy representations.

Proposition 1.7 (Almost slimness of the images of outer monodromy representations). Let $H \subseteq \text{Im}(\rho_{g,[r]}^\Sigma)$ be a closed subgroup of the image $\text{Im}(\rho_{g,[r]}^\Sigma)$. Then the following hold:

(i) If $\Sigma$ consists of exactly one prime number $l$, then $H$ is almost pro-$l$ (cf. the discussion entitled “Topological groups” in §0).

(ii) Suppose that $k$ is a generalized sub-$l$-adic field (cf. the discussion entitled “Numbers” in §0) for some $l \in \Sigma$ and that there exists a hyperbolic curve $X$ of type $(g,r)$ over a finite extension $k' \subseteq \overline{k}$ of $k$ such that $H$ contains the image $\text{Im}(\rho_{X/k'}^\Sigma)$.
Then $H$ is almost slim (cf. the discussion entitled “Topological groups” in §0). In particular, the images $\text{Im}(\rho_{g, r}^X)$, $\text{Im}(\rho_{g, [r]}^X)$, and $\text{Im}(\rho_{X/k'}^X)$ — where $X$ is a hyperbolic curve of type $(g, r)$ over a finite extension $k' \subseteq \overline{k}$ of $k$ — are almost slim.

Proof. First, we consider assertion (i). It follows from [2], Corollary 7, together with the fact that $\Delta_{g,r}$ is topologically finitely generated (cf. Remark 1.3.1) and $\rho_{g, [r]}^X$ is almost pro-$l$. Thus, $H$ is almost pro-$l$, as desired. This completes the proof of assertion (i).

Next, we consider assertion (ii). Suppose that there exists a hyperbolic curve $X$ of type $(g, r)$ over a finite extension $k' \subseteq \overline{k}$ of $k$ such that $H$ contains the image $\text{Im}(\rho_{X/k'}^X)$. Then since $\Delta_{g,r}$ is center-free (cf. Remark 1.3.1), it follows from [14], Theorem 4.12, together with [17], Corollary 1.5.7, that there exists a natural bijection

$$\text{Aut}(X \otimes_{k'} k) \xrightarrow{\sim} Z_{\text{Out}(\Delta_{g,r})}^\text{loc}(\text{Im}(\rho_{X/k'}^X))$$

(cf. the discussion entitled “Topological groups” in §0); in particular, $Z_{\text{Out}(\Delta_{g,r})}^\text{loc}(\text{Im}(\rho_{X/k'}^X))$ is finite. On the other hand, since $\text{Im}(\rho_{X/k'}^X) \subseteq H$, it follows that $Z_{\text{Out}(\Delta_{g,r})}^\text{loc}(\text{Out}(\Delta_{g,r}))(H) \subseteq Z_{\text{Out}(\Delta_{g,r})}^\text{loc}(\text{Out}(\Delta_{g,r}))(\text{Im}(\rho_{X/k'}^X))$ (cf. the discussion entitled “Topological groups” in §0) is finite. Therefore, it follows from Lemma 1.8 below that $H$ is almost slim. This completes the proof of assertion (ii).

Lemma 1.8 (Almost slimness and the finiteness of local center). Let $G$ be a profinite group. Then the following conditions are equivalent:

(i) $G$ is almost slim (cf. the discussion entitled “Topological groups” in §0).

(ii) The local centre $Z^\text{loc}(G)$ (cf. the discussion entitled “Topological groups” in §0) is finite.

Proof. First, to prove the implication

(i) $\implies$ (ii),

suppose that condition (i) is satisfied, i.e., there exists an open subgroup $H \subseteq G$ of $G$ that is slim. By replacing $H$ by a suitable open subgroup of $H$, we may assume without loss of generality that $H$ is normal in $G$. Now since $H$ is slim, it follows that $Z^\text{loc}(H) = Z^\text{loc}(G) \cap H = \{1\}$. Thus, the composite $Z^\text{loc}(G) \hookrightarrow G \twoheadrightarrow G/H$ is injective; in particular, $Z^\text{loc}(G)$ is finite. This completes the proof of the above implication. Finally, to prove the implication

(ii) $\implies$ (i),

suppose that condition (ii) is satisfied. Since $Z^\text{loc}(G) \subseteq G$ is finite, there exists an open subgroup $H \subseteq G$ of $G$ such that $Z^\text{loc}(G) \cap H = \{1\}$. On the other hand, since $Z^\text{loc}(H) = Z^\text{loc}(G) \cap H$, it follows that
$Z^{loc}(H) = \{1\}$, i.e., $H$ is slim. This completes the proof of the above implication. \hfill \Box

2. Monodromically full points and curves

In the present §, we define the notion of a $\Sigma$-monodromically full hyperbolic curve (cf. Definition 2.2 below), as well as the related notion of a $\Sigma$-monodromically full point (cf. Definition 2.1 below). In the present §, let $(g, r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$ and $\Sigma$ a nonempty set of prime numbers.

First, we define the notion of a $\Sigma$-monodromically full, strictly $\Sigma$-monodromically full, and quasi-$\Sigma$-monodromically full point.

\textbf{Definition 2.1.} Let $S$ be a scheme that is connected and of finite type over $k$, $X = (C, D \subseteq C)$ a hyperbolic curve of type $(g, r)$ over $S$, and $s \in S$ a closed point of $S$. Write $X_s$ for the hyperbolic curve over the residue field $k(s)$ of $S$ at $s$ obtained as the fiber of $X \to S$ at $s \in S$, i.e., $X_s = (C \times_S \text{Spec} \, k(s), D \times_S \text{Spec} \, k(s))$.

(i) We shall say that $s \in S$ is a $\Sigma$-monodromically full point with respect to $X/S$ if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ of $\text{Im}(\rho^{(l)}_{X/S})$ — here, $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ and $\text{Im}(\rho^{(l)}_{X/S})$ are determined up to $\text{Im}(\rho^{(l)}_{g,[r]}{-}\text{geom})$-conjugation — contains $\text{Im}(\rho^{(l)}_{X/S}) \cap \text{Im}(\rho^{(l)}_{g,[r]})$.

(ii) We shall say that $s \in S$ is a strictly $\Sigma$-monodromically full point with respect to $X/S$ if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ of $\text{Im}(\rho^{(l)}_{X/S})$ — here, $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ and $\text{Im}(\rho^{(l)}_{X/S})$ are determined up to $\text{Im}(\rho^{(l)}_{g,[r]}{-}\text{geom})$-conjugation — coincides with $\text{Im}(\rho^{(l)}_{X/S})$.

(iii) We shall say that $s \in S$ is a quasi-$\Sigma$-monodromically full point with respect to $X/S$ if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ of $\text{Im}(\rho^{(l)}_{X/S})$ — here, $\text{Im}(\rho^{(l)}_{X_s/k(s)})$ and $\text{Im}(\rho^{(l)}_{X/S})$ are determined up to $\text{Im}(\rho^{(l)}_{g,[r]}{-}\text{geom})$-conjugation — is an open subgroup of $\text{Im}(\rho^{(l)}_{X/S})$.

If $l$ is a prime number, then for simplicity, we write $l$-monodromically full (respectively, strictly $l$-monodromically full; quasi-$l$-monodromically full) instead of $(l)$-monodromically full (respectively, strictly $(l)$-monodromically full; quasi-$\{l\}$-monodromically full).

\textbf{Remark 2.1.1.} Let $S$ be a scheme that is connected and of finite type over $k$, $X$ a hyperbolic curve over $S$, and $s \in S$ a closed point of $S$. Consider the following conditions:

(i) $s \in S$ is strictly $\Sigma$-monodromically full with respect to $X/S$.
(ii) $s \in S$ is $\Sigma$-monodromically full with respect to $X/S$.
(iii) $s \in S$ is quasi-$\Sigma$-monodromically full with respect to $X/S$.  

Then, as the terminologies suggest, it follows immediately from the various definitions involved that the implications

\[(i) \implies (ii) \implies (iii)\]

hold.

Next, we define the notion of a $\Sigma$-monodromically full, strictly $\Sigma$-monodromically full, and quasi-$\Sigma$-monodromically full hyperbolic curve. Roughly speaking, a $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full) hyperbolic curve is a hyperbolic curve corresponding to a $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full) point of the moduli stack with respect to the universal curve.

**Definition 2.2.** Let $X$ be a hyperbolic curve of type $(g,r)$ over $k$.

1. We shall say that $X$ is strictly $\Sigma$-monodromically full if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho_{X/k}^{(l)})$ — which is determined up to $\text{Im}(\rho_{g,[r]}^{(l)-\text{geom}})$-conjugation — of $\text{Im}(\rho_{g,[r]}^{(l)})$ contains $\text{Im}(\rho_{g,r}^{(l)})$.

2. We shall say that $X$ is $\Sigma$-monodromically full if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho_{X/k}^{(l)})$ — which is determined up to $\text{Im}(\rho_{g,[r]}^{(l)-\text{geom}})$-conjugation — of $\text{Im}(\rho_{g,r}^{(l)})$ contains $\text{Im}(\rho_{g,[r]}^{(l)-\text{geom}})$, or, equivalently, the closed subgroup $\text{Im}(\rho_{X/k}^{(l)})$ of $\text{Im}(\rho_{g,[r]}^{(l)})$ coincides with $\text{Im}(\rho_{g,r}^{(l)})$.

3. We shall say that $X$ is quasi-$\Sigma$-monodromically full if, for any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho_{X/k}^{(l)})$ — which is determined up to $\text{Im}(\rho_{g,[r]}^{(l)-\text{geom}})$-conjugation — of $\text{Im}(\rho_{g,[r]}^{(l)})$ is an open subgroup of $\text{Im}(\rho_{g,r}^{(l)})$.

If $l$ is a prime number, then for simplicity, we write $l$-monodromically full (respectively, strictly $l$-monodromically full; quasi-$l$-monodromically full) instead of $\{l\}$-monodromically full (respectively, strictly $\{l\}$-monodromically full; quasi-$\{l\}$-monodromically full).

**Remark 2.2.1.** Let $X$ be a hyperbolic curve over $k$. Consider the following conditions:

1. $X$ is strictly $\Sigma$-monodromically full.
2. $X$ is $\Sigma$-monodromically full.
3. $X$ is quasi-$\Sigma$-monodromically full.

Then, as the terminologies suggest, it follows immediately from the various definitions involved that the implications

\[(i) \implies (ii) \implies (iii)\]

hold.
**Remark 2.2.2.** Let $X$ be a hyperbolic curve over $k$ and $\Sigma_1, \Sigma_2$ nonempty sets of prime numbers. Suppose that $\Sigma_2 \subseteq \Sigma_1$. Consider the following conditions:

(i) $X$ is $\Sigma_1$-monodromically full (respectively, strictly $\Sigma_1$-monodromically full; quasi-$\Sigma_1$-monodromically full).

(ii) $X$ is $\Sigma_2$-monodromically full (respectively, strictly $\Sigma_2$-monodromically full; quasi-$\Sigma_2$-monodromically full).

Then it follows immediately from the various definitions involved that the implication

(i) \[\implies\] (ii)

holds.

**Remark 2.2.3.** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Suppose that $r \leq 1$. Consider the following conditions:

(i) $X$ is $\Sigma$-monodromically full.

(ii) $X$ is strictly $\Sigma$-monodromically full.

Then it follows immediately from the various definitions involved that the equivalence

(i) \[\iff\] (ii)

holds.

**Remark 2.2.4.** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Suppose that $r \geq 2$. Consider the following conditions:

(i) $X$ is strictly $\Sigma$-monodromically full.

(ii) $X$ is not split (cf. Definition 1.5, (i)).

Then it follows immediately from Remark 1.5.1, (i), that the implication

(i) \[\implies\] (ii)

holds.

**Remark 2.2.5.** Let $X_1$ be a hyperbolic curve over $k$ and $X_2$ a hyperbolic partial compactification of $X_1$ (cf. Definition 1.5, (ii)). Consider the following conditions:

(i) $X_1$ is $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full).

(ii) $X_2$ is $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full).

Then it follows immediately from Remark 1.5.1, (ii), that the implication

(i) \[\implies\] (ii)

holds.

**Remark 2.2.6.** Let $X$ be a hyperbolic curve over $k$, and $k' \subseteq \overline{k}$ a finite extension of $k$. Consider the following conditions:

(i) $X$ is a quasi-$\Sigma$-monodromically full hyperbolic curve over $k$. 
(ii) $X \otimes_k k'$ is a quasi-$\Sigma$-monodromically full hyperbolic curve over $k'$.

Then it follows immediately from the various definitions involved that the equivalence

$$(i) \iff (ii)$$

holds.

**Remark 2.2.7.** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Consider the following conditions:

(i) $X$ is split and $\Sigma$-monodromically full.

(ii) For any $l \in \Sigma$, the closed subgroup $\text{Im}(\rho^{(l)}_{X/k})$ — which is determined up to $\text{Im}(\rho^{(l)}_{g,r,\text{geom}})$-conjugation — of $\text{Im}(\rho^{(l)}_{g,r})$ coincides with $\text{Im}(\rho^{(l)}_{g,r})$.

Then it follows immediately from Remark 1.5.1, (i), together with the various definition involved, that the equivalence

$$(i) \iff (ii)$$

holds.

**Remark 2.2.8.** Let $S$ be a scheme that is connected and of finite type over $k$, $X$ a hyperbolic curve over $S$, and $s \in S$ a closed point of $S$. Write $k(s)$ for the residue field of $S$ at $s$ and $X_s$ for the hyperbolic curve over $k(s)$ obtained as the fiber of $X \to S$ at $s \in S$ (cf. Definition 2.1). Consider the following conditions:

(i) $X_s$ is a $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full) hyperbolic curve over $k(s)$.

(ii) $s \in S$ is a $\Sigma$-monodromically full (respectively, strictly $\Sigma$-monodromically full; quasi-$\Sigma$-monodromically full) point with respect to $X/S$.

Then it follows immediately from the various definitions involved that the implication

$$(i) \implies (ii)$$

holds.

The following result is a result essentially obtained in [11] (cf. [11], Theorem 1.2). Note that in [11], the following theorem in the case where $\Sigma$ is of cardinality one, and $k$ is a number field was proven. However, by a similar argument used in the proof of [11], Theorem 1.2, one may prove the following theorem.

**Theorem 2.3 (Existence of many monodromically full points).** Let $k$ be a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0); $\overline{k}$ an algebraic closure of $k$; $S$ a scheme that is connected, regular, of finite type, and separated
over $k$; $X$ a hyperbolic curve over $S$ (cf. Definition 1.1, (ii)); $\Sigma$ a nonempty finite set of prime numbers; $S^\text{MF} \subseteq S(\overline{k})$ the subset of $S(\overline{k})$ consisting of closed points of $S$ which are strictly $\Sigma$-monodromically full with respect to $X/S$ (cf. Definition 2.1, (ii)). Fix an inclusion $\overline{k} \hookrightarrow \mathbb{C}$; in particular, we obtain an inclusion $S(\overline{k}) \hookrightarrow S(\mathbb{C})$. Then the subset $S^\text{MF} \subseteq (S(\overline{k}) \subseteq S(\mathbb{C}))$ is dense with respect to the complex topology of $S(\mathbb{C})$. If, moreover, $S$ is rational (i.e., there exists an open subscheme of $S$ which is isomorphic to an open subscheme of $\mathbb{P}^n_k$ for some positive integer $n$), then the complement $S(k) \setminus (S(k) \cap S^{\text{MF}})$ in $S(k)$ of $S(k) \cap S^{\text{MF}}$ forms a thin set in $S(k)$ in the sense of Hilbert’s irreducibility theorem.

Proof. This follows from the fact that a finitely generated field of characteristic zero is Hilbertian, together with a similar argument to the argument used in the proof of [11], Theorem 1.2, by replacing [11], Lemma 3.1 (respectively, [11], Lemma 3.3) by Lemma 2.4 (respectively, Lemma 2.5) below.

Lemma 2.4 (Existence of a certain open subgroup). Let $G$ be a profinite group, $\Sigma$ a nonempty finite set of prime numbers, and for each $l \in \Sigma$, $G \rightarrow Q_l$ a quotient of $G$ which is topologically finitely generated and almost pro-$l$ (cf. the discussion entitled “Topological groups” in §0). Then there exists a normal open subgroup $N \subseteq G$ of $G$ satisfying the following condition: If $H$ is a profinite group and $H \rightarrow G$ is a continuous homomorphism such that the composite $H \rightarrow G \rightarrow G/N$ is surjective, then the composite $H \rightarrow G \rightarrow Q_l$ is surjective for each $l \in \Sigma$.

Proof. If $\Sigma$ is of cardinality one, then Lemma 2.4 follows from [11], Lemma 3.1; in particular, for each $l \in \Sigma$, there exists a normal open subgroup $N_l \subseteq G$ satisfying the following condition: If $H$ is a profinite group and $H \rightarrow G$ is a continuous homomorphism such that the composite $H \rightarrow G \rightarrow G/N_l$ is surjective, then the composite $H \rightarrow G \rightarrow Q_l$ is surjective. Now write $N \overset{\text{def}}{=} \bigcap_{l \in \Sigma} N_l \subseteq G$. Then it is immediate that this normal open subgroup $N$ of $G$ satisfies the condition in the statement of Lemma 2.4. This completes the proof of Lemma 2.4.

Lemma 2.5 (Finitely generatedness of the images of outer monodromy representations). Let $k$ be a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0), $S$ a scheme that is connected and of finite type over $k$, $X$ a hyperbolic curve over $S$, and $l$ a prime number. Suppose that $S$ is regular and separated over $k$. Then the quotient $\text{Im}(\rho^{(l)}_{X/S})$ of $\pi_1(S)$ is topologically finitely generated.

Proof. To verify Lemma 2.5, it is immediate that by replacing $k$ by a finite extension of $k$, we may assume without loss of generality that
$S$ is geometrically connected over $k$ and that $S$ has a $k$-rational point $s \in S(k)$. Then we have an exact sequence

$$1 \rightarrow \pi_1(S \otimes_k \overline{k}) \rightarrow \pi_1(S) \rightarrow G_k \rightarrow 1.$$ 

Since $\pi_0(S \otimes_k \overline{k})$ is topologically finitely generated (cf. [7], Exposé II, Théorème 2.3.1), to verify Lemma 2.5, it suffices to show that the image of the composite

$$G_k \rightarrow \pi_1(S) \xrightarrow{\rho^{(l)}_{X/S}} \text{Out}(\Delta^{(l)}_{g,r})$$

— where the first arrow is the homomorphism (which is determined up to $\pi_1(S \otimes_k \overline{k})$-inner automorphism) induced by $s \in S(k)$, and $(g, r)$ is the type of the hyperbolic curve $X$ over $S$ — is topologically finitely generated; in particular, since the above composite coincides with the pro-$l$ outer monodromy representation $\rho^{(l)}_{X_s/k}$ associated to the hyperbolic curve $X_s$ over $k$ obtained as the fiber of $X \rightarrow S$ at $s \in S(k)$, to verify Lemma 2.5 — by replacing $X$ by $X_s$ — we may assume without loss of generality that $S = \text{Spec } k$.

Since $k$ is finitely generated field of characteristic zero, there exist a finite extension $k' \subseteq \overline{k}$ of $k$, a subfield $k_0 \subseteq k'$ of $k'$, and a scheme $V_0$ over $k_0$ satisfying the following conditions:

(i) $k_0$ is a number field (cf. the discussion entitled “Numbers” in §0).

(ii) $V_0$ is regular, separated, geometrically connected, and of finite type over $k_0$.

(iii) $V_0$ has a $k_0$-rational point $v \in V_0(k_0)$.

(iv) The function field of $V_0$ is isomorphic to $k_0$.

(v) The hyperbolic curve $X \otimes_k k'$ over $k'$ extends to a hyperbolic curve $X_0$ over $V_0$.

Now since the natural homomorphism $\pi_1(\text{Spec } k') \rightarrow \pi_1(V_0)$ (cf. (iv)) is surjective (cf. (ii)), and the pro-$l$ outer monodromy representation $\rho^{(l)}_{X_0/V_0}$ factors through $\rho^{(l)}_{X_{0,v}/V_0}$ (cf. (v)), to verify Lemma 2.5, it suffices to show that the image $\text{Im}(\rho^{(l)}_{X_0/V_0})$ is topologically finitely generated. Moreover, by the existence of the exact sequence (cf. (ii))

$$1 \rightarrow \pi_1(V_0 \otimes_{k_0} \overline{k}_0) \rightarrow \pi_1(V_0) \rightarrow \text{Gal}(\overline{k}_0/k_0) \rightarrow 1$$

— where $\overline{k}_0$ is the algebraic closure of $k_0$ determined by $\overline{k}$ — together with the fact that $\pi_1(V_0 \otimes_{k_0} \overline{k}_0)$ is topologically finitely generated (cf. [7], Exposé II, Théorème 2.3.1), to verify Lemma 2.5, it suffices to show that the image of the composite

$$\text{Gal}(\overline{k}_0/k_0) \rightarrow \pi_1(V_0) \xrightarrow{\rho^{(l)}_{X_{0,v}/V_0}} \text{Out}(\Delta^{(l)}_{g,r})$$

— where the first arrow is the homomorphism (which is determined up to $\pi_1(V_0 \otimes_{k_0} \overline{k}_0)$-inner automorphism) induced by $v \in V_0(k_0)$ (cf. (iii))
is topologically finitely generated. On the other hand, since $k_0$ is a number field (cf. (i)), it follows from [11], Lemma 3.1, that the image of the above composite is topologically finitely generated, as desired. This completes the proof of Lemma 2.5.

By Theorem 2.3, we obtain the following result.

**Corollary 2.6 (Existence of many monodromically full hyperbolic curves).** Let $k$ be a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0), $\overline{k}$ an algebraic closure of $k$, $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$, $M_{g,[r]}$ the moduli stack of hyperbolic curves of type $(g, r)$ over $k$ (cf. Definition 1.2, (ii)), $M_{g,[r]}$ the coarse moduli space associated to $M_{g,[r]}$, and $\Sigma$ a nonempty finite set of prime numbers. Fix an inclusion $\overline{k} \leftarrow \mathbb{C}$. Then the subset of $M_{g,[r]}(\mathbb{C})$ of $\mathbb{C}$-valued points $s \in M_{g,[r]}(\mathbb{C})$ satisfying the following condition $(*)_{MF}$ is dense with respect to the complex topology of $M_{g,[r]}(\mathbb{C})$: 

$$(*)_{MF}: \text{There exists a subfield } k' \subseteq \overline{k} (\subseteq \mathbb{C}) \text{ containing } k \text{ and a morphism } s_{k'}: \text{Spec } k' \to M_{g,[r]} \text{ such that the hyperbolic curve corresponding to } s_{k'} \text{ is a } \Sigma\text{-monodromically full hyperbolic curve over } k' \text{ (cf. Definition 2.2, (i))},$$

and, moreover, $s: \text{Spec } \mathbb{C} \to M_{g,[r]}$ factors through the composite $\text{Spec } k' \xrightarrow{s_{k'}} M_{g,[r]} \to M_{g,[r]}$.

3. **Relationship between monodromic fullness and certain properties of hyperbolic curves**

In the present §3, we consider the relationship between monodromic fullness and certain properties of hyperbolic curves (cf. Propositions 3.4, 3.6, 3.8 below). In the present §3, let $(g, r)$ be a pair of nonnegative integers such that $2g - 2 + r > 0$.

**Definition 3.1.** We shall write

$$G_{g,r} \overset{\text{def}}{=} \left\{ \begin{array}{ll} \{1\} & \text{(if } 2g - 2 + r \geq 3) \\
\mathbb{Z}/2\mathbb{Z} & \text{(if } (g, r) = (1, 1), (1, 2), \text{ or } (2, 0)) \\
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{(if } (g, r) = (0, 4)) \\
\mathfrak{S}_3 & \text{(if } (g, r) = (0, 3)). \end{array} \right.$$ 

It seems to the author that the following proposition is likely to be well-known.

**Proposition 3.2 (Automorphisms of general hyperbolic curves).** Suppose that $k$ is algebraically closed. Then the following hold:

(i) If $X = (C, D \subseteq C)$ is a hyperbolic curve of type $(g, r)$ over $k$, then $G_{g,r}$ is isomorphic to a subgroup of the group $\text{Aut}_k(X)$ of automorphisms of $X$ over $k$. 

(ii) There exists a hyperbolic curve \( X = (C, D \subseteq C) \) of type \((g, r)\) over \(k\) such that the group \(\text{Aut}_k(X)\) of automorphisms of \(X\) over \(k\) is isomorphic to \(G_{g,r}\).

Proof. First, we verify assertion (i). If \(2g - 2 + r \geq 3\), then assertion (i) is immediate. If \((g, r) = (0, 3)\) or \((0, 4)\), then assertion (i) may be verified by the fact that \(\text{Aut}_k(C)\) — note that \(C\) is isomorphic to \(\mathbb{P}^1_k\) over \(k\) — is isomorphic to \(\text{PGL}_2(k)\), together with a straightforward calculation. (Note that if \((g, r) = (0, 4)\), i.e., \(X = (C, D \subseteq C)\) is isomorphic to \((\mathbb{P}^1_k, \{0, 1, \infty, x\} \subseteq \mathbb{P}^1_k)\) for some \(x \in k \setminus \{0, 1\}\), then the following two automorphisms generate a subgroup of \(\text{Aut}_k(X)\) which is isomorphic to \(G_{0,4} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\):

\[
C \simeq \mathbb{P}^1_k \xrightarrow{t/s} \mathbb{P}^1_k \simeq C, \\
C \simeq \mathbb{P}^1_k \xrightarrow{t/s} \mathbb{P}^1_k \simeq C.
\]

Next, suppose that \((g, r) = (1, 1)\) or \((1, 2)\). If \((g, r) = (1, 1)\) (respectively, \((1, 2)\)), then write \(\{o\}\) (respectively, \(\{o, x\}\) \(\subseteq C(k)\) for the set of the marked divisor “\(D\)” of the hyperbolic curve \(X = (C, D \subseteq C)\). Then since \(g = 1\), by regarding the marked \(k\)-rational point \(o\) of \(C\) as an origin, one may regard \(C\) as an abelian group scheme over \(k\) whose identity section is the section determined by the \(k\)-rational point \(o\). Thus, we have an automorphism

\[
\begin{cases}
C \ni t \mapsto -t \in C & \text{(if } r = 1) \\
C \ni t \mapsto x - t \in C & \text{(if } r = 2)
\end{cases}
\]

over \(k\) of order 2 that preserves \(D = \{o\}\) (respectively, \(= \{o, x\}\) \(\subseteq C\); in particular, \(G_{g,r} = \mathbb{Z}/2\mathbb{Z}\) is isomorphic to a subgroup of \(\text{Aut}_k(X)\). Next, suppose that \((g, r) = (2, 0)\). Then since the proper curve \(C\) is hyperelliptic, we have an automorphism of \(C\) of order 2; in particular, \(G_{2,0} = \mathbb{Z}/2\mathbb{Z}\) is isomorphic to a subgroup of \(\text{Aut}_k(X) = \text{Aut}_k(C)\). This completes the proof of assertion (i).

Finally, we verify assertion (ii). If \(2g - 2 + r \geq 3\), then assertion (ii) follows immediately from [12], Theorem C. If \((g, r) = (0, 3)\) or \((0, 4)\), then assertion (ii) may be verified by the fact that \(\text{Aut}_k(C)\) — note that \(C\) is isomorphic to \(\mathbb{P}^1_k\) over \(k\) — is isomorphic to \(\text{PGL}_2(k)\), together with a straightforward calculation. Next, suppose that \((g, r) = (1, 1)\) or \((1, 2)\). Then since \(g = 1\), one may regard \(C\) as an abelian group scheme over \(k\). Moreover, as is well-known, there exists a hyperbolic curve \(X = (C, D \subseteq C)\) of type \((1, 1)\) (respectively, \((1, 2)\)) over \(k\) such that \(\text{Aut}_k(C)\) is isomorphic to \(C(k) \times \{\pm 1\}\) — where the action of \(\{\pm 1\}\) on \(C(k)\) is the natural action of \(\{\pm 1\}\) on an abelian group \(C(k)\). Now assertion (ii) in the case where \((g, r) = (1, 1)\) or \((1, 2)\) follows from this fact that \(\text{Aut}_k(C)\) is isomorphic to \(C(k) \times \{\pm 1\}\), together with a straightforward calculation. Next, suppose that \((g, r) = (2, 0)\).
Then the assertion follows from, for example, [21], Theorem 1. This completes the proof of assertion (ii).

**Definition 3.3.** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Then we shall say that $X$ has no special symmetry if the group $\text{Aut}(\mathcal{X}(X \otimes_k \bar{k}))$ of automorphisms of $X \otimes_k \bar{k}$ over $\bar{k}$ is isomorphic to $G_{g,r}$.

**Proposition 3.4** (Quasi-monodromic fullness and automorphisms of hyperbolic curves). Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Suppose that $X$ is quasi-$\Sigma$-monodromically full for a nonempty set of prime numbers $\Sigma$ and that $k$ is a generalized sub-$l$-adic field (cf. the discussion entitled “Numbers” in §0) for some $l \in \Sigma$. Then $X$ has no special symmetry.

**Proof.** Let $X_0$ be a hyperbolic curve of type $(g, r)$ over a finite extension $k_0 \subseteq \bar{k}$ of $k$ such that $\text{Aut}(\mathcal{X}(X_0 \otimes_{k_0} \bar{k})) \simeq G_{g,r}$ (cf. Lemma 3.2, (ii)). Then since $\Delta_{g,r}^\Sigma$ is center-free (cf. Remark 1.3.1), it follows from [14], Theorem 4.12, together with [17], Corollary 1.5.7, that there exist natural bijections

$$\text{Aut}(\mathcal{X}(X \otimes_k \bar{k})) \sim Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X/k}^\Sigma)) ;$$

$$G_{g,r} \sim \text{Aut}(\mathcal{X}(X_0 \otimes_{k_0} \bar{k})) \sim Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X_0/k_0}^\Sigma)).$$

On the other hand, since $X$ is quasi-$\Sigma$-monodromically full, it follows immediately from the definition of the term “quasi-$\Sigma$-monodromically full” that

$$Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X/k}^\Sigma)) = Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X_0/k_0}^\Sigma))$$

(cf. the discussion entitled “Topological groups” in §0). Thus, since $\text{Im}(\rho_{X_0/k_0}^\Sigma) \subseteq \text{Im}(\rho_{X/k}^\Sigma)$, we obtain that

$$\text{Aut}(\mathcal{X}(X \otimes_k \bar{k})) \sim Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X/k}^\Sigma)) = Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X_0/k_0}^\Sigma))$$

$$\subseteq Z^{\text{loc}}_{\text{Out}(\Delta_{g,r}^\Sigma)}(\text{Im}(\rho_{X_0/k_0}^\Sigma)) \simeq G_{g,r}$$

(cf. the discussion entitled “Topological groups” in §0); in particular, it follows immediately from Lemma 3.2, (i), that $X$ has no special symmetry. This completes the proof of Proposition 3.4.

**Definition 3.5.** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$ and $\Sigma$ a nonempty set of prime numbers. Suppose that $g \neq 0$. Then we shall say that $X$ is of $\Sigma$-AIJ-type (where the “AIJ” stands for “absolutely irreducible Jacobian”) if the following condition is satisfied: For any prime number $l \in \Sigma$ and finite extension $k' \subseteq \bar{k}$ of $k$ such that $X(k') \neq \emptyset$, the $l$-adic Tate module of the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k k'$ is irreducible as a $G_{k'}$-module.
Remark 3.5.1. It follows immediately from the definition of the term “of AIJ-type” that if a hyperbolic curve $X$ over $k$ is of $\Sigma$-AIJ-type for some nonempty set of prime numbers $\Sigma$, then the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$ is simple.

Proposition 3.6 (Quasi-monodromic fullness and the absolute irreducibility of Jacobian variety). Let $X$ be a hyperbolic curve of type $(g,r)$ over $k$ and $\Sigma$ a nonempty set of prime numbers. Suppose that $k$ is finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0), that $g \neq 0$, and that $X$ is quasi-$\Sigma$-monodromically full. Then $X$ is of $\Sigma$-AIJ-type. In particular, the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$ is simple (cf. Remark 3.5.1).

Proof. To prove Proposition 3.6, it follows from the definition of the term “of AIJ-type” that we may assume without loss of generality that $\Sigma$ is of cardinality one. Write $H^\Sigma_{g,r}$ for the abelian quotient of $\Delta^\Sigma_{g,r}$ by the normal closed subgroup generated by the cuspidal inertia subgroups of $\Delta^\Sigma_{g,r}$ and the closure of the commutator subgroup of $\Delta^\Sigma_{g,r}$.

(Thus, if $g \geq 2$, then $H^\Sigma_{g,r}$ is naturally isomorphic to $(\Delta^\Sigma_{g,0})^{\text{ab}}$.) Now it follows from a similar argument to the argument used in Remark 1.5.1, (ii), that the pro-$\Sigma$ outer representation $\rho^\Sigma_{g,r} : \pi_1(M_{g,[r]}) \to \text{Out}(\Delta^\Sigma_{g,r})$ induces a pro-$\Sigma$ representation $\rho : \pi_1(M_{g,[r]}) \to \text{Aut}(H^\Sigma_{g,r})$. Moreover, as is well-known, the following holds (cf. also Remark 1.3.1):

Let $k' \subseteq \overline{k}$ be a finite extension of $k$ such that $X(k') \neq \emptyset$.
Then there exists an isomorphism of $H^\Sigma_{g,r}$ with the $\Sigma$-adic Tate module of the Jacobian variety of the compactification of $X \otimes_k k'$ such that, under this isomorphism, the action of $G_{k'}$ on $H^\Sigma_{g,r}$ determined by $\rho$, $s_{X/k'}$ (cf. Definition 1.3, (iii)) and the natural action of $G_{k'}$ on the pro-$\Sigma$ Tate module coincide.

Therefore, Proposition 3.6 follows from the definition of the term “quasi-monodromically full”, together with the existence of a hyperbolic curve of $\Sigma$-AIJ-type over a number field (cf. e.g., [4], the proof of Proposition 4, also [4], Remark 5, (iv), (v)).

Definition 3.7. Let $X$ be a hyperbolic curve of type $(g,r)$ over $k$. Suppose that $g \neq 0$. Then we shall say that $X$ has a JCM-component (where the “JCM” stands for “Jacobian complex multiplication”) if there exist a nontrivial simple abelian variety $A$ over $\overline{k}$ such that $\text{End}_{\overline{k}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to a number field of degree $2\dim(A)$ and a nontrivial morphism over $\overline{k}$ from $A$ to the Jacobian variety of the compactification of the hyperbolic curve $X \otimes_k \overline{k}$.

Remark 3.7.1. Let $X$ be a hyperbolic curve of type $(1,1)$ over $k$. Then it follows from the various definitions involved that $X$ has a JCM-component if and only if the elliptic curve determined by $X$ admits
complex multiplication over $\overline{k}$ — i.e., the ring of endomorphisms of the elliptic curve determined by $X$ over $\overline{k}$ is isomorphic to an order of an imaginary quadratic field.

**Proposition 3.8 (Quasi-monodromic fullness and complex multiplication).** Let $X$ be a hyperbolic curve of type $(g, r)$ over $k$. Suppose that $k$ is finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0), that $g \neq 0$, and that $X$ is quasi-$\Sigma$-monodromically full for a nonempty set of prime numbers $\Sigma$. Then $X$ does not have a JCM-component.

**Proof.** This follows immediately from Proposition 3.6, together with [23], Corollary 2 to Theorem 5.

---

4. **Moduli stacks of hyperbolic curves of genus zero**

In the present §, we consider the moduli stacks of hyperbolic curves of genus zero. In the present §, let $r \geq 3$ be an integer and $l$ a prime number.

**Lemma 4.1 (Moduli stacks of hyperbolic curves of genus zero).**

(i) The moduli stack $\mathcal{M}_{0,r}$ is isomorphic to the $(r - 3)$-rd configuration space of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ over $k$, i.e., the open subscheme of the fiber product over $k$ of $r - 3$ copies of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ obtained as the complement of the various diagonal divisors.

(ii) The natural homomorphism $\mathcal{S}_r \rightarrow \text{Aut}_k(\mathcal{M}_{0,r})$ determined by the $\mathcal{S}_r$-covering $\mathcal{M}_{0,r} \rightarrow \mathcal{M}_{0,[r]}$ is surjective. In particular, any automorphism $\phi$ of $\mathcal{M}_{0,r}$ over $k$ is an automorphism over $\mathcal{M}_{0,[r]}$, i.e., there exists a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{0,r} & \xrightarrow{\phi} & \mathcal{M}_{0,r} \\
\downarrow & & \downarrow \\
\mathcal{M}_{0,[r]} & \rightarrow & \mathcal{M}_{0,[r]}
\end{array}
\]

— where the vertical arrows are natural morphisms, and the lower horizontal arrow is the identity automorphism of $\mathcal{M}_{0,[r]}$.

**Proof.** Assertions (i), (ii) are well-known. (Concerning assertion (ii), see [17], discussion following Theorem A in §0.)

**Lemma 4.2 (Universal geometric monodromy outer representations of genus zero).**

(i) The quotient $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k}) \rightarrow \text{Im}(\rho_{0,r}^{[1]}_{\text{geom}})$ of $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k})$ coincides with the maximal pro-$l$ quotient of $\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k})$. In particular, there exists a natural homomorphism

\[
\text{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r})) \rightarrow \text{Aut}_{\text{Im}(\rho_{0,r}^{[1]}_{\text{geom}})(\text{Im}(\rho_{0,r}^{[1]}_{\text{geom}})))}
\]
(cf. Lemma 1.6, (i)).

(ii) The abelianization of $\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})$ is a free $\mathbb{Z}_l$-module of rank $(r - 2)(r + 1)/2$.

Proof. Assertion (i) follows from [3], Remark following the proof of Theorem 1, together with Lemma 4.1, (i). Assertion (ii) follows immediately from [18], Corollary 2.5, together with Lemma 4.1, (i). Indeed, it follows [18], Corollary 2.5, together with Lemma 4.1, (i), that $\text{rank}\mathbb{Z}_l(\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})) = \sum_{i=3}^{r} \text{rank}\mathbb{Z}_l((\Delta_{0,i})^{ab}) = \sum_{i=3}^{r} (i - 1) = (r - 2)(r + 1)/2$.

Lemma 4.3 (Universal monodromy outer representations of genus zero). Suppose that $k$ is a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §6). Then the following hold:

(i) The image $\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})$ is pro-$l$ and slim.

(ii) The image $\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})$ is slim. If, moreover, $k$ contains a primitive $l$-th root of unity, then the image $\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})$ is pro-$l$.

(iii) The composite of natural homomorphisms

$$
\text{Aut}_k(\mathcal{M}_{0,r}) \longrightarrow \text{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r}))/\text{Inn}(\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k}))
$$

$$
\longrightarrow \text{Aut}_{\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})}(\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}))/\text{Inn}(\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}))
$$

(cf. Lemma 4.2, (i)) is bijective (cf. Remark 4.3.1).

(iv) The composite of natural maps

$$
\mathcal{M}_{0,r}(k) \longrightarrow \text{Hom}_{\text{Gal}(k)}(G_k, \pi_1(\mathcal{M}_{0,r}))/\text{Inn}(\pi_1(\mathcal{M}_{0,r} \otimes_k \overline{k}))
$$

$$
\longrightarrow \text{Hom}_{\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})}(G_k, \text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}))/\text{Inn}(\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}))
$$

(cf. Lemma 4.2, (i)) is injective.

(v) The composite of natural maps

$$
\mathcal{M}_{0,[r]}(k) \longrightarrow \text{Hom}_{\text{Gal}(k)}(G_k, \pi_1(\mathcal{M}_{0,[r]}))/\text{Inn}(\pi_1(\mathcal{M}_{0,[r]} \otimes_k \overline{k}))
$$

$$
\longrightarrow \text{Hom}_{\text{Im}(\rho_{0,[r]}^{[l]}_{\text{geom}})}(G_k, \text{Im}(\rho_{0,[r]}^{[l]}_{\text{geom}}))/\text{Inn}(\text{Im}(\rho_{0,[r]}^{[l]}_{\text{geom}}))
$$

(cf. Lemma 4.2, (i)) is injective.

Proof. Assertion (i) follows from [16], Proposition 2.2, (ii), together with Lemmas 4.1, (i); 4.2, (i).

Next, we verify assertion (ii). Since we have an exact sequence

$$
1 \longrightarrow \text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}) \longrightarrow \text{Im}(\rho_{0,r}^{[l]}_{\text{geom}}) \longrightarrow \text{Im}(\rho_{0,3}^{[l]}_{\text{geom}}) \longrightarrow 1
$$

(cf. Lemma 1.6, (i)), it follows from assertion (i) that to verify the fact that $\text{Im}(\rho_{0,r}^{[l]}_{\text{geom}})$ is slim (respectively, pro-$l$), it suffices to show that $\text{Im}(\rho_{0,3}^{[l]}_{\text{geom}})$ is slim (respectively, pro-$l$). Now we prove the fact that $\text{Im}(\rho_{0,3}^{[l]}_{\text{geom}})$ is slim. It follows from a similar argument to the argument
used in the proof of Proposition 1.7, (ii), together with Lemma 4.1, (i),
that we obtain a natural bijection
\[
\text{Aut}(\mathbb{P}_T \setminus \{0, 1, \infty\}) \xrightarrow{\sim} Z^{\text{loc}}_{\text{Out}(\Delta_{0,3}^{(1)})}(\text{Im}(\rho_{0,3}^{(l)})).
\]

Therefore, by comparing the natural actions of \(\text{Aut}(\mathbb{P}_T \setminus \{0, 1, \infty\})\) and \(\text{Im}(\rho_{0,3}^{(l)})\) on the set of the conjugacy classes of the cuspidal inertia subgroups of \(\Delta_{0,3}^{(l)}\) (cf. Remark 1.3.1), it follows that the intersection
\[
Z^{\text{loc}}_{\text{Out}(\Delta_{0,3}^{(1)})}(\text{Im}(\rho_{0,3}^{(l)})) \cap \text{Im}(\rho_{0,3}^{(l)})
\]
is trivial; in particular, the local center \(Z^{\text{loc}}_{\text{Out}(\Delta_{0,3}^{(1)})}(\text{Im}(\rho_{0,3}^{(l)}))\) of \(\text{Im}(\rho_{0,3}^{(l)})\) is trivial. This completes the proof of the fact that \(\text{Im}(\rho_{0,3}^{(l)})\) is slim. On the other hand, it follows immediately from [1], Theorems A, B, that if \(k\) contains a primitive \(l\)-th root of unity, then \(\text{Im}(\rho_{0,3}^{(l)})\) is pro-\(l\). This completes the proof of assertion (ii).

Next, we prove assertion (iii). By considering the action of \(\text{Aut}_k(\mathcal{M}_{0,r})\) on the set of the conjugacy classes of the cuspidal inertia subgroups of \(\text{Im}(\rho_{0,r}^{(l)}_{-\text{geom}})\), the injectivity of the composite in question follows immediately from Lemmas 4.1, (i), (ii); 4.2, (i), together with Remark 1.3.1. Now we verify the surjectivity of the composite in question by induction on \(r\). If \(r = 3, 4\), then the surjectivity of the composite in question follows from [14], Theorem 4.12, together with Lemmas 4.1, (i); 4.2, (i).

Suppose that \(r \geq 5\) and that the composite of natural homomorphisms
\[
\text{Aut}_k(\mathcal{M}_{0,r-1}) \longrightarrow \text{Aut}_{G_k}(\pi_1(\mathcal{M}_{0,r-1})) / \text{Im}(\rho_{0,r-1}^{(l)})
\]
is bijective. Let \(\alpha\) be an automorphism of \(\text{Im}(\rho_{0,r}^{(l)})\) over \(\text{Im}(\rho_{0,3}^{(l)})\). Then it follows immediately from [17], Theorem 3.1.13 (note that [17], Theorem 3.1.13, is valid for a finitely generated field of characteristic zero, even though in [17], this result for a number field is only stated), that — by compositing a suitable automorphism of \(\text{Im}(\rho_{0,r}^{(l)})\) over \(\text{Im}(\rho_{0,3}^{(l)})\) arising from an element of \(\text{Aut}_k(\mathcal{M}_{0,r})\) — we may assume without loss of generality that \(\alpha\) preserves the kernel \(\Delta^{(l)}_{0,r-1} \subseteq \text{Im}(\rho_{0,r}^{(l)}_{-\text{geom}})\) of the natural surjection \(\text{Im}(\rho_{0,r}^{(l)}) \twoheadrightarrow \text{Im}(\rho_{0,r-1}^{(l)})\) (cf. Lemmas 4.1, (i); 4.2, (i)). Moreover, it follows immediately from the above induction hypothesis that — again by compositing a suitable automorphism of \(\text{Im}(\rho_{0,r}^{(l)})\) over \(\text{Im}(\rho_{0,3}^{(l)})\) arising from an element of \(\text{Aut}_k(\mathcal{M}_{0,r})\) — we may assume without loss of generality that the automorphism of \(\text{Im}(\rho_{0,r-1}^{(l)})\) induced by \(\alpha\) is the identity automorphism of \(\text{Im}(\rho_{0,r-1}^{(l)})\), i.e., we obtain
a commutative diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & \Delta_{0,r-1}^{(l)} & \longrightarrow & \text{Im}(\rho_{0,r}^{(l)}) & \longrightarrow & \text{Im}(\rho_{0,r-1}^{(l)}) & \longrightarrow & 1 \\
\downarrow & & \alpha \downarrow & & & & & & \\
1 & \longrightarrow & \Delta_{0,r-1}^{(l)} & \longrightarrow & \text{Im}(\rho_{0,r}^{(l)}) & \longrightarrow & \text{Im}(\rho_{0,r-1}^{(l)}) & \longrightarrow & 1
\end{array}
\]
— where the horizontal sequences are exact, and the right-hand vertical arrow is the identity automorphism. Therefore, it follows immediately from [14], Theorem 4.12, together with Lemma 4.4, (ii), below, that \( \alpha \) arises from an automorphism of \( M_{0,r} \) over \( k \). This completes the proof of assertion (iii).

Assertion (iv) follows immediately from [13], Theorem C, together with Lemmas 4.1, (i); 4.2, (i). Assertion (v) follows from [14], Remark following Theorem 4.12 (cf. also the proof of [13], Theorem C).

Remark 4.3.1. In [17], Theorem A, the bijectivity of the composite of natural homomorphisms
\[
\text{Aut}_k(M_{0,r}) \longrightarrow \text{Aut}_{G_k}(\pi_1(M_{0,r})) / \text{Inn} \left( \pi_1(M_{0,r} \otimes_k \overline{k}) \right)
\longrightarrow \text{Aut}_{\text{Im} \left( \rho_{0,r}^{(l)} \right)} \left( \text{Im} \left( \rho_{0,r}^{(l)} \right) / \text{Inn} \left( \text{Im} \left( \rho_{0,r}^{(l)} \right) \right) \right)
\]
in the case where \( p \) is odd was proven.

Lemma 4.4. Let \( S \) be a connected normal scheme and \( \eta_S \to S \) the generic point of \( S \). Then the following hold:

(i) Let \( T \to S \) be a scheme that is finite over \( S \). Then the natural morphism \( \text{Hom}_S(S,T) \to \text{Hom}_S(\eta_S,T) \) is bijective.

(ii) Let \( X_1, X_2 \) be hyperbolic curves over \( S \). Then the natural morphism
\[
\text{Isom}_S(X_1,X_2) \longrightarrow \text{Isom}_{\eta_S}(X_1 \times_S \eta_S, X_2 \times_S \eta_S)
\]
is bijective.

Proof. First, we consider assertion (i). The injectivity of the morphism in question follows immediately from the fact that the natural morphism \( \eta_S \to S \) is scheme-theoretically dense. To verify the surjectivity of the morphism in question, let \( \phi : \eta_S \to T \) be a morphism over \( S \). Write \( F \subseteq T \) for the scheme-theoretic image of \( \phi \). Then it follows immediately from the various definitions involved that \( F \) is integral, and the composite \( F \to T \to S \) is birational and finite. Thus, since \( S \) is normal, it follows from Zariski’s main theorem (cf. [6], Corollaire 4.4.9) that the composite \( F \to T \to S \) is an isomorphism; in particular, \( \phi \) extends to a morphism \( S \to T \) over \( S \).

Finally, we consider assertion (ii). It follows from, for example, [5], Theorem 1.11, that the functor \( \text{Isom}_S(X_1,X_2) \) is represented by a scheme that is finite and unramified over \( S \). Thus, assertion (ii) follows from assertion (i).
5. A Grothendieck conjecture-type lemma for certain images of the universal monodromy

In the present §5, we prove a Grothendieck conjecture-type lemma for certain images of the universal monodromy (cf. Lemma 5.2 below). In the present §5, let $r \geq 3$ be an integer and $l$ a prime number. Suppose, moreover, that $k$ is a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0). Let us fix an isomorphism $\text{Im}(\rho_{0,[r]}^{(l)}/\text{Im}(\rho_{0,r}^{(l)}) \cong \mathcal{S}_r$ (cf. Lemma 1.6, (iii)).

For $i = 1, 2$, let

$$H_i \subseteq \text{Im}(\rho_{0,[r]}^{(l)})$$

be an open subgroup of $\text{Im}(\rho_{0,[r]}^{(l)})$ that contains the normal open subgroup $\text{Im}(\rho_{0,r}^{(l)}) \subseteq \text{Im}(\rho_{0,[r]}^{(l)})$,

$$\text{Im}(\rho_{0,[r]}^{(l)}) \to \text{Im}(\rho_{0,r}^{(l)})$$

the image of the composite $H_i \hookrightarrow \text{Im}(\rho_{0,[r]}^{(l)}) \to \text{Im}(\rho_{0,r}^{(l)})/\text{Im}(\rho_{0,r}^{(l)}) \cong \mathcal{S}_r$, and

$$H_i^{\text{geom}} \subseteq H_i$$

the kernel of the composite $H_i \hookrightarrow \text{Im}(\rho_{0,[r]}^{(l)}) \to \text{Im}(\rho_{0,3}^{(l)})$, i.e., $H_i^{\text{geom}} \overset{\text{def}}{=} H_i \cap \text{Im}(\rho_{0,[r]}^{(l)})$ (cf. Lemma 1.6, (ii)). Thus, $H_i$ fits into the following exact sequences:

$$1 \to H_i^{\text{geom}} \to H_i \to \text{Im}(\rho_{0,3}^{(l)}) \to 1;$$

$$1 \to \text{Im}(\rho_{0,r}^{(l)}) \to H_i \to Q_i (\subseteq \mathcal{S}_r) \to 1. $$

(Here, the surjectivity of $H_i \to \text{Im}(\rho_{0,3}^{(l)})$ follows from Lemma 1.6, (i).)

By the various definitions involved, this open subgroup $H_i \subseteq \text{Im}(\rho_{0,[r]}^{(l)})$ corresponds to the intermediate connected finite étale covering

$$[\mathcal{M}_{0,r}/Q_1] \to \mathcal{M}_{0,[r]}$$

def of the $\mathcal{S}_r$-covering $\mathcal{M}_{0,r} \to [\mathcal{M}_{0,r}/\mathcal{S}_r] = \mathcal{M}_{0,[r]}$ — where $[\mathcal{M}_{0,r}/(-)]$ is the quotient of $\mathcal{M}_{0,r}$ by “$(-)$” in the sense of stacks. Now we shall write

$$\text{Aut}_k^{Q_1,Q_2}(\mathcal{M}_{0,r})$$

for the set of automorphisms of $\mathcal{M}_{0,r}$ over $k$ which is compatible with the respective actions $Q_1 \hookrightarrow \mathcal{S}_r \to \text{Aut}_k(\mathcal{M}_{0,r})$ and $Q_2 \hookrightarrow \mathcal{S}_r \to \text{Aut}_k(\mathcal{M}_{0,r})$ relative to an isomorphism $Q_1 \sim Q_2$ of finite groups, i.e., the subset of $\text{Aut}_k(\mathcal{M}_{0,r})$ consisting of automorphisms $\phi$ of $\mathcal{M}_{0,r}$ over $k$ which fit into a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_{0,r} & \xrightarrow{\phi} & \mathcal{M}_{0,r} \\
\downarrow & & \downarrow \\
[\mathcal{M}_{0,r}/Q_1] & \to & [\mathcal{M}_{0,r}/Q_2]
\end{array}$$
— where the vertical arrows are natural morphisms, and the horizontal arrows are isomorphisms over $k$. Then we define a map

$$\Phi: \text{Aut}_k^{Q_1, Q_2}(\mathcal{M}_{0, r}) \longrightarrow \text{Isom}_{\text{Im}(\rho_{0,3})}(H_1, H_2)/\text{Inn}(H_2^{\text{geom}})$$

as follows: Let $\phi \in \text{Aut}_k^{Q_1, Q_2}(\mathcal{M}_{0, r})$. Then it follows from the definition of $\text{Aut}_k^{Q_1, Q_2}(\mathcal{M}_{0, r})$ that $\phi$ induces a diagram

$$\begin{array}{ccc}
\pi_1(\mathcal{M}_{0, r}) & \longrightarrow & \pi_1(\mathcal{M}_{0, r}) \\
\downarrow & & \downarrow \\
\pi_1([\mathcal{M}_{0, r}/Q_1]) & \longrightarrow & \pi_1([\mathcal{M}_{0, r}/Q_2])
\end{array}$$

— where the top horizontal arrow is the $\pi_1(\mathcal{M}_{0, r} \otimes_k \overline{k})$-conjugacy class of the automorphism of $\pi_1(\mathcal{M}_{0, r})$ induced by $\phi$, and this diagram commutes up to $\pi_1([\mathcal{M}_{0, r}/Q_2] \otimes_k \overline{k})$-inner automorphism. Thus, by considering the $H_2^{\text{geom}}$-conjugacy class of the isomorphism

$$H_1 = \pi_1([\mathcal{M}_{0, r}/Q_1])/\text{Ker}(\rho_{0, r}^{(l)}) \sim \pi_1([\mathcal{M}_{0, r}/Q_2])/\text{Ker}(\rho_{0, r}^{(l)}) = H_2$$

induced by the lower horizontal arrow in the above diagram (note that by Lemma 4.2, (i), the top horizontal arrow in the above diagram preserves $\text{Ker}(\rho_{0, r}^{(l)}) \subseteq \pi_1(\mathcal{M}_{0, r}))$, we obtain an element $\Phi(\phi)$ of $\text{Isom}_{\text{Im}(\rho_{0,3})}(H_1, H_2)/\text{Inn}(H_2^{\text{geom}})$, as desired.

The purpose of the present § is to prove the surjectivity of this map $\Phi$ under the assumption that

$$(*): \text{prime, } l \text{ is prime to the orders of } Q_1 \text{ and } Q_2.$$

In the rest of the present §, suppose that the above condition $(*)$ is satisfied.

**Lemma 5.1 (Preserving the $\mathcal{M}_{0, r}$-parts).** Let $\phi: H_1 \sim H_2$ be an isomorphism over $\text{Im}(\rho_{0, r}^{(l)})$. Then $\phi(\text{Im}(\rho_{0, r}^{(l)-\text{geom}})) = \text{Im}(\rho_{0, r}^{(l)-\text{geom}})$. If, moreover, $k$ contains a primitive $l$-th root of unity, then $\phi(\text{Im}(\rho_{0, r}^{(l)})) = \text{Im}(\rho_{0, r}^{(l)})$.

**Proof.** It follows immediately from Lemma 4.3, (i), together with the assumption that the condition $(*)$ is satisfied (cf. the discussion preceding Lemma 5.1), that $\text{Im}(\rho_{0, r}^{(l)-\text{geom}}) \subseteq H_i^{\text{geom}}$ is the maximal pro-$l$ closed subgroup of $H_i^{\text{geom}}$; therefore, it follows that $\phi(\text{Im}(\rho_{0, r}^{(l)-\text{geom}})) = \text{Im}(\rho_{0, r}^{(l)-\text{geom}})$. If, moreover, $k$ contains a primitive $l$-th root of unity, then it follows from Lemma 4.3, (ii), together with the assumption that the condition $(*)$ is satisfied (cf. the discussion preceding Lemma 5.1), that $\text{Im}(\rho_{0, r}^{(l)}) \subseteq H_i$ is the maximal pro-$l$ closed subgroup of $H_i$; therefore, it follows that $\phi(\text{Im}(\rho_{0, r}^{(l)})) = \text{Im}(\rho_{0, r}^{(l)})$. \qed

Next, we shall write

$$\tilde{\Phi}: \text{Aut}_k^{Q_1, Q_2}(\mathcal{M}_{0, r}) \longrightarrow \text{Isom}_{\text{Im}(\rho_{0,3}^{(l)})}(H_1, H_2)/\text{Inn}(\text{Im}(\rho_{0, r}^{(l)-\text{geom}}))$$
for the map defined as follows: Let $\phi \in \text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r})$. Then $\phi$ determines an $\text{Im}(\rho_{0,r}^{(l)})$-conjugacy class of an automorphism of $\text{Im}(\rho_{0,3}^{(l)})$ over $\text{Im}(\rho_{0,3}^{(l)})$. Moreover, by the definition of $\text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r})$, this $\text{Im}(\rho_{0,r}^{(l)})$-conjugacy class is compatible with the respective outer actions of $Q_1$ and $Q_2$ on $\text{Im}(\rho_{0,r}^{(l)})$ relative to an isomorphism $Q_1 \sim Q_2$. Therefore, since $\text{Im}(\rho_{0,r}^{(l)})$ is center-free (cf. Lemma 4.3, (ii)), we obtain an $\text{Im}(\rho_{0,r}^{(l)})$-conjugacy class $\Phi^{Q_1, Q_2}(\phi)$ of an isomorphism

$$H_1 \simeq \text{Im}(\rho_{0,r}^{(l)}) \times Q_1 \xrightarrow{\sim} \text{Im}(\rho_{0,r}^{(l)}) \times Q_2 \simeq H_2$$

(cf. the discussion entitled “Topological groups” in §0) over $\text{Im}(\rho_{0,3}^{(l)})$.

Note that by the various definitions involved, the diagram

$$\begin{array}{ccc}
\text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r}) & \xrightarrow{\Phi} & \text{Isom}_{\text{Im}(\rho_{0,3}^{(l)})}(H_1, H_2)/\text{Inn}(\text{Im}(\rho_{0,r}^{(l)}-\text{geom}))) \\
\downarrow & & \downarrow \\
\text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r}) & \xrightarrow{\Phi} & \text{Isom}_{\text{Im}(\rho_{0,3}^{(l)})}(H_1, H_2)/\text{Inn}(H_{2}^{\text{geom}})
\end{array}$$

— where the right-hand vertical arrow is the natural surjection — commutes.

**Lemma 5.2 (A Grothendieck conjecture-type lemma for certain images of the universal monodromy).** In the above diagram, the following hold:

(i) $\Phi$ is injective.

(ii) $\Phi$ is surjective.

(iii) $\Phi$ is surjective. Moreover, for $\phi, \phi' \in \text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r})$, it holds that $\Phi(\phi) = \Phi(\phi')$ if and only if $\phi' \circ \phi^{-1} \in \text{Aut}_{k}(\mathcal{M}_{0,r})$ is an element of the image of the composite $Q_2 \hookrightarrow \mathfrak{S}_r \twoheadrightarrow \text{Aut}(\mathcal{M}_{0,r})$.

**Proof.** First, we consider assertion (i). To prove the injectivity of $\Phi$ — by replacing $k$ by a finite extension of $k$ — we may assume without loss of generality that $k$ contains a primitive $l$-th root of unity (cf. Remark 5.2.1 below). Now we have a commutative diagram

$$\begin{array}{ccc}
\text{Aut}_{k}^{Q_1, Q_2}(\mathcal{M}_{0,r}) & \xrightarrow{\Phi} & \text{Isom}_{\text{Im}(\rho_{0,3}^{(l)})}(H_1, H_2)/\text{Inn}(\text{Im}(\rho_{0,r}^{(l)}-\text{geom}))) \\
\downarrow & & \downarrow \\
\text{Aut}_{k}(\mathcal{M}_{0,r}) & \xrightarrow{\Phi} & \text{Aut}_{\text{Im}(\rho_{0,3}^{(l)})}(\text{Im}(\rho_{0,r}^{(l)}))/\text{Inn}(\text{Im}(\rho_{0,r}^{(l)}-\text{geom})))
\end{array}$$

— where the left-hand vertical arrow is the natural inclusion, the right-hand vertical arrow is the map obtained by restricting elements of $\text{Isom}_{\text{Im}(\rho_{0,3}^{(l)})}(H_1, H_2)/\text{Inn}(\text{Im}(\rho_{0,r}^{(l)}-\text{geom})))$ to $\text{Im}(\rho_{0,r}^{(l)}) \subseteq H_i$ (cf. Lemma 5.1), and the lower horizontal arrow is the homomorphism obtained in Lemma 4.2,
(i). Thus, since the lower horizontal arrow is injective (cf. Lemma 4.3, (iii)), it follows that $\tilde{\Phi}$ is injective. This completes the proof of assertion (i).

Next, we consider assertion (ii). To prove the surjectivity of $\tilde{\Phi}$, it follows from assertion (i), together with Galois descent, by replacing $k$ by a finite extension of $k$, we may assume without loss of generality that $k$ contains a primitive $l$-th root of unity (cf. Remark 5.2.1 below). Let $\phi: H_1 \sim H_2$ be an isomorphism over $\text{Im}(\rho_{0,\Gamma}^{(l)})$. Then it follows from Lemma 5.1 that we obtain a commutative diagram

$$
1 \longrightarrow \text{Im}(\rho_{0,r}^{(l)}) \longrightarrow H_1 \longrightarrow Q_1 \longrightarrow 1
$$

$$
\phi \downarrow \quad \phi \downarrow \quad \downarrow \tilde{\phi}
$$

$$
1 \longrightarrow \text{Im}(\rho_{0,r}^{(l)}) \longrightarrow H_2 \longrightarrow Q_2 \longrightarrow 1
$$

where the horizontal sequences are exact, and the vertical arrows are isomorphisms. Now it follows from Lemma 4.3, (iii), that the $\text{Im}(\rho_{0,r}^{(l)\text{-geom}})$-conjugacy class of the left-hand vertical arrow arises from an automorphism $\phi$ of $\mathcal{M}_{0,r}$ over $k$; moreover, since the above diagram commutes, it follows from assertion (i) that this automorphism $\phi$ is compatible with the respective actions $Q_1 \hookrightarrow \mathfrak{S}_r \rightarrow \text{Aut}_k(\mathcal{M}_{0,r})$ and $Q_2 \hookrightarrow \mathfrak{S}_r \rightarrow \text{Aut}_k(\mathcal{M}_{0,r})$ relative to the isomorphism $\tilde{\phi}: Q_1 \sim \tilde{Q}_2$, i.e., $\tilde{\phi}$ is an element of $\text{Aut}_{Q_1, Q_2}(\mathcal{M}_{0,r})$. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii), together with the various definitions involved.

\[\square\]

Remark 5.2.1. Let $\zeta_l \in \overline{k}$ be a primitive $l$-th root of unity. Then it follows from [1], Theorems A, B, that $\text{Ker}(G_k \rightarrow \text{Im}(\rho_{0,\Gamma}^{(l)})) \subseteq G_{k(\zeta_l)}$. Therefore, if we write

$$(H_i)_{k(\zeta_l)} = H_i \cap \rho_{0,\{\Gamma\}}^{(l)}(\pi_1(\mathcal{M}_{0,\{\Gamma\} \otimes_k k(\zeta_l)})) \subseteq \text{Im}(\rho_{0,\{\Gamma\}}^{(l)})$$

and

$$(H_i)^{\text{geom}}_{k(\zeta_l)} = (H_i)_{k(\zeta_l)} \cap H_i^{\text{geom}},$$

then $(H_i)^{\text{geom}}_{k(\zeta_l)} = H_i^{\text{geom}}$; in particular, $(H_i)_{k(\zeta_l)}$ fits into similar exact sequences

$$1 \longrightarrow (H_i)^{\text{geom}}_{k(\zeta_l)} (= H_i^{\text{geom}}) \longrightarrow (H_i)_{k(\zeta_l)} \longrightarrow \rho_{0,\{\Gamma\}}^{(l)}(\pi_1(\mathcal{M}_{0,\{\Gamma\} \otimes_k k(\zeta_l)})) \longrightarrow 1;$$

$$1 \longrightarrow \rho_{0,\{\Gamma\}}^{(l)}(\pi_1(\mathcal{M}_{0,\{\Gamma\} \otimes_k k(\zeta_l)})) \longrightarrow (H_i)_{k(\zeta_l)} \longrightarrow Q_i (\subseteq \mathfrak{S}_r) \longrightarrow 1$$

to the exact sequences

$$1 \longrightarrow H_i^{\text{geom}} \longrightarrow H_i \longrightarrow \text{Im}(\rho_{0,\{\Gamma\}}^{(l)}) \longrightarrow 1;$$

$$1 \longrightarrow \text{Im}(\rho_{0,\{\Gamma\}}^{(l)}) \longrightarrow H_i \longrightarrow Q_i (\subseteq \mathfrak{S}_r) \longrightarrow 1.$$
6. Proof of the main result

In the present §, we prove that the isomorphism class of an \( l \)-monodromically full hyperbolic curve of genus zero over a finitely generated field of characteristic zero is completely determined by the kernel of the associated pro-\( l \) outer Galois representation (cf. Theorem 6.1 below).

**Theorem 6.1 (Galois-theoretic characterization of isomorphism classes of monodromically full hyperbolic curves of genus zero).**

Let \( l \) be a prime number; \( k \) a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0); \( \overline{k} \) an algebraic closure of \( k \); \( G_k = \text{Gal}(\overline{k}/k) \); \( X_1 = (C_1, D_1 \subseteq C_1) \), \( X_2 = (C_2, D_2 \subseteq C_2) \) hyperbolic curves (cf. Definition 1.1, (ii)) of genus zero over \( k \) which are \( l \)-monodromically full (cf. Definition 2.2, (i)). Suppose that the following condition (\( y \)) is satisfied:

\( y \) prime:

There exists a finite Galois extension \( k' \subseteq \overline{k} \) of \( k \) of extension degree is prime to \( l \), such that \( X_1 \otimes_k k' \) and \( X_2 \otimes_k k' \) are split (cf. Definition 1.5, (i)).

(For example, if one of the following conditions is satisfied, then the above condition (\( y \)) is satisfied:

- \( X_1 \) and \( X_2 \) are split.
- If we write \( r_i \) for the number of the cusps of \( X_i \) — i.e., if \( X_i \) is of type \((0, r_i)\) — then \( l \) is prime to \( r_1 \) and \( r_2 \) — or, equivalently, \( r_1, r_2 < l \).

Then the following conditions are equivalent:

- (i) \( X_1 \) is isomorphic to \( X_2 \) over \( k \).
- (ii) For \( i = 1, 2 \), write
  \[ \rho^{(l)}_{X_i/k} : G_k \to \text{Out}(\pi_1((C_i \setminus D_i) \otimes_k \overline{k})^{(l)}) \]
  for the pro-\( l \) outer Galois representation associated to \( X_i \). Then \( \text{Ker}(\rho^{(l)}_{X_1/k}) = \text{Ker}(\rho^{(l)}_{X_2/k}) \).

**Proof.** The implication

\( (i) \implies (ii) \)

is immediate; thus, to verify Theorem 6.1, it suffices to show the implication

\( (ii) \implies (i) \).

Suppose that condition (ii) is satisfied. Let us write \( N \overset{\text{def}}{=} \text{Ker}(\rho^{(l)}_{X_1/k}) = \text{Ker}(\rho^{(l)}_{X_2/k}) \subseteq G_k \) (cf. condition (ii)) and \( r_i \) for the number of the cusps of the hyperbolic curve \( X_i \), i.e., \( X_i \) is a hyperbolic curve of type \((0, r_i)\).

Now it follows immediately that the bijection of sets \( \phi : \text{Im}(\rho^{(l)}_{X_1/k}) \to \text{Im}(\rho^{(l)}_{X_2/k}) \) obtained as the composite

\[ \text{Im}(\rho^{(l)}_{X_1/k}) \overset{\sim}{\twoheadrightarrow} G_k/N \overset{\sim}{\rightarrow} \text{Im}(\rho^{(l)}_{X_2/k}) \]
— where the “$\sim$” and “$\simeq$” are natural isomorphisms — is an isomorphism of profinite groups; moreover, it follows from Lemma 1.6, (ii), that this isomorphism $\phi$ is an isomorphism over $\text{Im}(\rho_{0,i})$. Thus, since $X_1$ and $X_2$ are $l$-monodromically full, and the condition $(\dagger)$prime is satisfied, it follows from a similar argument to the argument used in the proof of Lemma 5.1 (cf. the condition $(*)$prime in the discussion preceding Lemma 5.1) that $\phi$ maps $\text{Im}(\rho_{0,i})_{\text{geom}} \subseteq \text{Im}(\rho_{X_1/k})$ bijectively onto $\text{Im}(\rho_{X_2/k})$. In particular, it follows immediately from Lemma 4.2, (ii), that $r_1 = r_2$.

Write $r \overset{\text{def}}{=} r_1 = r_2$, $Q_i$ for the image of the composite $\text{Im}(\rho_{X_i/k}) \hookrightarrow \text{Im}(\rho_{0,i}) \twoheadrightarrow \text{Im}(\rho_{0,i})/\text{Im}(\rho_{0,r}) \simeq \mathcal{G}_r$ — cf. Lemma 1.6, (iii)), and $[\mathcal{M}_{0,r}/Q_i] \to \mathcal{M}_{0,r}$ for the intermediate connected finite étale covering of the $\mathcal{G}_r$-covering $\mathcal{M}_{0,r} \to [\mathcal{M}_{0,r}/\mathcal{G}_r] = \mathcal{M}_{0,r}$ corresponding to the image $\text{Im}(\rho_{X_i/k}) \subseteq \text{Im}(\rho_{0,i})$. Then it follows from Lemma 5.2, (iii), together with the assumption that the condition $(\dagger)$prime is satisfied (cf. the condition $(*)$prime in the discussion preceding Lemma 5.1), that the isomorphism obtained as the composite

$$\phi \sim \text{Im}(\rho_{X_2/k}) = \pi_1([\mathcal{M}_{0,r}/Q_2])/\ker(\rho_{0,r})$$

arises from the lower horizontal arrow in a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_{0,r} & \longrightarrow & \mathcal{M}_{0,r} \\
\downarrow & & \downarrow \\
[\mathcal{M}_{0,r}/Q_1] & \longrightarrow & [\mathcal{M}_{0,r}/Q_2]
\end{array}$$

— where the vertical arrows are natural morphisms, and the horizontal arrows are isomorphisms over $k$. Therefore, it follows from Lemma 4.1, (ii), together with the various definitions involved, that if we write $\tilde{s}_{X_i} \in \mathcal{M}_{0,r}(k)$ for the classifying morphism of $X_i$, then the elements of

$$\text{Hom}_{\text{Im}(\rho_{0,i})_{\text{geom}}}(G_k, \text{Im}(\rho_{0,i}))/\text{Inn}(\text{Im}(\rho_{0,i}))$$

determined by $\tilde{s}_{X_1}$ and $\tilde{s}_{X_2}$, respectively, coincide. Thus, it follows from Lemma 4.3, (v), that $X_1$ is isomorphic to $X_2$ over $k$, as desired. This completes the proof of the above implication. \hfill $\square$

7. EXAMPLE I: HYPERBOLIC CURVES OF TYPE $(0, 4)$ OVER NUMBER FIELDS

In the present §, we consider the monodromic fullness of hyperbolic curves of type $(0, 4)$ over number fields. In particular, we obtain sufficient conditions for such a hyperbolic curve to be monodromically full (cf. Theorem 7.8 and Corollaries 7.10, 7.11 below). Moreover, as an
application of these sufficient conditions, we obtain a Galois-theoretic characterization of the isomorphism classes of certain hyperbolic curves of type \((0, 4)\) over number fields (cf. Corollary 7.12 below). In the present \(\S\), suppose that \(k\) is a \textit{number field} (cf. the discussion entitled "Numbers" in \(\S 0\)), and let \(\mathcal{O}_k \subseteq k\) be the ring of integers of \(k\) and \(\lambda \in k \setminus \{0, 1\}\) an element of \(k \setminus \{0, 1\}\). Moreover, in the present \(\S\), if \(k' \subseteq \overline{k}\) is a finite extension of \(k\), and \(p\) is a prime number, then write \(\mathfrak{P}(k'; p)\) for the set of nonarchimedean primes of \(k'\) whose residue characteristic are \(p\).

\textbf{Definition 7.1}. Let \(l\) be an odd prime number and \(\zeta_l \in \overline{k}\) a primitive \(l\)-th root of unity.

(i) We shall write \(k_l \subseteq \overline{k}\) for the \textit{finite} Galois extension of \(k(\zeta_l)\) corresponding to the quotient

\[ G_{k(\zeta_l)} \twoheadrightarrow \rho_{0, 3}^{(l)}(G_{k(\zeta_l)}) \twoheadrightarrow \rho_{0, 3}^{(l)}(G_{k(\zeta_l)})^{ab} \otimes \mathbb{Z}_l \mathbb{F}_l \]

(cf. Lemma 7.2, (i), below).

(ii) We shall write \(\pi_1(\mathbb{P}_{k(\zeta_l)}^1 \setminus \{0, 1, \infty\}) \rightarrow Q_l\) for the quotient of \(\pi_1(\mathbb{P}_{k(\zeta_l)}^1 \setminus \{0, 1, \infty\})\) obtained as the composite

\[ \pi_1(\mathbb{P}_{k(\zeta_l)}^1 \setminus \{0, 1, \infty\}) \simeq \pi_1(\mathcal{M}_{0, 4} \otimes_k k(\zeta_l)) \twoheadrightarrow \rho_{0, 3}^{(l)}(\pi_1(\mathcal{M}_{0, 4} \otimes_k k(\zeta_l))) \]

\[ \twoheadrightarrow \rho_{0, 3}^{(l)}(\pi_1(\mathcal{M}_{0, 4} \otimes_k k(\zeta_l)))^{ab} \otimes \mathbb{Z}_l \mathbb{F}_l \]

where the first arrow is the isomorphism obtained by an isomorphism \(\mathbb{P}_{k(\zeta_l)}^1 \setminus \{0, 1, \infty\} \simeq \mathcal{M}_{0, 4}\) over \(k\) (cf. Lemma 4.1, (i)).

(iii) We shall write \(X_l \rightarrow \mathbb{P}_{k}^1 \setminus \{0, 1, \infty\}\) for the connected \textit{finite étale} covering of \(\mathbb{P}_{k}^1 \setminus \{0, 1, \infty\}\) corresponding to the \textit{open} subgroup (cf. Lemma 7.2, (ii), below) obtained as the kernel of \(\pi_1(\mathbb{P}_{k(\zeta_l)}^1 \setminus \{0, 1, \infty\}) \rightarrow Q_l\).

(iv) We shall write

\[ Y_l \overset{\text{def}}{=} \text{Spec } k[s^\pm 1, t^\pm 1]/(s^l + t^l - 1) \]

\[ \rightarrow \text{Spec } k[u^\pm 1, 1/(u - 1)] = \mathbb{P}_{k}^1 \setminus \{0, 1, \infty\} \]

where \(s, t, u\) are indeterminates — for the connected finite étale covering of \(\mathbb{P}_{k}^1 \setminus \{0, 1, \infty\}\) determined by the homomorphism of algebras over \(k\)

\[ k[u^\pm 1, 1/(u - 1)] \rightarrow k_l[s^\pm 1, t^\pm 1]/(s^l + t^l - 1) \]

\[ u \leftrightarrow s^l . \]

\textbf{Lemma 7.2 (Properties of certain extensions and étale coverings)}. Let \(l\) be an odd prime number and \(\zeta_l \in \overline{k}\) a primitive \(l\)-th root of unity. Then the following hold:

(i) \(k_l\) is a \textit{finite abelian extension} of \(k(\zeta_l)\) of degree a power of \(l\); moreover, the extension \(k_l\) of \(k\) is \textit{unramified} outside \(\mathfrak{P}(k; l)\).
Therefore, assertion (i) follows immediately from the fact that the extension \( \text{topologically nitely generated} \) proof of assertion (i).

Therefore, assertion (i) follows immediately from the fact that the extension \( k(\zeta_l) \) corresponding to the above quotient is \( \text{unramified outside} \) \( \mathcal{P}(k(\zeta_l); l) \). Therefore, assertion (i) follows immediately from the fact that the extension \( k(\zeta_l) \) of \( k \) is \( \text{unramified outside} \) \( \mathcal{P}(k; l) \). This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows from Lemma 2.5 that the quotient \( G_{k(\zeta_l)} \rightarrow \rho_{0,3}^{(l)}(G_{k(\zeta_l)}) \) is \( \text{topologically nitely generated} \) (respectively, \( \text{pro-l} \)). Moreover, it follows from [1], Theorems A, B, that the algebraic extension of \( k(\zeta_l) \) corresponding to the above quotient is \( \text{unramified outside} \) \( \mathcal{P}(k(\zeta_l); l) \). Therefore, assertion (i) follows immediately from the fact that the extension \( k(\zeta_l) \) of \( k \) is \( \text{unramified outside} \) \( \mathcal{P}(k; l) \). This completes the proof of assertion (i).

Finally, we verify assertion (iii). It follows immediately from the definition of the connected finite étale covering \( Y_l \rightarrow \mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\} \), together with Lemma 4.2, (i), that this covering is \( \text{Galois} \), and the quotient by the normal open subgroup \( \pi_1(Y_l) \subseteq \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\}) \) fits into an \( \text{exact} \) sequence

\[
1 \rightarrow \text{Im}(\rho_{0,4}^{(l)-\text{geom}})^{\text{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \rightarrow \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\})/\pi_1(Y_l) \rightarrow \rho_{0,3}^{(l)}(\text{Gal}(\overline{k}/k(\zeta_l)))^{\text{ab}} \otimes_{\mathbb{Z}_l} \mathbb{F}_l (= \text{Gal}(k_l/k(\zeta_l))) \rightarrow 1.
\]

Therefore, it follows immediately from the definition of the connected finite étale covering \( X_l \rightarrow \mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\} \) that the natural surjection \( \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\}) \rightarrow Q_l \) \( \text{factors through} \) \( \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\}) \rightarrow \pi_1(\mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\})/\pi_1(Y_l) \). This completes the proof of assertion (iii).

**Lemma 7.3 (Fibers and monodromic fullness).** Let \( l \) be an odd prime number and \( \zeta_l \in \overline{k} \) a primitive \( l \)-th root of unity. Consider the following four conditions:

(i) The fiber of \( X_l \rightarrow \mathbb{P}^1_k \setminus \{0, 1, \infty\} \) at the (image of the) \( k \)-rational point of \( \mathbb{P}^1_k \setminus \{0, 1, \infty\} \) corresponding to the element \( \lambda \in k \setminus \{0, 1\} \) is connected.
(ii) The composite

\[ \text{Gal}(\overline{k}/k(\zeta_l)) \longrightarrow \pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \longrightarrow Q_l \]

— where the first arrow is the homomorphism (which is determined up to \( \pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \)-inner automorphism) induced by the \( k(\zeta_l) \)-rational point of \( \mathbb{P}^1_{k(\zeta_l)} \setminus \{0, 1, \infty\} \) corresponding to the element \( \lambda \in k \setminus \{0, 1\} \subseteq k(\zeta_l) \setminus \{0, 1\} \) — is surjective.

(iii) The composite

\[ G_k \longrightarrow \pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \longrightarrow \pi_1(M_{0,4}) \longrightarrow \text{Im}(\rho_{0,4}^{(l)}) \]

— where the first arrow is the homomorphism (which is determined up to \( \pi_1(\mathbb{P}^1_k \setminus \{0, 1, \infty\}) \)-inner automorphism) induced by the \( k \)-rational point of \( \mathbb{P}^1_k \setminus \{0, 1, \infty\} \) corresponding to the element \( \lambda \in k \setminus \{0, 1\} \), and the second arrow is the isomorphism over \( G_k \) obtained by an isomorphism \( \mathbb{P}^1_k \setminus \{0, 1, \infty\} \simeq M_{0,4} \) over \( k \) (cf. Lemma 4.1, (i)) — is surjective.

(iv) The hyperbolic curve \( (\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subseteq \mathbb{P}^1_k) \) of type \( (0, 4) \) over \( k \) is \( l \)-monodromically full.

Then the following implication and equivalences hold:

(i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv)

Proof. The equivalences

(i) \( \iff \) (ii) ; (iii) \( \iff \) (iv)

follow immediately from the various definitions involved. Thus, to verify Lemma 7.3, it suffices to show the implication

(ii) \( \implies \) (iii).

Suppose that condition (ii) is satisfied. It follows from the fact that \( \text{Ker}(\rho_{0,4}^{(l)}) \subseteq G_{k(\zeta_l)} \) (cf. Remark 5.2.1) that to verify condition (iii) — by replacing \( k \) by \( k(\zeta_l) \) — we may assume without loss of generality that \( \zeta_l \in k \). Then it follows from Lemma 2.5 (respectively, Lemma 4.3, (ii)) that \( \text{Im}(\rho_{0,4}^{(l)}) \) is topologically finitely generated (respectively, pro-\( l \)). Therefore, it follows from [22], Lemma 2.8.7, (c); [22], Corollary 2.8.5, together with the definition of the quotient \( Q_l \), that condition (iii) is satisfied. This completes the proof of the above implication.

Definition 7.4.

(i) If \( p \) is a nonarchimedean prime of \( k \), then we shall write \( v_p : k \to \mathbb{Z} \) for the \( p \)-adic valuation such that if \( p \) is the residue characteristic of \( p \), then \( v_p(p) \) coincides with the absolute ramification index of the completion of \( k \) at \( p \).

(ii) If \( a \) is an element of \( k^* \), then we shall write \( [a]^+ \) (respectively, \( [a]_-; [a]_+ \)) for the (necessarily finite) set of nonarchimedean primes \( p \) of \( k \) such that \( v_p(a) \neq 0 \) (respectively, \( v_p(a) > 0; v_p(a) < 0 \)).
Lemma 7.5 (Zeros and poles of certain divisors).

(i) \([\lambda]_+ \cap [\lambda]_- \) is empty.
(ii) \([\lambda]_+ = [1 - \lambda]_- \).
(iii) \([\lambda]_+ \cap [1 - \lambda]_+ \) is empty.
(iv) \([\lambda]_+ \neq \emptyset, [1 - \lambda]_+ \neq \emptyset \) if and only if \([\lambda]_+ \not\subseteq [1 - \lambda]_+, [1 - \lambda]_+ \not\subseteq [\lambda]_+ \).
(v) Suppose that \(\{\lambda, 1 - \lambda, \lambda/(\lambda - 1)\} \cap o_k^* = \emptyset \).

Then there exists an element \(\lambda'\) of

\[
\{\lambda, 1/\lambda, 1 - \lambda, 1/(\lambda - 1), \lambda/(\lambda - 1), (\lambda - 1)/\lambda\}
\]

such that \([\lambda']_+ \not\subseteq [1 - \lambda']_+ \) and \([1 - \lambda']_+ \not\subseteq [\lambda']_+ \).

Proof. Assertion (i) follows from the definitions of \([-]_+\) and \([-]_-\). Assertions (ii) and (iii) follow immediately from a straightforward calculation. Assertion (iv) follows immediately from assertions (i), (ii), and (iii). Finally, we verify assertion (v). Suppose that any element of \(m_\lambda = \{\lambda, 1/\lambda, 1 - \lambda, 1/(\lambda - 1), \lambda/(\lambda - 1), (\lambda - 1)/\lambda\}\) does not satisfy the desired condition. Now since \(m_\lambda \cap o_k^* = \emptyset\), any element \(\lambda' \in m_\lambda\) satisfies either \([\lambda']_+ \neq \emptyset\) or \([1/\lambda']_+ \neq \emptyset\); thus, it follows from assertion (iv) that — by replacing \(\lambda\) by an element of \(m_\lambda\) — we may assume without loss of generality that

\[
[\lambda]_+ = \emptyset; [1/\lambda]_+ \neq \emptyset; [(\lambda - 1)/\lambda]_+ = \emptyset;
\]

\[
[\lambda/(\lambda - 1)]_+ \neq \emptyset; [1/(1 - \lambda)]_+ = \emptyset; [1 - \lambda]_+ \neq \emptyset.
\]

Therefore, it follows that \(1/\lambda, \lambda/(\lambda - 1), 1 - \lambda \in o_k\); in particular, we obtain that \(\lambda/(\lambda - 1) \in o_k^*\) — in contradiction to the assumption that \(m_\lambda \cap o_k^* = \emptyset\). This completes the proof of assertion (v). \(\square\)

Definition 7.6. Let \(l\) be an odd prime number. Then we shall say that \(l\) satisfies the condition \((\dagger_{\lambda \in k})\) if there exist nonarchimedean primes \(p_0\) and \(q_0\) of \(k\) satisfying the following conditions:

(i) \(p_0 \not\in \mathfrak{P}(k; l), p_0 \in [\lambda]_+, p_0 \not\subseteq [1 - \lambda]_+, \) and \(l\) is prime to \(v_{p_0}(\lambda)\).

(ii) \(q_0 \not\in \mathfrak{P}(k; l), q_0 \not\in [\lambda]_+, q_0 \in [1 - \lambda]_+, \) and \(l\) is prime to \(v_{q_0}(1 - \lambda)\).

Remark 7.6.1. It is easily verified that if \([\lambda]_+ \not\subseteq [1 - \lambda]_+ \) and \([1 - \lambda]_+ \not\subseteq [\lambda]_+ \), then there exists a cofinite set \(\Sigma\) of prime numbers — i.e., a (necessarily infinite) set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that if \(l \in \Sigma\), then \(l\) satisfies the condition \((\dagger_{\lambda \in k})\).

Lemma 7.7 (Connectedness of a fiber). Let \(l\) be an odd prime number, \(\zeta \in \overline{k}\) a primitive \(l\)-th root of unity, and \(\alpha_l \in \overline{k}\) (respectively, \(\beta_l \in \overline{k}\)) a solution of \(t^l - \lambda\) (respectively, \(t^l - (1 - \lambda)\)) — where \(t\) is an indeterminate. Suppose that the prime number \(l\) satisfies the condition \((\dagger_{\lambda \in k})\). Then the following hold:
The finite extension $k(\zeta_l, \alpha_l)$ (respectively, $k(\zeta_l, \beta_l)$) of $k$ is ramified at $p_0$ (respectively, $q_0$) — cf. Definition 7.6 — and unramified at $q_0$ (respectively, $p_0$).

(ii) The extension $k_l(\alpha_l, \beta_l)$ of $k_l$ is of degree $l^2$.

(iii) The fiber of $Y_l \to \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ at the (image of the) $k$-rational point of $\mathbb{P}^1_k \setminus \{0, 1\}$ corresponding to the element $\lambda \in k \setminus \{0, 1\}$ is connected. In particular, condition (i) of Lemma 7.3 is satisfied.

Proof. Assertion (i) follows from the definition of the condition $(\dagger_{\lambda \in k})$, together with, for example, [19], Chapter V, Lemma 3.3. Assertion (ii) follows immediately from Lemma 7.2, (i), together with assertion (i). Assertion (iii) follows from assertion (ii), together with Lemma 7.2, (iii).

\begin{flushright}
\Box
\end{flushright}

Theorem 7.8 (Monodromic fullness of certain split hyperbolic curves of type $(0, 4)$ over number fields). Let $l$ be an odd prime number, $k$ a number field (cf. the discussion entitled “Numbers” in §0), and $\lambda \in k \setminus \{0, 1\}$. Suppose that $l$ satisfies the condition $(\dagger_{\lambda \in k})$ (cf. Definition 7.6). Then the hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subseteq \mathbb{P}^1_k)$ of type $(0, 4)$ over $k$ is $l$-monodromically full (cf. Definition 2.2, (i)).

Proof. This follows from Lemma 7.3, together with Lemma 7.7, (iii).

\begin{flushright}
\Box
\end{flushright}

Definition 7.9. Let $X$ be a hyperbolic curve of type $(0, 4)$ over $k$. Then it follows immediately that there exists an element $\lambda_X \in \overline{k} \setminus \{0, 1\}$ such that the hyperbolic curve $X \otimes_k \overline{k}$ is isomorphic over $\overline{k}$ to the hyperbolic curve

$$(\mathbb{P}^1_{\overline{k}}, \{0, 1, \lambda_X, \infty\} \subseteq \mathbb{P}^1_{\overline{k}})$$

of type $(0, 4)$ over $\overline{k}$. Now we shall write

$$m_X \overset{\text{def}}{=} \{\lambda_X, 1/\lambda_X, 1 - \lambda_X, 1/(1 - \lambda_X), \lambda_X/(\lambda_X - 1), (\lambda_X - 1)/\lambda_X\} \subseteq \overline{k}.$$ 

Note that, as is well-known, $m_X$ depends only on (and completely determines!) the isomorphism class of the hyperbolic curve $X \otimes_k \overline{k}$ over $\overline{k}$.

Corollary 7.10 (Monodromic fullness of certain hyperbolic curves of type $(0, 4)$ over number fields). Let $k$ be a number field (cf. the discussion entitled “Numbers” in §0), $\overline{k}$ an algebraic closure of $k$, $\sigma_{\overline{k}}$ the ring of integers of $\overline{k}$, and $X$ a hyperbolic curve (cf. Definition 1.1, (ii)) of type $(0, 4)$ over $k$. If $m_X \cap \sigma_{\overline{k}} = \emptyset$ (cf. Definition 7.9), then there exists a cofinite set $\Sigma$ of prime numbers — i.e., a (necessarily infinite) set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the
hypothesis curve $X$ over $k$ is $\Sigma$-monodromically full (cf. Definition 2.2, (i)).

In particular, if $\sigma_k \subseteq k$ is the ring of integers of $k$, and $\lambda \in k \setminus \{0, 1\}$ is an element of $k \setminus \{0, 1\}$ such that

$$\left\{ \lambda, 1 - \lambda, \lambda / (\lambda - 1) \right\} \cap \sigma_k^* = \emptyset,$$

then there exists a cofinite set $\Sigma$ of prime numbers such that the hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1_k)$ of type $(0, 4)$ over $k$ is $\Sigma$-monodromically full.

**Proof.** This follows from Theorem 7.8 together with Lemma 7.5, (v); Remark 7.6.1. $\square$

**Corollary 7.11** (Monodromic fullness of split hyperbolic curves of type $(0, 4)$ over the field of rational numbers or certain imaginary quadratic fields). Let $d$ be a square-free positive integer such that $d \neq 1, 3$. Write $k$ for the field of rational numbers $\mathbb{Q}$ or the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. If $\lambda \in k$, then the following conditions are equivalent:

(i) The hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1_k)$ — of type $(0, 3)$ or $(0, 4)$ — over $k$ is not isomorphic to the hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, -1, \infty\} \subset \mathbb{P}^1_k)$ of type $(0, 4)$ over $k$.

(ii) $\lambda$ is not equal to $-1, 2, 1/2$.

(iii) There exists a cofinite set $\Sigma$ of prime numbers — i.e., a (necessarily infinite) set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1_k)$ of type $(0, 3)$ or $(0, 4)$ — over $k$ is $\Sigma$-monodromically full (cf. Definition 2.2, (i)).

(iv) There exists a prime number $l$ such that the hyperbolic curve $(\mathbb{P}^1_k, \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1_k)$ of type $(0, 3)$ or $(0, 4)$ — over $k$ is $l$-monodromically full.

**Proof.** The implication

(i) $\implies$ (ii)

is immediate. The implication

(ii) $\implies$ (iii)

follows from Theorem 7.10, together with the fact that $\sigma_k^* = \{\pm 1\}$. The implication

(iii) $\implies$ (iv)

is immediate. Finally, we verify the implication

(iv) $\implies$ (i).
It is easily verified that \( \mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, -1, \infty\} \) has some special symmetry — i.e., \( \text{Aut}(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, -1, \infty\}) \) is not isomorphic to \( G_{0,4} \) (cf. Definition 3.1). Therefore, the above implication follows from Proposition 3.4.

\[ \text{Corollary 7.12 (Galois-theoretic characterization of isomorphism classes of certain hyperbolic curves of type (0, 4) over number fields). Let } k \text{ be a number field (cf. the discussion entitled "Numbers" in §0) over } \mathbb{Q}, \text{ then there exist a finite extension } k_0 / k \text{ of } k, \text{ a number field } k_1 / k, \text{ and a hyperbolic curve } X_0 \text{ over } k_0 \text{ such that } X \otimes_{k_0} k_1 \text{ is isomorphic to } X \otimes_{k_0} k_1\). \]

\[ \text{Corollary 8.2 (Monodromic fullness of nonisotrivial hyperbolic curves of type (0, 4)). Let } k \text{ be a finitely generated field of characteristic zero (cf. the discussion entitled "Numbers" in §0) and } X \text{ a hyperbolic curve (cf. Definition 1.1, (ii)) of type (0, 4) over } k \text{ which is not NF-isotrivial (cf. Definition 8.1). Then there exists a cofinite}\]
set $\Sigma$ of prime numbers — i.e., a (necessarily infinite) set of prime numbers obtained as the complement of a finite set of prime numbers in the set of all prime numbers — such that the hyperbolic curve $X$ over $k$ is $\Sigma$-monodromically full (cf. Definition 2.2, (i)).

Proof. It is immediate that to verify Corollary 8.2 — by replacing $k$ by a suitable finite extension of $k$ — we may assume without loss of generality that $X$ is split. Now since $k$ is finitely generated field of characteristic zero, there exist a subfield $k_0 \subseteq k$ of $k$ and a scheme $V_0$ over $k_0$ satisfying the following conditions:

(i) $k_0$ is a number field (cf. the discussion entitled “Numbers” in §0).
(ii) $V_0$ is regular, separated, geometrically connected, and of finite type over $k_0$.
(iii) The function field of $V_0$ is isomorphic to $k$.
(iv) The split hyperbolic curve $X$ over $k$ extends to a split hyperbolic curve $X_0$ over $V_0$.

Now since the natural homomorphism $\pi_1(\text{Spec } k) \to \pi_1(V_0)$ (cf. (iii)) is surjective (cf. (ii)), and the pro-$\Sigma$ outer monodromy representation $\rho_{X/k}^\Sigma$ factors through $\rho_{X_0/V_0}^\Sigma$ (cf. (iv)), it follows from the definition of the term “monodromically full” that, to verify Corollary 8.2, it suffices to show that there exists a closed point $v \in V_0$ of $V_0$ such that the hyperbolic curve $(X_0)_v$ over the residue field at $v \in V_0$ obtained as the fiber of the hyperbolic curve $X_0$ over $V_0$ at $v \in V_0$ is $\Sigma$-monodromically full for some cofinite set $\Sigma$ of prime numbers.

Write $\overline{s}_{X_0/V_0} : V_0 \to \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ for the classifying morphism of the split hyperbolic curve $X_0$ over $V_0$ (cf. (iv), together with Lemma 4.1, (i)). Then since $X$ is not NF-isotrivial, and $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ is of dimension one, it follows that the image of the morphism $\overline{s}_{X_0/V_0}$ is open; in particular, there exists a closed point $\overline{\tau}$ of $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ contained in the image of $\overline{s}_{X_0/V_0}$ such that if $\lambda \in \overline{k}_0 \setminus \{0, 1\}$ is an element of $\overline{k}_0 \setminus \{0, 1\}$ naturally corresponding to $\overline{\tau} \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$, then $\{\lambda, 1 - \lambda, \lambda/(\lambda - 1)\} \cap \overline{\omega}_k^\Sigma = \emptyset$ — where $\overline{k}_0$ is an algebraic closure of $k_0$ and $\overline{\omega}_k^\Sigma$ is the ring of integers of $k_0$ (cf. (i)). Let $v \in V_0$ be a closed point of $V_0$ whose image via $\overline{s}_{X_0/V_0}$ is $\overline{\tau}$. Then it follows immediately from Corollary 7.10 that the hyperbolic curve $(X_0)_v$ over the residue field at $v \in V_0$ obtained as the fiber of the hyperbolic curve $X_0$ over $V_0$ at $v \in V_0$ is $\Sigma$-monodromically full for some cofinite set $\Sigma$ of prime numbers. This completes the proof of Corollary 8.2.

Remark 8.2.1. It is immediate that Corollary 8.2 implies the following assertion:

Let $k$ be a finitely generated field of characteristic zero (cf. the discussion entitled “Numbers” in §0) and $X$ a hyperbolic curve (cf. Definition 1.1, (ii)) of type
(0, 4) over k. Suppose that there exists an infinite set 
\[ \Sigma \] of prime numbers such that if \( l \in \Sigma \), then \( X \) is not 
\( l \)-monodromically full (cf. Definition 2.2, (i)). Then 
\( X \) is NF-isotrivial (cf. Definition 8.1).

On the other hand, if, in the above assertion, one replaces “(0, 4)” 
by “(0, r)” for some \( r \geq 5 \), then the conclusion no longer holds in 
general. A counter-example is as follows: Let \( k_0 \) be a number field, 
\[ S \overset{\text{def}}{=} \mathbb{P}^1_{k_0} \setminus \{0, 1, -1, \infty\} \], and \( k \) the function field of \( S \). Then the 
natural open immersion

\[ S \hookrightarrow \mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\} \]

and the composite

\[ S \rightarrow \text{Spec} k_0 \hookrightarrow \mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\} \]

— where the first arrow is the structure morphism of \( S \), and the second 
arrow is the \( k_0 \)-rational point corresponding to \(-1 \in k_0 \setminus \{0, 1\} \) — 
determine a morphism over \( k \) from \( S \) to the second configuration space of 
\( \mathbb{P}^1_{k_0} \setminus \{0, 1, \infty\} \); in particular, it follows immediately from Lemma 4.1, 
(i), that we obtain a split hyperbolic curve \( X \) over \( k \) of type (0, 5). Now 
since \( X \) may be embedded as an open subscheme of \( \mathbb{P}^1_k \setminus \{0, 1, -1, \infty\} \), 
it follows immediately from Proposition 3.4 (cf. also the argument 
used in the proof of the implication (iv) \( \Rightarrow \) (i) in the proof of Corollary 7.11), 
together with Remark 2.2.5, that, for any prime number \( l \), the 
hyperbolic curve \( X \) over \( k \) is not \( l \)-monodromically full. On the 
other hand, it follows immediately from the definition of \( X \) that \( X \) is 
not NF-isotrivial.

Corollary 8.3 (Galois-theoretic characterization of isomorphism 
classes of nonisotrivial hyperbolic curves of type (0, 4)). Let \( k \) 
be a finitely generated field of characteristic zero (cf. the dis-
scussion entitled “Numbers” in §0); \( \overline{k} \) an algebraic closure of \( k \); \( G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k) \); \( X_1 = (C_1, D_1 \subseteq C_1) \), \( X_2 = (C_2, D_2 \subseteq C_2) \) hyperbolic curves 
(cf. Definition 1.1, (ii)) of type (0, 4) over \( k \) which are not NF-
isotrivial (cf. Definition 8.1). Then the following conditions are 
equivalent:

(i) \( X_1 \) is isomorphic to \( X_2 \) over \( k \).

(ii) There exists an infinite set \( \Sigma \) of prime numbers such that, for 
any \( l \in \Sigma \), if we write

\[ \rho_{X_1/k}^{(l)} : G_k \rightarrow \text{Out} \left( \pi_1((C_1 \setminus D_1) \otimes_k \overline{k})^{(l)} \right) \]

for the pro-\( l \) outer Galois representation associated to \( X_1 \), then 
\( \text{Ker}(\rho_{X_1/k}^{(l)}) = \text{Ker}(\rho_{X_2/k}^{(l)}) \).

Proof. The implication

(i) \( \implies \) (ii)
is immediate; on the other hand, the implication

(ii) $\implies$ (i)

follows immediately from Theorem 6.1, together with Corollary 8.2. □

APPENDIX. Ramification of Outer Galois Representations
and Isomorphism Classes of Hyperbolic Curves

In the present §, we prove finiteness results, which are related to the main result of the present paper (cf. Theorem A.3, Corollary A.4 below). It seems to the author that the results appearing in the present § are likely to be well-known; since, however, the results could not be found in the literature, the author decided to give proofs of the results in the present §. In the present §, let $l$ be a prime number, $k$ a number field (cf. the discussion entitled “Numbers” in §0), and $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$.

Definition A.1. Let $N \subseteq G_k$ be a normal closed subgroup of $G_k$ and $\mathfrak{P}$ a set of primes of $k$. Then we shall write

$$\mathcal{I}\text{Gal}(l, k, g, r, N) \quad \text{(respectively, } \mathcal{I}\text{unr}(l, k, g, r, \mathfrak{P}))$$

for the set of the isomorphism classes over $k$ of hyperbolic curves $X = (C, D \subseteq C)$ of type $(g, r)$ over $k$ satisfying the following condition: If we write

$$\rho^{(l)}_{X/k} : G_k \longrightarrow \text{Out}\left(\pi_1((C \setminus D) \otimes_k \overline{k})^{(l)}\right)$$

for the pro-$l$ outer Galois representation associated to $X$, then the kernel of $\rho^{(l)}_{X/k}$ coincides with $N \subseteq G_k$ (respectively, then $\rho^{(l)}_{X/k}$ is unramified outside $\mathfrak{P}$).

Remark A.2. If $N \subseteq G_k$ is a normal closed subgroup of $G_k$ obtained as the kernel of the pro-$l$ outer Galois representation associated to a hyperbolic curve over $k$, then it is easily verified that there exists a finite set $\mathfrak{P}$ of primes of $k$ such that $\mathcal{I}\text{Gal}(l, k, g, r, N) \subseteq \mathcal{I}\text{unr}(l, k, g, r, \mathfrak{P})$.

The main purpose of the present § is to prove the following fact.

Theorem A.3. Let $l$ be a prime number, $k$ a number field (cf. the discussion entitled “Numbers” in §0), $\overline{k}$ an algebraic closure of $k$, $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$, $(g, r)$ a pair of nonnegative integers such that $2g - 2 + r > 0$, and $\mathfrak{P}$ a finite set of primes of $k$. Then the set $\mathcal{I}\text{unr}(l, k, g, r, \mathfrak{P})$ (cf. Definition A.1) is finite.

By Theorem A.3, together with Remark A.2, we obtain the following corollary.

Corollary A.4. Let $l$ be a prime number, $k$ a number field (cf. the discussion entitled “Numbers” in §0), $\overline{k}$ an algebraic closure of $k$, $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$, $(g, r)$ a pair of nonnegative integers such that $2g -$
2 + r > 0, and $N \subset G_k$ a normal closed subgroup of $G_k$. Then the set $\mathcal{I}^\text{Gal}(l,k,g,r,N)$ (cf. Definition A.1) is finite.

The rest of the present § is devoted to prove Theorem A.3.

**Lemma A.5.** Let $\mathfrak{P}$ be a finite set of primes of $k$ and $X = (C,D \subset C)$ a hyperbolic curve over $k$ whose isomorphism class over $k$ is in $\mathcal{I}^\text{fin}(l,k,g,r,\mathfrak{P})$. Then there exists a finite extension $k(l,k,r,\mathfrak{P}) \subset \overline{k}$ of $k$ that depends only on $l$ and $r$, and $\mathfrak{P}$ such that the hyperbolic curve $X \otimes_k k(l,k,r,\mathfrak{P})$ over $k(l,k,r,\mathfrak{P})$ is split (cf. Definition 1.5, (i)).

**Proof.** To prove Lemma A.5 — by replacing $\mathfrak{P}$ by a finite set of primes of $k$ containing $\mathfrak{P}$ and the set of the primes of $k$ whose residue characteristic are $l$ — we may assume without loss of generality that the set of primes of $k$ whose residue characteristic are $l$ is contained in $\mathfrak{P}$. Then it follows immediately from the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8) that any irreducible component of $D$ is isomorphic to the spectrum of a finite extension of $k$ which is unramified outside $\mathfrak{P}$. On the other hand, it follows immediately from a well-known theorem of Hermite-Minkowski that there are only finitely many isomorphism classes of finite extensions of $k$ of extension degree $\leq r$ which are unramified outside $\mathfrak{P}$. Therefore, if we write $k(l,k,r,\mathfrak{P})$ for the composite field of all extension fields (in $\overline{k}$) of extension degree $\leq r$ which are unramified outside $\mathfrak{P}$, then $k(l,k,r,\mathfrak{P})$ satisfies the desired condition. This completes the proof of Lemma A.5. $\square$

**Lemma A.6.** Let $k'$ be a finite extension of $k$ and $Y$ a hyperbolic curve over $k'$. Then there are only finitely many isomorphism classes over $k$ of hyperbolic curves $X$ over $k$ satisfying the following condition: $X \otimes_k k'$ is isomorphic to $Y$ over $k'$.

**Proof.** To verify Lemma A.6 — by replacing $k'$ by a finite extension of $k'$ — we may assume without loss of generality that the extension $k'$ of $k$ is Galois. Write $\mathcal{D}$ for the set of the isomorphism classes $[X, \phi: X \otimes_k k' \sim Y]$ of pairs $(X, \phi: X \otimes_k k' \sim Y)$ of hyperbolic curves $X$ over $k$ and isomorphisms $\phi: X \otimes_k k' \sim Y$ over $k'$ — where we shall say that a pair $(X_1, \phi_1: X_1 \otimes_k k' \sim Y)$ is isomorphic to a pair $(X_2, \phi_2: X_2 \otimes_k k' \sim Y)$ if there exists an isomorphism $\psi: X_1 \sim X_2$ over $k$ such that $\phi_2 \circ \psi = \phi_1$.

To verify Lemma A.6, it is immediate that it suffices to show that this set $\mathcal{D}$ is finite. Moreover, to verify the finiteness of $\mathcal{D}$, it is immediate that we may assume without loss of generality that $\mathcal{D}$ is nonempty. Let us fix an element $[X_0, \phi_0: X_0 \otimes_k k' \sim Y] \in \mathcal{D}$ of $\mathcal{D}$. Then we obtain a map

$$\mathcal{D} \rightarrow Z^1(\text{Gal}(k'/k), \text{Aut}_{k'}(Y))$$

$$[X, \phi: X \otimes_k k' \sim Y] \mapsto (g \mapsto \phi \circ g^{-1} \circ \phi^{-1} \circ \phi_0 \circ g \circ \phi_0^{-1})$$
where the action of $\text{Gal}(k'/k)$ on $\text{Aut}_{k'}(Y)$ is given by
\[
\begin{align*}
\text{Gal}(k'/k) & \longrightarrow \text{Aut}\left(\text{Aut}_{k'}(Y)\right) \\
g & \mapsto (f \mapsto \phi_0 \circ g^{-1} \circ \phi_0^{-1} \circ f \circ \phi_0 \circ g \circ \phi_0^{-1}).
\end{align*}
\]
Moreover, by Galois descent, this map is injective. Therefore, the finiteness of $\mathcal{D}$ follows from the finiteness of $\text{Gal}(k'/k)$ and $\text{Aut}_{k'}(Y)$.  

**Proof of Theorem A.3.** To prove Theorem A.3 — by replacing $\mathfrak{P}$ by a finite set of primes of $k$ containing $\mathfrak{P}$ and the set of the primes of $k$ whose residue characteristic are $l$ — we may assume without loss of generality that the set of primes of $k$ whose residue characteristic are $l$ is contained in $\mathfrak{P}$. Moreover, it follows from Lemma A.6 that to prove Theorem A.3, it suffices to verify that if we write $\mathcal{I}_1 \subseteq \mathcal{I}_{\text{unr}}(l, k, g, r; \mathfrak{P})$ for

the subset of $\mathcal{I}_{\text{unr}}(l, k, g, r; \mathfrak{P})$ consisting of the isomorphism classes over $k$ of hyperbolic curves which are split,

then $\mathcal{I}_1$ is finite. Now if $X = (C, D \subseteq C)$ is a hyperbolic curve over $k$ whose isomorphism class over $k$ is in $\mathcal{I}_1$, then it follows from the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8) that the proper curve $C$ admits good reduction at all primes outside $\mathfrak{P}$. Therefore, if $g \geq 1$ (respectively, if $g = 0$), then it follows from a well-known theorem of Faltings-Shafarevich (respectively, the fact that $X$ is split) that

the set consisting of the isomorphism classes over $k$ of the proper curves “$C$” appearing in the elements of $\mathcal{I}_1$ is finite. Thus, to prove Theorem A.3, it suffices to verify that for a hyperbolic curve $X_0 = (C_0, D_0 \subseteq C_0)$ over $k$ whose isomorphism class over $k$ is in $\mathcal{I}_1$, if we write $\mathcal{I}_2 \subseteq \mathcal{I}_1$ for

the subset of $\mathcal{I}_1$ consisting of the isomorphism classes over $k$ of hyperbolic curves $X = (C, D \subseteq C)$ over $k$ whose isomorphism classes over $k$ are in $\mathcal{I}_1$ such that the proper curves $C$ are isomorphic to the proper curve $C_0$ over $k$,

then $\mathcal{I}_2$ is finite. On the other hand, this follows immediately from two well-known theorems of Mahler-Siegel and Faltings-Mordell, together with the criterion of Oda-Tamagawa for good reduction of hyperbolic curves (cf. [25], Theorem 0.8). This completes the proof of Theorem A.3. \qed

**References**


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