Matematica – Existence for semilinear parabolic stochastic equations

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Abstract

The boundary value problem for semilinear parabolic stochastic equations of the form\[ dX - \Delta X \, dt + \beta(X) \, dt \ni \sqrt{Q} \, dW_t, \] where \( W_t \) is a Wiener process and \( \beta \) is a maximal monotone graph everywhere defined, is well posed.

Key words: Wiener process, mild solution, random differential equation.

Riassunto. Il problema ai limiti per l’equazione stocastica semilineare di forma\[ dX - \Delta X \, dt + \beta(X) \, dt \ni \sqrt{Q} \, dW_t, \] dove \( W_t \) è un processo Wiener e \( \beta \) è un grafico massimale monotono definito ovunque, è ben posto.

1 Introduction

Consider the stochastic differential equation
\[
\begin{align*}
\frac{dX}{dt} - \Delta X + \beta(X) \, dt & \ni \sqrt{Q} \, dW_t \quad \text{in } (0, T) \times \mathcal{O} = Q_T, \\
X(0) & = x \quad \text{in } \mathcal{O}, \\
X & = 0 \quad \text{on } (0, T) \times \partial \mathcal{O} = \Sigma_T.
\end{align*}
\]

Here, \( \mathcal{O} \) is an open and bounded subset of \( \mathbb{R}^d \) with smooth boundary \( \partial \mathcal{O} \), \( d \geq 1 \), and \( W_t \) is a cylindrical Wiener process in \( L^2(\mathcal{O}) = H \) defined by
\[ W_t = \sum_{k=1}^{\infty} e_k(\xi) \beta_k(t), \quad \xi \in \mathcal{O}, \ t \geq 0, \]
where \( \{\beta_k\}_k \) are mutually independent Brownian motions on a probability space \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) and \( \{e_k\} \) is an orthonormal basis in \( H \). The operator \( Q \in L(H, H) \) is self-adjoint, positive and of finite trace.

Finally, \( \beta : R \to 2^R \) is a maximal monotone graph (see [1]) everywhere defined on \( R \).

\[ ^{1}\text{Nella seduta del...} \]
The main result of this note is that, under suitable assumptions on $Q$ (see (H1) below), equation (1) has a unique strong(mild) solution (Theorem 2). A similar result was proven in [2] for the stochastic porous media equation.

Compared with standard existence theory for equation (1) (see [3], [4]), where the main assumption is that $\beta$ is continuous, monotonically increasing, here $\beta$ might be multivalued and, therefore, discontinuous. Also, as seen later on, $\beta$ might be a time dependent function $\beta = \beta(t, \cdot)$ measurable in $t \in [0, T]$.

Moreover, our existence results apply to multivalued graphs $\beta$ everywhere defined on $\mathbb{R}$. Such a graph (multivalued) arises naturally when in equation (1) the function $\beta$ is monotonically increasing and discontinuous in $\{r_j\}_{j=1}^\infty$.

Then, one redefines $\beta$ by

$$\tilde{\beta}(r) = \beta(r) \text{ for } r \neq r_j, \quad \tilde{\beta}(r_j) = [\beta(r_j), \beta(r_{j+1} - 0)]$$

and get a maximal monotone graph $\tilde{\beta}$. So, one might say that the existence result established here in Theorem 2 below applies as well to discontinuous monotonically increasing besides continuous functions $\beta$.

We shall denote by $C_W([0, T]; H)$ the space of all adapted processes $X \in C([0, T]; L^2(\Omega, \mathcal{F}, \mathbb{P}, H))$, $H = L^2(\mathcal{O})$ and by $L^2_W(0, T; H^1_0(\mathcal{O}))$ the space of all adapted processes $X \in L^2(0, T; L^2(\Omega, \mathcal{F}, \mathbb{P}, H^1_0(\mathcal{O}))$ (see [3]). Here, $H^1_0(\mathcal{O})$ is the standard Sobolev space.

We denote also by $W_A$ the stochastic convolution

$$W_A(t) = \int_0^t e^{-A(t-s)} \sqrt{Q} dW_s, \quad t \geq 0,$$

where $A = -\Delta$, $D(A) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})$. We recall that $W_A(t)$ is a Gaussian process and $E(|W_A(t)|^2) < \infty, \forall t \geq 0$ (see [3], p. 21).

## 2 The main result

The following hypotheses will be assumed.

(H1) $W_A(\cdot, \cdot)$ is continuous on $[0, T] \times \mathcal{O}$, $\mathbb{P}$-a.s..

(H2) $\beta : \mathbb{R} \rightarrow 2^\mathbb{R}$ is a maximal monotone graph such that $D(\beta) = \mathbb{R}$.

Here, $D(\beta) = \{r \in \mathbb{R}; \beta(r) \neq \emptyset\}$.

In particular, hypotheses (H2) holds if $\beta$ is a monotonically nondecreasing and continuous function.
As regards hypotheses (H1), we refer to [3], Theorem 2.13, for sufficient conditions on $Q$ under which it holds.

**Definition 1** By strong (or mild) solution to equation (1) we mean a process $X \in C([0, T]; H)$ which satisfies

\[
X(t) = e^{-At}x - \int_0^t e^{-A(t-s)}\eta(s)ds + W_A(t), \quad \mathbb{P}\text{-a.s., } t \in [0, T],
\]

where $\eta \in L^1((0, T) \times \mathcal{O} \times \Omega)$ is a process such that

\[
\eta(t, \xi) \in \beta(X(t, \xi)), \quad \text{a.e. } (t, \xi) \in Q_T, \quad \mathbb{P}\text{-a.s.}
\]

**Theorem 2** Under hypotheses (H1), (H2), for each $x \in H = L^2(\mathcal{O})$ there is a unique strong solution $X$ to equation (1), such that

\[
X \in L^2_W([0, T]; H^1_0(\mathcal{O})),
\]

\[
j(X), j^*(\eta) \in L^1(Q, T) \times \mathcal{O} \times \Omega).
\]

Here, $j$ is the subpotential associated with $\beta$, i.e., $\partial j = \beta$ and $j^*$ is the conjugate of $j$. (See the notation below.)

## 3 Proof of Theorem 2

**Existence.** By using a standard device, we shall reduce equation (1) to the random differential equation

\[
y_t - \Delta y + \beta(y + W_A) \geq 0, \quad (t, \xi) \in Q_T = (0, T) \times \mathcal{O},
\]

\[
y(0, \xi) = x(\xi), \quad \xi \in \mathcal{O},
\]

\[
y = 0 \quad \text{on } (0, T) \times \partial \mathcal{O} = \Sigma_T,
\]

where $y = X - W_A$.

We fix $\omega \in \Omega$ and approximate (6) by

\[
(y_\epsilon)_t - \Delta y_\epsilon + \beta_\epsilon(y_\epsilon + W_A) \geq 0, \quad (t, \xi) \in Q_T,
\]

\[
y_\epsilon(0, \xi) = x(\xi), \quad \text{in } \mathcal{O},
\]

\[
y = 0 \quad \text{on } \Sigma_T,
\]
where $\beta_\varepsilon = \frac{1}{\varepsilon} (1 - (1 + \varepsilon \beta)^{-1})$ is the Yosida approximation of $\beta$ (see, e.g., [1]). Since $\beta_\varepsilon$ is Lipschitzian, equation (7) has a unique solution $y_\varepsilon \in C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H^1_0(\mathcal{O}))$

\[ \sqrt{t}(y_\varepsilon)_t \in L^2(0, T; L^2(\mathcal{O})), \quad \sqrt{t} y_\varepsilon \in L^2(0, T; H^2(\mathcal{O})). \]

Denote by $j : R \to R$ the subpotential function corresponding to $\beta$, that is $\partial j = \beta$, where $\partial j$ is subdifferential of $\beta$ (see, e.g., [1], p. 53). Let $j^*$ be the conjugate of $j$, that is,

\[ j^*(p) = \sup \{ p \cdot r - j(r); \ r \in R \} \]

and recall that $p \in \partial \beta(r)$ if and only if

\[ j(r) + j^*(p) = rp. \]

We have also $\beta_\varepsilon = \nabla j_\varepsilon$, where

\[ j_\varepsilon(r) = \inf \left\{ \frac{|r - s|^2}{2\varepsilon} + j(s); \ s \in R \right\} \]

\[ = \frac{1}{2\varepsilon} ((1 + \varepsilon \beta)^{-1}r - r)^2 + j((1 + \varepsilon \beta)^{-1}r), \ \forall r \in R. \]

Multiplying (7) by $y_\varepsilon$ and integrating on $(0, T) \times \mathcal{O}$, we obtain that

\[ \frac{1}{2} \| y_\varepsilon(t) \|^2_{L^2(\mathcal{O})} + \int_0^t \| y_\varepsilon(s) \|^2_{H^1_0(\mathcal{O})} ds + \int_\mathcal{O} j_\varepsilon(y_\varepsilon + W_A) ds d\xi \]

\[ \leq \frac{1}{2} \| x \|^2_{L^2(\mathcal{O})} + \int_0^t \int_\mathcal{O} j_\varepsilon(W_A) ds d\xi \leq C, \]

\[ \forall t \in [0, T]. \]

Hence, on a subsequence $\varepsilon \to 0$, we have

(11) $y_\varepsilon \to y^*$ weakly in $L^2(0, T; H^1_0(\mathcal{O}))$ and weak-star in $L^\infty(0, T; L^2(\mathcal{O}))$.

Also, by (9)~(10), we see that, for $\varepsilon \to 0$,

(12) $(1 + \varepsilon \beta)^{-1}(y_\varepsilon + W_A) \to y^* + W_A$ weak-star in $L^\infty(0, T; L^2(\mathcal{O}))$. 

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By (8), we have

\[ j^*(\beta_\varepsilon(y_\varepsilon + W_A)) + j((1 + \varepsilon\beta)^{-1}(y_\varepsilon + W_A)) \]
\[ = (\beta_\varepsilon(y_\varepsilon + W_A))(1 + \varepsilon\beta)^{-1}(y_\varepsilon + W_A) \leq \beta_\varepsilon(y_\varepsilon + W_A)(y_\varepsilon + W_A). \]

This yields

\[ \int_{Q_T} j^*(\beta_\varepsilon(y_\varepsilon + W_A))d\xi dt \leq \int_{Q_T} \beta_\varepsilon(y_\varepsilon + W_A)y_\varepsilon d\xi dt \]
\[ - \int_{Q_T} \beta_\varepsilon(y_\varepsilon + W_A)W_A d\xi dt = -\frac{1}{2}\|y_\varepsilon(T)\|^2_{L^2(\mathcal{O})} + \frac{1}{2}\|x\|^2_{L^2(\mathcal{O})} \]
\[ -\|y_\varepsilon\|^2_{L^2([0,T];H^1_0(\mathcal{O}))} - \int \int_{Q_T} \beta_\varepsilon(y_\varepsilon + W_A)W_A d\xi dt. \]

Since \( D(\beta) = R \), we have that

\[ \lim_{|r| \to \infty} \frac{j^*(r)}{|r|} = +\infty. \]

Then, by (14) we obtain that for each \( n \) there is \( C_n > 0 \) such that

\[ j^*(\beta_\varepsilon(y_\varepsilon + W_A)) \geq n|\beta_\varepsilon(y_\varepsilon + W_A)| \]
\[ \text{a.e. on } \{(\xi, t); |\beta_\varepsilon(y_\varepsilon + W_A)(\xi, t)| \geq C_n\}. \]

We shall use this to prove that \( \{\beta_\varepsilon(y_\varepsilon + W_A)\}_{\varepsilon > 0} \) is weakly compact in \( L^1(Q_T) \).

To this purpose, it suffices to show that

\[ \int_{Q_T} |\beta_\varepsilon(t_\varepsilon + W_A)|d\xi dt \leq C, \ \forall \varepsilon > 0, \]

and that, for each \( \delta > 0 \), there is \( C(\delta) \) such that for any measurable subset \( Q^* \subset Q_T \) with the Lebesgue measure \( m(Q^*) \leq C_\delta \), we have

\[ \int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)|d\xi dt \leq \delta, \ \forall \varepsilon > 0, \]

\( (C_\delta \text{ independent of } \varepsilon) \).
Estimate (16) follows by (13) and (15). As regards (17), we start from the inequality
\[
\int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)| d\xi dt \leq \int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)| d\xi dt
+ nm(Q^*) \leq \frac{1}{n} \int_{Q^*} j_\varepsilon^*(\beta_\varepsilon(y_\varepsilon + W_A)) d\xi dt + nm(Q^*)
\leq \frac{1}{n} \|W_A\|_{L^\infty(Q_T)} \|\beta_\varepsilon(y_\varepsilon + W_A)\|_{L^1(Q_T)} \leq \frac{C}{n} + nm(Q^*).
\]
(Here, we have used (13), (15), (16) and (H1).)

Hence, for \( n \geq \frac{\delta}{2C} \) and \( m(Q^*) \leq \frac{\delta}{2n} \), we obtain (17), as claimed.

Then, by the Pettis theorem, \( \{\beta_\varepsilon(y_\varepsilon + W_A)\}_{\varepsilon > 0} \) is weakly compact in \( L^1(Q_T) \) and so, on a subsequence, again denoted \( \varepsilon \), we have
\[
\beta_\varepsilon(y_\varepsilon + W_A) \rightharpoonup \eta \text{ weakly in } L^1(Q_T).
\]
Inasmuch as \( \{\beta_\varepsilon(y_\varepsilon + W_A)\} \) is bounded in \( L^1(Q_T) \), it follows by (7) that \( \{y_\varepsilon\} \) is compact in \( C([0, T]; L^1(\mathcal{O})) \) and, therefore, for \( \varepsilon \to 0 \),
\[
y_\varepsilon \to y^* \text{ strongly in } C([0, T]; L^1(\mathcal{O}))
\]
and
\[
y^*_\varepsilon - \Delta y^* + \eta = 0 \quad \text{in } Q_T,
\]
\[
y^*(0) = x, \quad y^*(t) \in H_0^1(\mathcal{O}), \quad \text{a.e. } t \in [0, T].
\]

In order to conclude the proof of existence for equation (6), it remains to be proven that
\[
\eta(t, \xi) \in \beta(y^*(t, \xi) + W_A(t, \xi)), \quad \text{a.e. } (t, \xi) \in Q_T.
\]
To this end, we start from the inequality
\[
\int_{Q_0} \beta_\varepsilon(y_\varepsilon + W_A)(y_\varepsilon + W_A - z) d\xi dt
\geq \int_{Q_0} j_\varepsilon(y_\varepsilon + W_A) d\xi dt - \int_{Q_0} j_\varepsilon(z) d\xi dt, \quad \forall z \in L^\infty(Q_0),
\]
for any measurable subset \( Q_0 \subset Q_T \).
On the other hand, by (19), by Egorov Theorem, it follows that for each \( \delta > 0 \) there is \( Q_\delta \subset Q_T \) such that \( m(Q_T \setminus Q_\delta) \leq \delta \) and \( y_\varepsilon \to y^* \) uniformly on \( Q_\delta \) as \( \varepsilon \to 0 \). Taking \( Q_0 = Q_T \) in (22), we obtain

\[
\int_{Q_\delta} \eta(y^* + W_A - z) d\xi dt \geq \int_{Q_\delta} (j(y^* + W_A) - j(z)) d\xi dt, \quad \forall z \in L^\infty(Q_\delta).
\]

The latter implies by a standard device the pointwise inequality

\[
\eta(y^* + W_A - z) \geq j(y^* + W_A) - j(z), \quad \text{a.e. in } Q_\delta, \quad \forall z \in R,
\]

and, therefore, \( \eta \in \partial j(y^* + W_A) = \beta(y^* + W_A) \), a.e. in \( Q_\delta \), and since \( \delta \) is arbitrary, we obtain (21), as claimed.

Now, it is clearly seen that \( X(t) = y(t) + W_A \) is a solution to (1) in the sense precised in Definition 1. (The fact that the process \( X(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t) + W_A(t) \) is adapted is obvious because so is \( X_\varepsilon(t) = y_\varepsilon(t) + W_A(t) \).

By (10) and (13), it is also easily seen that \( j(X), j^*(\eta) \in L^1((0, T) \times \mathcal{O} \times \Omega) \).

This completes the proof of the existence.

**Uniqueness.** It is immediate, because if \( X_i, \ i = 1, 2, \) are solutions to (1) in the above sense, then \( y_i = X_i - W_A, \ i = 1, 2, \) are \( \mathbb{P} \)-a.s. solutions to equation (6), which clearly has a unique solution by monotonicity of \( \beta \).

**Remark 3** Theorem 2 remains true for time dependent maximal monotone graphs \( \beta = \beta(t, \cdot) \) which satisfy the following assumptions.

(H2)' For almost all \( t \in (0, T) \), \( \beta(t, \cdot) : R \to 2^R \) is maximal monotone, measurable in \( t \) and for each \( M > 0 \) there is \( C_M \) independent of \( t \) such that

\[
|\beta(t, r)| \leq C_M \quad \text{a.e. } t \in (0, T), \quad \forall r \in [-M, M].
\]

If \( \beta \) is independent of \( t \), (H2)' is implied by (H2). The proof is exactly the same as that of Theorem 2.

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References


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