PATHWISE UNIQUENESS FOR SINGULAR SDEs DRIVEN BY STABLE PROCESSES *

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Abstract

We prove pathwise uniqueness for stochastic differential equations driven by non-degenerate symmetric \( \alpha \)-stable Lévy processes with values in \( \mathbb{R}^d \) having a bounded and \( \beta \)-Hölder continuous drift term. We assume \( \beta > 1 - \alpha/2 \) and \( \alpha \in [1, 2) \). The proof requires analytic regularity results for the associated integro-differential operators of Kolmogorov type. We also study differentiability of solutions with respect to initial conditions and the homeomorphism property.

1 Introduction

In this paper we prove a pathwise uniqueness result for the following SDE

\[
X_t = x + \int_0^t b(X_s) \, ds + L_t, \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{1.1}
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) is bounded and \( \beta \)-Hölder continuous and \( L = (L_t) \) is a non-degenerate \( d \)-dimensional symmetric \( \alpha \)-stable Lévy process (\( L_0 = 0 \), \( P \)-a.s.) and \( d \geq 1 \).

Currently, there is a great interest in understanding pathwise uniqueness for SDEs when \( b \) is not Lipschitz continuous or, more generally, when \( b \) is singular enough so that the corresponding deterministic equation (1.1) with \( L = 0 \) is not well-posed. A remarkable result in this direction was proved by Veretennikov in [25] (see also [27] for \( d = 1 \)). He was able to prove uniqueness when \( b : \mathbb{R}^d \to \mathbb{R}^d \) is only Borel and bounded and \( L \) is a standard \( d \)-dimensional Wiener process. This result has been generalized in various directions in [9], [13], [26], [6], [7], [5], [8].

The situation changes when \( L \) is not a Wiener process but is a symmetric \( \alpha \)-stable process, \( \alpha \in (0, 2) \). Indeed, when \( d = 1 \) and \( \alpha < 1 \), Tanaka, Tsuchiya and Watanabe prove in [24, Theorem 3.2] that even a bounded and \( \beta \)-Hölder continuous \( b \) is not enough to ensure pathwise uniqueness if \( \alpha + \beta < 1 \) (they consider drifts like \( b(x) = \text{sign}(x) \, (|x|^\beta \wedge 1) \) and initial

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\( 1 \)
condition $x = 0$). On the other hand, when $d = 1$ and $\alpha \geq 1$, they show pathwise uniqueness for any continuous and bounded $b$.

In this paper we prove pathwise uniqueness in any dimension $d \geq 1$, assuming that $\alpha \geq 1$ and $b$ is bounded and $\beta$-Hölder continuous with $\beta > 1 - \alpha/2$. Our proof is different from the one in [24] and is inspired by [7]. The assumptions on the $\alpha$-stable Lévy process $L$ which we consider are collected in Section 2 (see in particular Hypothesis 1). Here we only mention two significant examples which satisfy our hypotheses. The first is when $L = (L_t)$ is a standard $\alpha$-stable process (symmetric and rotationally invariant), i.e., the characteristic function of the random variable $L_t$ is

$$E[e^{i\langle L_t, u \rangle}] = e^{-tc_\alpha|u|^\alpha}, \quad u \in \mathbb{R}^d, \quad t \geq 0,$$

where $c_\alpha$ is a positive constant. The second example is $L = (L^1_t, \ldots, L^d_t)$, where $L^1, \ldots, L^d$ are independent one-dimensional symmetric stable processes of index $\alpha$. In this case

$$E[e^{i\langle L_t, u \rangle}] = e^{-tk_\alpha(|u_1|^\alpha + \cdots + |u_d|^\alpha)}, \quad u \in \mathbb{R}^d, \quad t \geq 0,$$

where $k_\alpha$ is a positive constant. Martingale problems for SDEs driven by $(L^1_t, \ldots, L^d_t)$ have been recently studied (see [3] and references therein).

We prove the following result.

**Theorem 1.1.** Let $L$ be a symmetric $\alpha$-stable process with $\alpha \in [1, 2)$, satisfying Hypothesis 1 (see Section 2). Assume that $b \in C^\beta_b(\mathbb{R}^d; \mathbb{R}^d)$ for some $\beta \in (0, 1)$ such that

$$\beta > 1 - \frac{\alpha}{2}.$$ 

Then pathwise uniqueness holds for equation (1.1). Moreover, if $X^x = (X^x_t)$ denotes the solution starting at $x \in \mathbb{R}^d$, we have:

(i) for any $t \geq 0$, $p \geq 1$, there exists a constant $C(t, p) > 0$ (depending also on $\alpha$, $\beta$ and $L = (L_t)$) such that

$$E[\sup_{0 \leq s \leq t} |X^x_s - X^y_s|^p] \leq C(t, p) |x - y|^p, \quad x, y \in \mathbb{R}^d; \quad (1.4)$$

(ii) for any $t \geq 0$, the mapping: $x \mapsto X^x_t$ is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$, $P$-a.s.;

(iii) for any $t \geq 0$, the mapping: $x \mapsto X^x_t$ is a $C^1$-function on $\mathbb{R}^d$, $P$-a.s..

All these assertions require that $L$ is non-degenerate. Estimate (1.4) replaces the standard Lipschitz-estimate which holds without expectation $E$ when $b$ is Lipschitz continuous. Assertion (ii) is the so-called homeomorphism property of solutions (we refer to [1], [19] and [14]; see also [20] for the case of Log-Lipschitz coefficients). Note that existence of strong solutions for (1.1) follows easily by a compactness argument (see the comment before Lemma 4.1). On the other hand, existence of weak solutions when $b$ is only measurable and bounded is proved in [15]. Since $C^\beta_b(\mathbb{R}^d; \mathbb{R}^d) \subset C^\beta_b(\mathbb{R}^d, \mathbb{R}^d)$
when $0 < \beta \leq \beta'$, our uniqueness result holds true for any $\alpha \geq 1$ when $\beta \in (1/2, 1)$. Theorem 1.1 implies the existence of a stochastic flow (see Remark 4.4).

The proof of the main result is given in Section 4. As in [7] our method is based on an Itô-Tanaka trick which requires suitable analytic regularity results. Such results are proved in Section 3. They provide global Schauder estimates for the following resolvent equation on $\mathbb{R}^d$

$$\lambda u - \mathcal{L}u - b \cdot Du = g,$$

(1.5) where $\lambda > 0$ and $g \in C^\beta_b(\mathbb{R}^d)$ are given and we assume $\alpha \geq 1$ and $\alpha + \beta > 1$.

Here $\mathcal{L}$ is the generator of the Lévy process $L$ (see (2.5), [1] and [22]). If $L$ satisfies (1.2) then $\mathcal{L}$ coincides with the fractional Laplacian $-(-\Delta)^{\alpha/2}$ on infinitely differentiable functions $f$ with compact support (see [22, Example 32.7]), i.e., for any $x \in \mathbb{R}^d$,

$$-(-\Delta)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - 1_{|y| \leq 1} \cdot y \cdot Df(x) \right) \frac{\tilde{c}_\alpha}{|y|^{d+\alpha}} dy.$$  

(1.6)

It is simpler to prove Schauder estimates for (1.5) when $\alpha > 1$. In such a case, assuming in addition that $\mathcal{L} = -(-\Delta)^{\alpha/2}$, i.e., $L$ is a standard $\alpha$-stable process, these estimates can be deduced from the theory of fractional powers of sectorial operators (see [16]). We also mention [2, Section 7.3] where Schauder estimates are proved when $\alpha > 1$ and $\mathcal{L}$ has the form (1.6) but with variable coefficients, i.e., $\tilde{c}_\alpha = \tilde{c}_\alpha(x, y)$. The limit case $\alpha = 1$ in (1.5) requires a special attention even for the fractional Laplacian $\mathcal{L} = -(-\Delta)^{1/2}$. Indeed in this case $\mathcal{L}$ is of the “same order” of $b \cdot D$. To treat $\alpha = 1$, we use a localization procedure which is based on Theorem 3.3 where Schauder estimates are proved in the case of $b(x) = k$, for any $x \in \mathbb{R}^d$, showing that the Schauder constant is independent of $k$ (the case $\alpha < 1$ is discussed in Remark 3.5).

In order to prove Theorem 1.1, in Section 4 we apply Itô’s formula to $u(X_t)$, where $u \in C_0^{\alpha+\beta}$ comes from Schauder estimates for (1.5) when $g = b$ (in such case (1.5) must be understood componentwise). This is needed to perform the Itô-Tanaka trick and find a new equation for $X_t$ in which the singular term $\int_0^t b(X_s)ds$ of (1.1) is replaced by more regular terms. Then uniqueness and (1.4) follow by $L^p$-estimates for stochastic integrals. Such estimates require Lemma 4.1 and the condition $\alpha/2 + \beta > 1$. In addition, properties (ii) and (iii) are obtained transforming (1.1) into a form suitable for applying the results in [14].

We will use the letter $c$ or $C$ with subscripts for finite positive constants whose precise value is unimportant; the constants may change from proposition to proposition.

2 Preliminaries and notation

General references for this section are [1], [21, Chapter 2], [22] and [28].
Let $\langle u, v \rangle$ (or $u \cdot v$) be the euclidean inner product between $u$ and $v \in \mathbb{R}^d$, for any $d \geq 1$; moreover $|u| = \langle u, u \rangle^{1/2}$. If $D \subset \mathbb{R}^d$ we denote by $1_D$ the indicator function of $D$. The Borel $\sigma$-algebra of $\mathbb{R}^d$ will be indicated by $\mathcal{B}(\mathbb{R}^d)$. All the measures considered in the sequel will be positive and Borel. A measure $\gamma$ on $\mathbb{R}^d$ is called symmetric if $\gamma(D) = \gamma(-D)$, $D \in \mathcal{B}(\mathbb{R}^d)$.

Let us fix $\alpha \in (0, 2)$. In (1.1) we consider a $d$-dimensional symmetric $\alpha$-stable process $L = (L_t)$, $d \geq 1$, defined on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and $\mathcal{F}_t$-adapted; the stochastic basis satisfies the usual assumptions (see [1, page 72]). Recall that $L$ is a Lévy process (i.e., it is continuous in probability, it has stationary increments, càdlàg trajectories, $L_t - L_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$, and $L_0 = 0$) with the additional property that the characteristic function of $L_t$ verifies

$$E[e^{i \langle L_t, u \rangle}] = e^{-t \psi(u)}, \quad \psi(u) = -\int_{\mathbb{R}^d} \left( e^{i \langle u, y \rangle} - 1 - i \langle u, y \rangle 1_{\{|y| \leq 1\}}(y) \right) \nu(dy),$$

(2.1) $u \in \mathbb{R}^d$, $t \geq 0$, where $\nu$ is a measure such that

$$\nu(D) = \int_S \mu(d\xi) \int_0^\infty 1_D(r\xi) \frac{dr}{r^{1+\alpha}}, \quad D \in \mathcal{B}(\mathbb{R}^d),$$

(2.2) for some symmetric, non-zero finite measure $\mu$ concentrated on the unitary sphere $S = \{ y \in \mathbb{R}^d : |y| = 1 \}$ (see [22, Theorem 14.3]).

The measure $\nu$ is called the Lévy (intensity) measure of $L$ and (2.1) is the Lévy-Khintchine formula. The measure $\nu$ is a $\sigma$-finite measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty$, with $1 \wedge |.| = \min(1, |.|)$. Formula (2.2) implies that (2.1) can be rewritten as

$$\psi(u) = -\int_{\mathbb{R}^d} (\cos(\langle u, y \rangle) - 1) \nu(dy) = -\int_S \mu(d\xi) \int_0^\infty \cos(\langle u, r\xi \rangle) - 1 \frac{dr}{r^{1+\alpha}} = c_\alpha \int_S |\langle u, \xi \rangle|^\alpha \mu(d\xi), \quad u \in \mathbb{R}^d$$

(2.3)

(see also [22, Theorem 14.13]). The measure $\mu$ is called the spectral measure of the stable process $L$. In this paper we make the following non-degeneracy assumption (cf. [23] and [22, Definition 24.16]).

**Hypothesis 1.** The support of the spectral measure $\mu$ is not contained in a proper linear subspace of $\mathbb{R}^d$.

It is not difficult to show that Hypothesis 1 is equivalent to the following assertion: there exists a positive constant $C_\alpha$ such that, for any $u \in \mathbb{R}^d$,

$$\psi(u) \geq C_\alpha |u|^\alpha.$$  

(2.4)

Condition (2.4) is also assumed in [11, Proposition 2.1]. To see that (2.4) implies Hypothesis 1, we argue by contradiction: if $\text{Supp}(\mu) \subset (M \cap S)$ where $M$ is the hyperplane containing all vectors orthogonal to some $u_0 \neq 0$, then
Let us fix some notation on function spaces. We define $C^k_d$ and $\mu$ (see \[22, Theorem 14.14\]).

Let us consider the canonical basis in $L^2$ with respect to the Lebesgue measure. The spectral measure $\nu$ gives a uniform distribution on $\mathbb{S}$ (i.e., $\mu$ gives a uniform distribution on $\mathbb{S}$; see \[21, Section 2.5\] and \[22, Theorem 14.14\]).

The second example is $L = \{L_1^d, \ldots, L_d^d\}$, see (1.3). In this case $\psi(u) = k_\alpha(1|u|^\alpha + \cdots + |u_d|^\alpha)$ and the Lévy measure $\nu$ is more singular since it is concentrated on the union of the coordinates axes, i.e., $\nu$ has density

$$c_\alpha \left(\sum_{k=1}^d 1_{x_k=0} \right) \frac{1}{|x_1|^{1+\alpha}} + \cdots + 1_{x_d=0} \frac{1}{|x_d|^{1+\alpha}}$$

with respect to the Lebesgue measure. The spectral measure $\mu$ is a linear combination of Dirac measures, i.e., $\mu = \sum_{k=1}^d (\delta_{e_k} + \delta_{-e_k})$, where $(e_k)$ is the canonical basis in $\mathbb{R}^d$. The generator is

$$\mathcal{L}f(x) = \sum_{k=1}^d \int_{\mathbb{R}^d} \left[ f(x+se_k) - f(x) - 1_{|s|\leq 1} s \partial_{x_k} f(x) \right] \frac{c_\alpha}{|s|^{1+\alpha}} ds, \ f \in C^\infty_c(\mathbb{R}^d).$$

Let us fix some notation on function spaces. We define $C_b(\mathbb{R}^d; \mathbb{R}^k)$, for integers $k$, $d \geq 1$, as the set of all functions $f : \mathbb{R}^d \to \mathbb{R}^k$ which are bounded and continuous. It is a Banach space endowed with the supremum norm $\|f\|_0 = \sup_{x \in \mathbb{R}^d} |f(x)|$, $f \in C_b(\mathbb{R}^d; \mathbb{R}^k)$. Moreover, $C^\beta_b(\mathbb{R}^d; \mathbb{R}^k)$, $\beta \in (0, 1)$, is the subspace of all $\beta$-Hölder continuous functions $f$, i.e., $f$ verifies

$$[f]_\beta := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\beta}} < \infty. \quad (2.6)$$

$C^\beta_b(\mathbb{R}^d; \mathbb{R}^k)$ is a Banach space with the norm $\| \cdot \|_\beta = \| \cdot \|_0 + [\cdot]_\beta$. If $k = 1$, we set $C^0_b(\mathbb{R}^d; \mathbb{R}) = C^0_b(\mathbb{R})$. Let $C^\alpha_b(\mathbb{R}^d; \mathbb{R}^k) = C_b(\mathbb{R}^d; \mathbb{R}^k)$ and $[\cdot]_0 = \| \cdot \|_0$. For any $n \geq 1$, $\alpha \in [0, 1)$, we say that $f \in C^{n+\alpha}_b(\mathbb{R}^d)$ if $f \in C^{n+\alpha}_b(\mathbb{R}^d) \cap C^\alpha_b(\mathbb{R}^d)$ and, for all $j = 1, \ldots, n$, the (Fréchet) derivatives $D^j f \in C^\alpha_b(\mathbb{R}^d; (\mathbb{R}^d)^{\otimes(j+1)})$. 

\[\psi(u_0) = 0. \] To show the converse, note that Hypothesis 1 implies that for any $u \in \mathbb{R}^d$ with $|u| = 1$, we have $\psi(v) > 0$ (indeed, otherwise, we would have $\mu(\{\xi \in S : |\langle v, \xi \rangle| > 0\}) = 0$ and so $\text{Supp}(\mu) \subset \{\xi \in S : \langle v, \xi \rangle = 0\}$ which contradicts the hypothesis). By using a compactness argument, we deduce that (2.4) holds for any $u \in \mathbb{R}^d$ with $|u| = 1$. Then, writing, for any $u \in \mathbb{R}^d$, $u \neq 0$, $\int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi) = |u|^\alpha \int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi)$, we obtain easily (2.4).
The space $C_0^{n+\alpha}(\mathbb{R}^d)$ is a Banach space endowed with the norm $\|f\|_{n+\alpha} = \|f\|_0 + \sum_{k=1}^{n} \|D^k f\|_0 + [D^n f]_\alpha$, $f \in C_0^{n+\alpha}(\mathbb{R}^d)$. Finally, we will also consider the Banach space $C_0(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ of all continuous functions vanishing at infinity endowed with the norm $\|\cdot\|_0$.

Remark 2.1. Hypothesis 1 (or condition (2.4)) is equivalent to the following Picard’s type condition (see [17]): there exists $\alpha \in (0, 2)$ and $C_\alpha > 0$, such that the following estimate holds, for any $\rho > 0$, $u \in \mathbb{R}^d$ with $|u| = 1$,

$$\int_{\{|\langle u, y \rangle| \leq \rho\}} |\langle u, y \rangle|^2 \nu(dy) \geq C_\alpha \rho^{2-\alpha}.$$  

The equivalence follows from the computation

$$\int_{\{|\langle u, y \rangle| \leq \rho\}} |\langle u, y \rangle|^2 \nu(dy) = \int_{\mathbb{S}} |\langle u, \xi \rangle|^2 \mu(d\xi) \int_0^\infty 1_{\{|\langle u, \xi \rangle| \leq \frac{r}{2}\}} r^{1-\alpha} dr = \rho^{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi \rangle|^2 \mu(d\xi) \int_0^\infty \frac{ds}{s^{3-\alpha}} = \rho^{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi).$$

The Picard’s condition is usually imposed on the Lévy measure $\nu$ of a non-necessarily stable Lévy process $L$ in order to ensure that the law of $L_t$, for any $t > 0$, has a $C^\infty$-density with respect to the Lebesgue measure.

### 3 Some analytic regularity results

In this section we prove existence of regular solutions to (1.5). This will be achieved through Schauder estimates and will be important in Section 4 to prove uniqueness for (1.1).

We will use the following three properties of the $\alpha$-stable process $L$ (in the sequel $\mu_t$ denotes the law of $L_t$, $t \geq 0$).

(a) $\mu_t(A) = \mu_1(t^{-1/\alpha} A)$, for any $A \in \mathcal{B}(\mathbb{R}^d)$, $t > 0$ (this scaling property follows from (2.1) and (2.3));

(b) $\mu_t$ has a density $p_t$ with respect to the Lebesgue measure, $t > 0$; moreover $p_t \in C^1(\mathbb{R}^d)$ and its spatial derivative $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ (this is a consequence of Hypothesis 1);

(c) for any $\sigma > \alpha$, we have by (2.2)

$$\int_{\{x| \leq 1\}} |x|^\sigma \nu(dx) < \infty. \quad (3.1)$$

The fact that (b) holds can be deduced by an argument of [23, Section 3]. Actually, Hypothesis 1 implies the following stronger result.

**Lemma 3.1.** For any $\alpha \in (0, 2)$, $t > 0$, the density $p_t \in C^\infty(\mathbb{R}^d)$ and all derivatives $D^k p_t$ are integrable on $\mathbb{R}^d$, $k \geq 1$.  

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Proof. We only show that \( p_t \in C^\infty(\mathbb{R}^d) \) and \( Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d) \), following [23]; arguing in a similar way one can obtain the full assertion. By (2.4), we know that \( e^{-t\psi(u)} \leq e^{-C_{\alpha} |u|^\alpha}, \ u \in \mathbb{R}^d \), and so by the inversion formula of Fourier transform (see [22, Proposition 2.5]) \( \mu_t \) has a density \( p_t \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d), \)

\[
p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(x,z)} e^{-t\psi(z)} dz, \ x \in \mathbb{R}^d, \ t > 0. \tag{3.2}
\]

Note that (a) implies that \( p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x) \). Thanks to (2.4) one can differentiate infinitely many times under the integral sign and verifies that \( p_t \in C^\infty(\mathbb{R}^d) \). Let us fix \( j = 1, \ldots, d \) and check that the partial derivative \( \partial_{x_j} p_t \in L^1(\mathbb{R}^d) \). By the scaling property (a) it is enough to consider \( t = 1 \).

By writing \( \psi = \psi_1 + \psi_2 \),

\[
\psi_1(u) = -\int_{\{|y| \leq 1\}} (\cos((u,y)) - 1) \nu(dy), \ \psi_2 = \psi - \psi_1.
\]

\[
\partial_{x_j} p_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(x,z)} (-iz_j e^{-\psi_1(z)}) e^{-\psi_2(z)} dz, \ x \in \mathbb{R}^d.
\]

We find easily that \( \psi_1 \in C^\infty(\mathbb{R}^d) \) and so, using also (2.4) we deduce that \(-iz_j e^{-\psi_1(z)}\) is in the Schwartz space \( S(\mathbb{R}^d) \). In particular, there exists \( f_1 \in L^1(\mathbb{R}^d) \) such that the Fourier transform \( \hat{f}_1(z) = (-iz_j)e^{-\psi_1(z)} \). On the other hand (see [22, Section 8]), there exists an infinitely divisible probability measure \( \gamma \) on \( \mathbb{R}^d \) such that the Fourier transform \( \hat{\gamma}(z) = e^{-\psi_2(z)} \). By [22, Proposition 2.5] we infer that \( \hat{f}_1 \ast \gamma = \hat{f}_1 \cdot \hat{\gamma} \). By the inversion formula we deduce that \( \partial_{x_j} p_1(x) = (f_1 \ast \gamma)(x) \) and this proves that \( \partial_{x_j} p_t \in L^1(\mathbb{R}^d) \). 

Remark that (c) implies that the expression of \( \mathcal{L} f \) in (2.5) is meaningful for any \( f \in C_b^{1+\gamma}(\mathbb{R}^d) \) if \( 1 + \gamma > \alpha \). Indeed \( \mathcal{L} f(x) \) can be decomposed into the sum of two integrals, over \( \{|y| > 1\} \) and over \( \{|y| \leq 1\} \) respectively. The first integral is finite since \( f \) is bounded. To treat the second one, we can use the estimate

\[
|f(y + x) - f(x) - y \cdot Df(x)| \leq \int_0^1 |Df(x + ry) - Df(x)| |y| dr \leq |Df|_\gamma |y|^{1+\gamma}, \ |y| \leq 1.
\]

Note that \( \mathcal{L} f \in C_b(\mathbb{R}^d) \) if \( f \in C_b^{1+\gamma}(\mathbb{R}^d) \) and \( 1 + \gamma > \alpha \).

The next result is a maximum principle. A related result is in [10, Section 4.5]. This will be used to prove uniqueness of solutions to (1.5) as well as to study existence.

**Proposition 3.2.** Let \( \alpha \in (0,2) \). If \( u \in C_b^{1+\gamma}(\mathbb{R}^d) \), \( 1 + \gamma > \alpha \), is a solution to \( \lambda u - Lu - b \cdot Du = g \), with \( \lambda > 0 \) and \( g \in C_b(\mathbb{R}^d) \), then

\[
\|u\|_0 \leq \frac{1}{\lambda} \|g\|_0, \ \lambda > 0.
\]
Let us define the operator \( L \in \mathcal{Y} \). Let

To this purpose let

Equation (3.5) is meaningful for test function \( u(x) \). Indeed since

\[ |u(x) - u(y)| \leq \epsilon u(x) \]

by \( u \) without the hypothesis \( \alpha \). Without the hypothesis \( \alpha \) it is not restrictive in the study of pathwise uniqueness for (1.1).

Indeed, since \( c > 0 \), the condition \( \alpha + \beta > 1 \) which we impose is needed to have a regular \( C^1 \)-solution \( u \). On the other hand, the next result holds more generally without the hypothesis \( \alpha + \beta < 2 \). This is assumed just to simplify the proof and it is not restrictive in the study of pathwise uniqueness for (1.1). Indeed since \( C^\alpha_b (\mathbb{R}^d, \mathbb{R}^d) \subset C^\beta_b (\mathbb{R}^d, \mathbb{R}^d) \) when \( 0 < \beta \leq \beta' \), it is enough to study uniqueness when \( \beta \) satisfies \( \beta < 2 - \alpha \).

**Theorem 3.3.** Assume Hypothesis 1. Let \( \alpha \in (0, 2) \) and \( \beta \in (0, 1) \) be such that \( 1 < \alpha + \beta < 2 \). Then, for any \( \lambda > 0 \), \( k \in \mathbb{R}^d \), \( g \in C^\beta_b (\mathbb{R}^d) \), there exists a unique solution \( u = u_\lambda \in C^\alpha_b (\mathbb{R}^d) \) to the equation

\[
\lambda u - \mathcal{L} u - k \cdot Du = g
\]  

(3.5)
on \( \mathbb{R}^d \) (\( \mathcal{L} \) is defined in (2.5)). In addition there exists a constant \( c \) independent of \( g \), \( u \), \( k \) and \( \lambda > 0 \) such that

\[
\lambda \| u \|_0 + \lambda^{\frac{\alpha + \beta - 1}{\alpha}} \| Du \|_0 + \| Du \|_{\alpha + \beta - 1} \leq c \| g \|_\beta.
\]  

(3.6)

**Proof.** Equation (3.5) is meaningful for \( u \in C^\alpha_b (\mathbb{R}^d) \) with \( \alpha + \beta > 1 \) thanks to (3.3). Moreover, uniqueness follows from Proposition 3.2.

To prove the result, we use the semigroup approach as in [4]. To this purpose, we introduce the \( \alpha \)-stable Markov semigroup \( (P_t) \) acting on \( C_b (\mathbb{R}^d) \) and associated to \( \mathcal{L} + k \cdot Du \), i.e.,

\[
P_t f(x) = \int_{\mathbb{R}^d} f(z + tk) p_t(z - x) dz, \quad t > 0, \quad f \in C_b (\mathbb{R}^d), \quad x \in \mathbb{R}^d,
\]  

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where \( p_t \) is defined in (3.2), and \( P_0 = I \). Then we consider the bounded function \( u = u_\lambda \),

\[
  u(x) = \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^d. \tag{3.7}
\]

We are going to show that \( u \) belongs to \( C^{\alpha+\beta}_b(\mathbb{R}^d) \), verifies (3.6) and solves (3.5).

\( I \) Part. We prove that \( u \in C^{\alpha+\beta}_b(\mathbb{R}^d) \) and that (3.6) holds.

First note that \( \lambda \|u\|_0 \leq \|g\|_0 \) since \( (P_t) \) is a contraction semigroup. Then, using the scaling property \( p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x) \), we arrive at

\[
  |Dp_t f(x)| \leq \frac{t^{-1/\alpha}}{t^{d/\alpha}} \int_{\mathbb{R}^d} |f(z + tk)| |Dp_1(t^{-1/\alpha} z - t^{-1/\alpha} x)| dz \leq \frac{c_0 \|f\|_0}{t^{1/\alpha}}, \tag{3.8}
\]

\( t > 0, f \in C_b(\mathbb{R}^d) \), where \( c_0 = \|Dp_1\|_{L^1(\mathbb{R}^d)} \), and so we find the estimate

\[
  \|Dp_t f\|_0 \leq \frac{c_0}{t^{1/\alpha}} \|f\|_0, \quad f \in C_b(\mathbb{R}^d), \quad t > 0. \tag{3.9}
\]

By interpolation theory we know that \( (C_b(\mathbb{R}^d), C^1_b(\mathbb{R}^d))_{\beta,\infty} = C^d_b(\mathbb{R}^d), \beta \in (0,1) \), see for instance [16, Chapter 1]; interpolating the previous estimate with the estimate \( \|Dp_t f\|_0 \leq \|f\|_0, t \geq 0, f \in C^1_b(\mathbb{R}^d) \), we obtain

\[
  \|Dp_t f\|_0 \leq \frac{c_1}{t^{1-(1-\beta)/\alpha}} \|f\|_{\beta}, \quad t > 0, \quad f \in C^\beta_b(\mathbb{R}^d), \tag{3.10}
\]

with \( c_1 = c_1(c_0, \beta) \). In a similar way, we also find

\[
  \|D^2p_t f\|_0 \leq \frac{c_2}{t^{2-\beta/\alpha}} \|f\|_{\beta}, \quad t > 0, \quad f \in C^\beta_b(\mathbb{R}^d). \tag{3.11}
\]

Using (3.10) and the fact that \( \frac{1-\beta}{\alpha} < 1 \), we can differentiate under the integral sign in (3.7) and prove that there exists \( Du(x) = Du_\lambda(x), x \in \mathbb{R}^d \). Moreover \( Du_\lambda \) is bounded on \( \mathbb{R}^d \) and we have, for any \( \lambda > 0 \) with \( \tilde{c} \) independent of \( \lambda, u, k \) and \( g \),

\[
  \lambda^{\frac{\alpha+1}{\alpha}} \|Du\|_0 \leq \tilde{c} \|g\|_{\beta}
\]

(we have used that \( \int_0^\infty e^{-\lambda t} t^{-\sigma} dt = \frac{\tilde{c}}{\lambda^{\sigma-1}} \), for \( \sigma < 1 \) and \( \lambda > 0 \)).

It remains to prove that \( Du \in C^d_b(\mathbb{R}^d, \mathbb{R}^d) \), where \( \theta = \alpha - 1 + \beta \in (0,1) \).

We proceed as in the proof of [2, Proposition 4.2] and [18, Theorem 4.2].

Using (3.10), (3.11) and the fact that \( 2 - \beta > \alpha \), we find, for any \( x, x' \in \mathbb{R}^d, x \neq x' \),

\[
  |Du(x) - Du(x')| \leq C \|g\|_{\beta} \int_0^{\|x-x'\|^{\alpha}} \frac{1}{t^{1-\beta/\alpha}} dt + \int_{\|x-x'\|^{\alpha}}^\infty \frac{1}{t^{(2-\beta)/\alpha}} dt
  \leq c_3 \|g\|_{\beta} |x - x'|^\theta,
\]

and so \( |Du|_{\alpha+1+\beta} \leq c_3 \|g\|_{\beta} \), where \( c_3 \) is independent of \( g, u, k \) and \( \lambda \).
II Part. We prove that $u$ solves (3.5), for any $\lambda > 0$.

We use the fact that the semigroup $(P_t)$ is strongly continuous on the Banach space $C_0(\mathbb{R}^d)$; see [1, Section 6.7] and [22, Section 31].

Let $A : D(A) \subset C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ be its generator. By [22, Theorem 31.5]) $C_0^2(\mathbb{R}^d) \subset D(A)$ and moreover $Af = Lf + k \cdot Df$ if $f \in C_0^2(\mathbb{R}^d)$ (we say that $f$ belongs to $C_0^2(\mathbb{R}^d)$ if $f \in C_0^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and all its first and second partial derivatives belong to $C_0(\mathbb{R}^d)$).

We first show the assertion assuming in addition that $g \in C_0^2(\mathbb{R}^d)$.

It is easy to check that $u$ belongs to $C_0^2(\mathbb{R}^d)$ as well. To this purpose, one can use the estimates $\|D^kP_tg\|_0 \leq \|D^kg\|_0$, $t \geq 0$, $k = 1, 2$, and the dominated convergence theorem. On the other hand, by the Hille-Yosida theorem we know that $u \in D(A)$ and $\lambda u - Au = g$. Thus we have found that $u$ solves (3.5).

Let us prove the assertion when $g \in C_0^2(\mathbb{R}^d)$.

Note that also $u \in C_0^2(\mathbb{R}^d)$. We consider a function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(0) = 1$ and introduce $g_n(x) = \psi(x/n)g(x)$. $x \in \mathbb{R}^d$, $n \geq 1$. It is clear that $g_n$, $u_n \in C_0^2(\mathbb{R}^d)$ ($u_n$ is given in (3.7) when $g$ is replaced by $g_n$). We know that

$$\lambda u_n(x) - Lu_n(x) - k \cdot Du_n(x) = g_n(x), \quad x \in \mathbb{R}^d. \quad (3.12)$$

It is easy to see that there exists $C > 0$ such that $\|g_n\|_2 \leq C$, $n \geq 1$, and moreover $g_n$ and $Dg_n$ converge pointwise to $g$ and $Dg$ respectively. It follows that also $\|u_n\|_2$ is uniformly bounded and moreover $u_n$ and $Du_n$ converge pointwise to $u$ and $Du$ respectively. Using also (3.3), we can apply the dominated convergence theorem and deduce that

$$\lim_{n \to \infty} Lu_n(x) = Lu(x), \quad x \in \mathbb{R}^d.$$

Passing to the limit in (3.12), we obtain that $u$ is a solution to (3.5).

Let now $g \in C_0^2(\mathbb{R}^d)$.

Take any $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Define $\phi_n(x) = n^d \phi(nx)$ and $g_n = g \ast \phi_n$. Note that $(g_n) \subset C_0^\infty(\mathbb{R}^d) = \cap_{k \geq 1} C_k^\infty(\mathbb{R}^d)$ and $\|g_n\|_{\beta} \leq \|g\|_{\beta}, \quad n \geq 1$. Moreover, possibly passing to a subsequence still denoted by $(g_n)$, we may assume that

$$g_n \to g \quad \text{in} \quad C^\beta(K). \quad (3.13)$$

for any compact set $K \subset \mathbb{R}^d$ and $0 < \beta' < \beta$ (see page 37 in [12]). Let $u_n$ be given in (3.7) when $g$ is replaced by $g_n$. By the first part of the proof, we know that

$$\|u_n\|_{\alpha + \beta} \leq C\|g_n\|_{\beta} \leq C\|g\|_{\beta},$$

where $C$ is independent of $n$. It follows that, possibly passing to a subsequence still denoted with $(u_n)$, we have that $u_n \to u$ in $C^{\alpha + \beta'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $\beta' > 0$ such that $1 < \alpha + \beta' < \alpha + \beta$. Arguing as before, we can pass to the limit in $\lambda u_n(x) - Lu_n(x) - k \cdot Du_n(x) = g_n(x)$ and obtain that $u$ solves (3.5). The proof is complete.
Now we extend Theorem 3.3 to the case in which $b$ is Hölder continuous. We can only do this when $\alpha \geq 1$ (see also Remark 3.5). To prove the result when $\alpha = 1$ we adapt the localization procedure which is well known for second order uniformly elliptic operators with Hölder continuous coefficients (see [12]). This technique works in our situation since in estimate (3.6) the constant is independent of $k \in \mathbb{R}^d$.

We also need the following interpolatory inequalities (see [12, page 40, (3.3.7)]): for any $t \in [0, 1)$, $0 \leq s \leq r < 1$, there exists $N = N(d, k, r, t)$ such that if $f \in C^{\alpha \cdot s + s'}_b(\mathbb{R}^d, \mathbb{R}^k)$, then

$$[f]_{s+t} \leq N[f]_{s/r+t} [f]_{t}^{1-s/r},$$

(3.14)

where $[f]_{s+t}$ is defined as in (2.6) if $0 < s + t < 1$, $[f]_0 = \|f\|_0$, $[f]_1 = \|Df\|_0$, and $[f]_{s+t} = [Df]_{s+t-1}$ if $1 < s + t < 2$. By (3.14) we deduce, for any $\epsilon > 0$,

$$[f]_{s+t} \leq \tilde{N} \epsilon^{s-r}[f]_{r+t} + \tilde{N} \epsilon^{s-t}[f]_{t}, \quad f \in C^{\alpha \cdot s + s'}_b(\mathbb{R}^d, \mathbb{R}^k).$$

(3.15)

**Theorem 3.4.** Assume Hypothesis 1. Let $\alpha \geq 1$ and $\beta \in (0, 1)$ be such that $1 < \alpha + \beta < 2$. Then, for any $\lambda > 0$, $g \in C^{\beta}_b(\mathbb{R}^d)$, there exists a unique solution $u = u_\lambda \in C^{\alpha + \beta}_b(\mathbb{R}^d)$ to the equation

$$\lambda u - \mathcal{L} u - b \cdot Du = g$$

(3.16)

on $\mathbb{R}^d$. Moreover, for any $\omega > 0$, there exists $c = c(\omega)$, independent of $g$ and $u$, such that

$$\lambda \|u\|_0 + [Du]_{\alpha + \beta - 1} \leq c\|g\|_\beta, \quad \lambda \geq \omega.$$  

(3.17)

Finally, we have $\lim_{\lambda \to \infty} \|Du_\lambda\|_0 = 0$.

**Proof.** Uniqueness and estimate $\lambda \|u\|_0 \leq \|g\|_0, \lambda > 0$, follow from the maximum principle (see Proposition 3.2). Moreover, the last assertion follows from (3.17) using (3.14). Indeed, with $t = 0$, $s = 1$, $r = \alpha + \beta$, we obtain, for $\lambda \geq \omega$,

$$[Du_\lambda]_0 = [u_\lambda]_1 \leq N[Du_\lambda]_{\alpha + \beta - 1} \frac{1}{\alpha + \beta} [u_\lambda]_0^{1 - \frac{1}{\alpha + \beta}} \leq N \tilde{c} \lambda^{-\frac{\alpha + \beta - 1}{\alpha + \beta}} \|g\|_\beta,$$

where $\tilde{c} = \tilde{c}(\omega)$. Letting $\lambda \to \infty$, we get the assertion.

Let us prove existence and estimate $[Du]_{\alpha + \beta - 1} \leq c\|g\|_\beta$, for $\lambda \geq \omega$, with $\omega > 0$ fixed. We treat $\alpha > 1$ and $\alpha = 1$ separately.

**I Part (the case $\alpha > 1$).** In the sequel we will use the estimate

$$\|lf\|_\theta \leq \|l\|_0 \|f\|_\theta + \|f\|_0 |\theta|, \quad l, f \in C^{\theta}_b(\mathbb{R}^d), \quad \theta \in (0, 1).$$

(3.18)

Writing $\lambda u(x) - \mathcal{L} u(x) = g(x) + b(x) \cdot Du(x)$, and using (3.6) and (3.18), we obtain the following a-priori estimate (assuming that $u \in C^{\alpha + \beta}_b(\mathbb{R}^d)$ is a solution to (3.16))

$$[Du]_{\alpha + \beta - 1} \leq C\|g\|_\beta + C\|b \cdot Du\|_\beta$$

$$\leq C\|g\|_\beta + C\|b\|_\beta [Du]_0 + C\|b\|_0 [Du]_\beta,$$

(3.19)
where $C$ is independent of $\lambda > 0$. Combining the interpolatory estimates (see (3.15) with $t = 0, s = 1 + \beta, r = \alpha + \beta$)

$$[Du]_\beta \leq \tilde{N} \epsilon^{\alpha - 1}[Du]_{\alpha + \beta - 1} + \tilde{N} \epsilon^{-(1 + \beta)}\|u\|_0, \quad \epsilon > 0,$$

and $\|Du\|_0 \leq \tilde{N} \epsilon^{\alpha + \beta - 1}[Du]_{\alpha + \beta - 1} + \tilde{N} \epsilon^{-1}\|u\|_0$ (recall that $\alpha + \beta > 1 + \beta$) with the maximum principle, we get for $\epsilon$ small enough the a-priori estimate

$$[Du]_{\alpha + \beta - 1} \leq c_1 (\|g\|_\beta + C(\epsilon)\|u\|_0) \leq c_1 (\|g\|_\beta + \frac{C(\epsilon)}{\omega}\|g\|_\beta) \leq C_1 \|g\|_\beta,$$

(3.20)

for any $\lambda \geq \omega$. Now to prove the existence of a $C^{\alpha + \beta}_b$-solution, we use the continuity method (see, for instance, [12, Section 4.3]). Let us introduce

$$\lambda u(x) - L u(x) - \delta b(x) \cdot Du(x) = g(x), \quad (3.21)$$

$x \in \mathbb{R}^d$, where $\delta \in [0, 1]$ is a parameter. Let us define $\Gamma = \{\delta \in [0, 1] : \text{there is a unique solution } u = u_\delta \in C^{\alpha + \beta}_b(\mathbb{R}^d), \text{for any } g \in C^{\beta}_b(\mathbb{R}^d)\}$. Clearly $\Gamma$ is not empty since $0 \in \Gamma$. Fix $\delta_0 \in \Gamma$ and rewrite (3.21) as

$$\lambda u(x) - L u(x) - \delta b(x) \cdot Du(x) = g(x) + (\delta - \delta_0) b(x) \cdot Du(x).$$

Introduce the operator $S : C^{\alpha + \beta}_b(\mathbb{R}^d) \to C^{\alpha + \beta}_b(\mathbb{R}^d)$. For any $v \in C^{\alpha + \beta}_b(\mathbb{R}^d)$, $u = Sv$ is the unique $C^{\alpha + \beta}_b$-solution to $\lambda u(x) - L u(x) - \delta b(x) \cdot Du(x) = g(x) + (\delta - \delta_0) b(x) \cdot Du(x)$. By using (3.20), we get $\|Sv_1 - Sv_2\|_{\alpha + \beta} \leq 2|\delta - \delta_0| \cdot c_1 \|b\|_\beta \|v_1 - v_2\|_{\alpha + \beta}$. By choosing $|\delta - \delta_0|$ small enough, $S$ becomes a contraction and it has a unique fixed point which is the solution to (3.21). A compactness argument shows that $\Gamma = [0, 1]$. The assertion is proved.

**II Part (the case $\alpha = 1$).** As before, we establish the existence of a $C^{1 + \beta}_b(\mathbb{R}^d)$-solution, by using the continuity method. This requires the a-priori estimate (3.20) for $\alpha = 1$.

Let $u \in C^{1 + \beta}_b(\mathbb{R}^d)$ be a solution. Let $r > 0$. Consider a function $\xi \in C^\infty_c(\mathbb{R}^d)$ such that $\xi(x) = 1$ if $|x| \leq r$ and $\xi(x) = 0$ if $|x| > 2r$.

Let now $x_0 \in \mathbb{R}^d$ and define $\rho(x) = \xi(x - x_0)$, $x \in \mathbb{R}^d$, and $v = u \rho$. One can easily check that

$$Lv(x) = \rho(x)Lu(x) + u(x)L\rho(x) \quad (3.22)$$

$$+ \int_{\mathbb{R}^d} (\rho(x + y) - \rho(x)) (u(x + y) - u(x)) \nu(dy), \quad x \in \mathbb{R}^d.$$

We have

$$\lambda v(x) - L v(x) - b(x_0) \cdot Dv(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x), \quad x \in \mathbb{R}^d,$$

where

$$f_1(x) = \rho(x)g(x), \quad f_2(x) = (b(x) - b(x_0)) \cdot Dv(x),$$

We have

$$\lambda v(x) - L v(x) - b(x_0) \cdot Dv(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x), \quad x \in \mathbb{R}^d,$$
\[ f_3(x) = -u(x)[L \rho(x) + b(x) \cdot D \rho(x)], \]
\[ f_4(x) = -\int_{\mathbb{R}^d} (\rho(x+y) - \rho(x)) \left( u(x+y) - u(x) \right) \nu(dy), \quad x \in \mathbb{R}^d. \]

By Theorem 3.3 we know that
\[ [Dv]_\beta \leq C_1(\|f_1\|_\beta + \|f_2\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta), \]
where the constant \( C_1 \) is independent of \( x_0 \) and \( \lambda \). Let us consider the crucial term \( f_2 \). By (3.18) we find
\[
\|f_2\|_\beta \leq \left( \sup_{x \in B(x_0,2r)} |b(x) - b(x_0)| \right) [Dv]_\beta + \|Dv\|_0 b_\|b\|_\beta.
\]

Let us fix \( r \) small enough such that \( C_1 \sup_{x \in B(x_0,2r)} |b(x) - b(x_0)| < 1/2 \). We get
\[ [Dv]_\beta \leq 2C_1(\|f_1\|_\beta + \|Dv\|_0 b_\|b\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta). \]  
(3.24)

Note that \( \|f_1\|_\beta \leq C(r) \|g\|_\beta \). By the interpolatory estimates (3.15) and the maximum principle, arguing as in (3.20), we arrive at
\[ [Dv]_\beta \leq C_2(\|g\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta), \]
for any \( \lambda \geq \omega \). Let us estimate \( f_4 \). To this purpose we introduce the following non-local linear operator \( T \)
\[ T f(x) = \int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(f(x+y) - f(x)) \nu(dy), \quad f \in C_b^1(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \]

One can easily check that \( T \) is continuous from \( C_b^1(\mathbb{R}^d) \) into \( C_b(\mathbb{R}^d) \) and from \( C_b^{1+\beta}(\mathbb{R}^d) \) into \( C_b^1(\mathbb{R}^d) \). To this purpose we only remark that, for any \( x \in \mathbb{R}^d \),
\[
|DT f(x)| \leq 5 \|\rho\|_2 \|f\|_1 \left( \int_{\{|y| \leq 1\}} |y|^2 \nu(dy) + \int_{\{|y| > 1\}} \nu(dy) \right)
+ 5 \|\rho\|_1 \|f\|_{1+\beta} \left( \int_{\{|y| \leq 1\}} |y|^{1+\beta} \nu(dy) + \int_{\{|y| > 1\}} \nu(dy) \right), \quad f \in C_b^{1+\beta}(\mathbb{R}^d).
\]

By interpolation theory we know that
\[ \left( C_b^1(\mathbb{R}^d), C_b^{1+\beta}(\mathbb{R}^d) \right)_{\beta,\infty} = C_b^{1+\beta}(\mathbb{R}^d), \]

see [16, Chapter 1], and so we get that \( T \) is continuous from \( C_b^{1+\beta}(\mathbb{R}^d) \) into \( C_b^1(\mathbb{R}^d) \) (see [16, Theorem 1.1.6]). Since \( f_4 = -Tu \), we obtain the estimate
\[ \|f_4\|_\beta \leq C_3 \|u\|_{1+\beta^2}. \]

We have \( \|f_4\|_\beta + \|f_3\|_\beta \leq c_3(r) \|u\|_{1+\beta^2} \) and so
\[ [Dv]_\beta \leq C_4(\|g\|_\beta + \|u\|_{1+\beta^2}), \]
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where $C_4$ is independent of $\lambda \geq \omega$. It follows that $|D\nu|_{C^3(B(x_0,r))} \leq C_4(|g|_\beta + \|u\|_{1+\beta^2})$, where $B(x_0, r)$ is the ball of center $x_0$ and radius $r > 0$. Since $C_4$ is independent of $x_0$, we obtain

$$[D\nu]_{\beta} \leq C_4(|g|_\beta + \|u\|_{1+\beta^2}),$$

for any $\lambda \geq \omega$. Using again (3.15) and the maximum principle, we get the a-priori estimate (3.20) for $\alpha = 1$. The proof is complete.

**Remark 3.5.** In contrast with Theorem 3.3, in Theorem 3.4 we can not show existence of $C_{\theta}^{\alpha+\beta}$-solutions to (3.16) when $\alpha < 1$. The difficulty is evident from the a-priori estimate (3.19). Indeed, starting from

$$[D\nu]_{\alpha+\beta-1} \leq C|g|_\beta + C\|b\|_\beta \|Du\|_0 + C\|b\|_0 |Du|_\beta,$$

we cannot continue, since $\alpha < 1$ gives $Du \in C^\theta_0$ with $\theta = \alpha + \beta - 1 < \beta$. Roughly speaking, when $\alpha < 1$, the perturbation term $b \cdot Du$ is of order larger than $L$ and so we are not able to prove the desired a-priori estimates.

### 4 The main result

We briefly recall basic facts about Poisson random measures which we use in the sequel (see also [1], [14], [19], [28]). The Poisson random measure $N$ associated with the $\alpha$-stable process $L = (L_t)$ in (1.1) is defined by

$$N((0,t] \times U) = \sum_{0 \leq s \leq t} 1_{t}(\triangle L_s) = \{0 < s \leq t : \triangle L_s \in U\},$$

for any Borel set $U$ in $\mathbb{R}^d \setminus \{0\}$, i.e., $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $t > 0$. Here $\triangle L_s = L_s - L_{s-}$ denotes the jump size of $L$ at time $s > 0$. The compensated Poisson random measure $\tilde{N}$ is defined by $\tilde{N}((0, t] \times U) = N((0, t] \times U) - t \nu(U)$, where $\nu$ is given in (2.2). Recall the Lévy-Itô decomposition of the process $L$ (see [1, Theorem 2.4.16] or [14, Theorem 2.7]). This says that

$$L_t = \bar{b} t + \int_0^t \int_{|x| \leq 1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x| > 1} xN(ds, dx), \quad t \geq 0, \quad (4.1)$$

where $\bar{b} = E[L_1 - \int_0^1 \int_{|x| > 1} xN(ds, dx)]$. Note that in our case, since $\nu$ is symmetric, we have $\bar{b} = 0$.

The stochastic integral $\int_0^t \int_{|x| \leq 1} x \tilde{N}(ds, dx)$ is the compensated sum of small jumps and is an $L^2$-martingale. The process $\int_0^t \int_{|x| > 1} xN(ds, dx) = \int_{(0,t]} \int_{|x| > 1} xN(ds, dx)$ is a compound Poisson process.

Let $T > 0$. The predictable $\sigma$-field $\mathcal{P}$ on $\Omega \times [0,T]$ is generated by all left-continuous adapted processes (defined on the same stochastic basis fixed in Section 2). Let $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. In the sequel, we will always consider a $\mathcal{P} \times \mathcal{B}(U)$-measurable mapping $F : [0,T] \times U \times \Omega \to \mathbb{R}^d$. 


If 0 \notin \bar{U}, then \( \int_0^T \int_U F(s,x) N(ds,dx) = \sum_{0<s\leq T} F(s,\Delta L_s)1_U(\Delta L_s) \) is a random finite sum.

If \( E\int_0^T ds \int_U |F(s,x)|^2 \nu(dx) < \infty \), then one can define the stochastic integral
\[
Z_t = \int_0^t \int_U F(s,x) \tilde{N}(ds,dx), \quad t \in [0,T]
\]
(here we do not assume 0 \notin \bar{U}). The process \( Z = (Z_t) \) is an \( L^2 \)-martingale with a càdlàg modification. Moreover, \( E|Z_t|^2 = E\int_0^t ds \int_U |F(s,x)|^2 \nu(dx) \) (see [14, Lemma 2.4]). We will use the following \( L^p \)-estimates (see [14, Theorem 2.11] or the proof of Proposition 6.6.2 in [1]); for any \( p \geq 2 \), there exists \( c(p) > 0 \) such that
\[
E\left[ \sup_{0<s\leq t} |Z_s|^p \right] \leq c(p) E\left[ \left( \int_0^t ds \int_U |F(s,x)|^2 \nu(dx) \right)^{p/2} \right] + c(p) E\left[ \int_0^t ds \int_U |F(s,x)|^{p \nu}(dx) \right], \quad t \in [0,T]
\]
(4.2) (the inequality is obvious if the right-hand side is infinite).

Let us recall the concept of (strong) solution which we consider. A solution to the SDE (1.1) is a càdlàg \( \mathcal{F}_t \)-adapted process \( X^x = (X^x_t) \) (defined on \((\Omega,\mathcal{F},(\mathcal{F}_t)_{t \geq 0},P) \) fixed in Section 2) which solves (1.1) \( P \)-a.s., for \( t \geq 0 \).

It is easy to show the existence of a solution to (1.1) using the fact that \( b \) is bounded and continuous. We may argue at \( \omega \) fixed. Let us first consider \( t \in [0,1] \). By introducing \( v(t) = X_t - L_t \), we get the equation
\[
v(t) = x + \int_0^t b(v(s) + L_s)ds.
\]
Approximating \( b \) with smooth drifts \( b_n \) we find solutions \( v_n \in C([0,1];\mathbb{R}^d) \). By the Ascoli-Arzela theorem, we obtain a solution to (1.1) on \([0,1] \). The same argument works also on the time interval \([1,2] \) with a random initial condition. Iterating this procedure we can construct a solution for all \( t \geq 0 \).

The proof of Theorem 1.1 requires some lemmas. We begin with a deterministic result.

**Lemma 4.1.** Let \( \gamma \in [0,1] \) and \( f \in C_b^{1+\gamma}(\mathbb{R}^d) \). Then for any \( u, v \in \mathbb{R}^d \), \( x \in \mathbb{R}^d \), with \(|x| \leq 1 \), we have
\[
|f(u + x) - f(u) - f(v + x) + f(v)| \leq c_\gamma \|f\|_{1+\gamma} |u - v| |x|^\gamma, \quad \text{with} \ c_\gamma = 3^{1-\gamma}.
\]

**Proof.** For any \( x \in \mathbb{R}^d \), \(|x| \leq 1 \), define the linear operator \( T_x : C_b^1(\mathbb{R}^d) \rightarrow C_b^1(\mathbb{R}^d) \),
\[
T_x f(u) = f(u + x) - f(u), \quad f \in C_b^1(\mathbb{R}^d), \ u \in \mathbb{R}^d.
\]
Since \( \|T_x f\|_0 \leq \|D f\|_0 |x| \) and \( \|D(T_x f)\|_0 \leq 2 \|D f\|_0 \), it follows that \( T_x \) is continuous and \( \|T_x f\|_1 \leq (2 + |x|) \|f\|_1 \), \( f \in C_b^1(\mathbb{R}^d) \). Similarly, \( T_x \) is continuous from \( C_b^2(\mathbb{R}^d) \) into \( C_b^1(\mathbb{R}^d) \) and
\[
\|T_x f\|_1 \leq |x| \|f\|_2, \quad f \in C_b^2(\mathbb{R}^d).
\]
Lemma 4.2. Let $X$ and $b$ be bounded. The next two results hold for SDEs of type (1.1) when $\lambda > 0$ for some $t > 0$ and moreover $x(0) = 0$. The proof is complete.

Proof. First note that the stochastic integral in (4.4) is meaningful thanks to the estimate

$$|f(u + x) - f(u) - f(v + x) + f(v)| = |T_x f(u) - T_x f(v)| \leq \|DT_x f\|_0 |u - v|.$$ 

The proof is complete.

In the sequel we will consider the following resolvent equation on $\mathbb{R}^d$

$$\lambda u - Lu - Du \cdot b = b,$$  

(4.3)

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is given in (1.1), $L$ in (2.5) and $\lambda > 0$ (the equation must be understood componentwise, i.e., $\lambda u_i - Lu_i - b \cdot Du_i = b_i$, $i = 1, \ldots, d$). The next two results hold for SDEs of type (1.1) when $b$ is only continuous and bounded.

Lemma 4.2. Let $\alpha \in (0, 2)$ and $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ in (1.1). Assume that, for some $\lambda > 0$, there exists a solution $u \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (4.3) with $\gamma \in [0, 1]$, and moreover

$$1 + \gamma > \alpha.$$ 

Let $X = (X_t)$ be a solution of (1.1) starting at $x \in \mathbb{R}^d$. We have, P-a.s.,

$$t \geq 0, u(X_t) - u(x)$$

(4.4)

$$= x - X_t + L_t + \lambda \int_0^t u(X_s)ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \nu(ds, dx).$$

Proof. First note that the stochastic integral in (4.4) is meaningful thanks to the estimate

$$E \int_0^t ds \int_{\mathbb{R}^d} |u(X_{s-} + x) - u(X_{s-})|^2 \nu(dx)$$

(4.5)

$$\leq 4t \|u\|^2_2 \int_{\{|x| > 1\}} \nu(dx) + t \|D^2 u\|^2_2 \int_{\{|x| \leq 1\}} |x|^2 \nu(dx) < \infty.$$

The assertion is obtained applying Itô’s formula to $u(X_t)$ (for more details on Itô’s formula see [1, Theorem 4.4.7] and [14, Section 2.3]).

Let us fix $i = 1, \ldots, d$ and set $u_i = f$. A difficulty is that Itô’s formula is usually stated assuming that $f \in C^2(\mathbb{R}^d)$. However, in the present situation in which $L$ is $\alpha$-stable, using (3.1), one can show that Itô’s formula holds for $f(X_t)$ when $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. We give a proof of this fact.

We assume that $\gamma > 0$ (the proof with $\gamma = 0$ is similar). By convolution with mollifiers, as in (3.13) we obtain a sequence $(f_n) \subset C_{\infty}^\gamma(\mathbb{R}^d)$ such that $f_n \to f$ in $C^{1+\gamma'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $0 < \gamma' < \gamma$. 
Moreover, \( \| f_n \|_{1+\gamma} \leq \| f \|_{1+\gamma}, n \geq 1 \). Let us fix \( t > 0 \). By Itô’s formula for \( f_n(X_t) \) we find, \( P \)-a.s.,

\[
\begin{align*}
&f_n(X_t) - f_n(x) \\
&= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left[ f_n(X_{s-} + x) - f_n(X_{s-}) \right] \tilde{N}(ds, dx) \\
&\quad + \int_0^t ds \int_{\mathbb{R}^d} \left[ f_n(X_{s-} + x) - f_n(X_{s-}) - 1_{\{|x| \leq 1\}} x \cdot Df_n(X_{s-}) \nu(dx) \right] \\
&\quad + \int_0^t b(X_s) \cdot Df_n(X_s) ds.
\end{align*}
\] (4.6)

It is not difficult to pass to the limit as \( n \to \infty \); we show two arguments which are needed. To deal with the integral involving \( \nu \), one can apply the dominated convergence theorem, thanks to the following estimate similar to (3.3),

\[
|f_n(X_{s-} + x) - f_n(X_{s-}) - x \cdot Df_n(X_{s-})| \leq |Df|_{\gamma}|x|^{1+\gamma}, \quad |x| \leq 1
\]

(recall that \( \int_{\{|x| \leq 1\}} |x|^{1+\gamma} \nu(dx) < \infty \) since \( 1 + \gamma > \alpha \)). To pass to the limit in the stochastic integral with respect to \( \tilde{N} \), one uses the isometry formula

\[
E \left| \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left[ f_n(X_{s-} + x) - f_n(X_{s-}) - f(X_{s-} + x) + f(X_{s-}) \right] \tilde{N}(ds, dx) \right|^2
\]

(4.7)

Arguing as in (4.5), since \( \| f_n \|_{1+\gamma} \leq \| f \|_{1+\gamma}, n \geq 1 \), we can apply the dominated convergence theorem in (4.7). Letting \( n \to \infty \) in (4.7) we obtain 0. Finally, we pass to the limit in probability in (4.6) and obtain Itô’s formula when \( f \in C_b^{1+\gamma}(\mathbb{R}^d) \).

Noting that, for any \( i = 1, \ldots, d, \)

\[
L u_i(y) = \int_{\mathbb{R}^d} [u_i(y + x) - u_i(y) - 1_{\{|x| \leq 1\}} x \cdot Du_i(y)] \nu(dx), \quad y \in \mathbb{R}^d,
\]

and using that \( u \) solves (4.3), i.e., \( L u + b \cdot Du = \lambda u - b \), we can replace in the Itô formula for \( u(X_t) \) the term

\[
\int_0^t L u(X_s) ds + \int_0^t Du(X_s) b(X_s) ds
\]

\[
= \sum_{i=1}^d \left( \int_0^t L u_i(X_s) ds + \int_0^t Du_i(X_s) \cdot b(X_s) ds \right) e_i
\]

with \(- \int_0^t b(X_s) ds + \lambda \int_0^t u(X_s) ds = x - X_t + L_t + \lambda \int_0^t u(X_s) ds \) and obtain the assertion. 

\[ \blacksquare \]
The proof of Theorem 1.1 will be a consequence of the following result.

**Theorem 4.3.** Let \( \alpha \in (0, 2) \) and \( b \in C_b(\mathbb{R}^d, \mathbb{R}^d) \) in (1.1). Assume that, for some \( \lambda > 0 \), there exists a solution \( u = u_\lambda \in C_{k+\gamma}^\mathbb{R}(\mathbb{R}, \mathbb{R}^d) \) to the equation (4.3) with \( \gamma \in [0, 1] \), such that \( c_\lambda = \|Du_\lambda\|_0 < 1/3 \). Moreover, assume that

\[
2\gamma > \alpha.
\]

Then the SDE (1.1), for every \( x \in \mathbb{R}^d \), has a unique solution \( (X_t^x) \).

Moreover, assertions (i), (ii) and (iii) of Theorem 1.1 hold.

**Proof.** Note that \( 2\gamma > \alpha \) implies the condition \( 1 + \gamma > \alpha \) of Lemma 4.2.

We provide a direct proof of pathwise uniqueness and assertion (i). This uses Lemmas 4.2 and 4.1 together with \( L^p \)-estimates for stochastic integrals (see (4.2)). Statements (ii) and (iii) will be obtained by transforming (1.1) in a form suitable for applying the results in [14, Chapter 3].

Let us fix \( t > 0 \), \( p \geq 2 \) and consider two solutions \( X \) and \( Y \) of (1.1) starting at \( x \) and \( y \in \mathbb{R}^d \) respectively. Note that \( X_t \) is not in \( L^p \) if \( p \geq \alpha \) (compare with [14, Theorem 3.2]) but the difference \( X_t - Y_t \) is a bounded process. Pathwise uniqueness and (1.4) (for any \( p \geq 1 \)) follow if we prove

\[
E[ \sup_{0 \leq s \leq t} |X_s - Y_s|^p ] \leq C(t) |x - y|^p, \quad x, y \in \mathbb{R}^d,
\]

with a positive constant \( C(t) \) independent of \( x \) and \( y \). Indeed in the special case of \( x = y \) estimate (4.8) gives uniqueness of solutions.

We have from Lemma 4.2, \( P \)-a.s.,

\[
X_t - Y_t = [x - y] + [u(x) - u(y)] + [u(Y_t) - u(X_t)] + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx)
\]

\[
+ \lambda \int_0^t [u(X_s) - u(Y_s)] ds.
\]

Since \( \|Du\|_0 \leq 1/3 \), we have \( |u(X_t) - u(Y_t)| \leq \frac{1}{3} |X_t - Y_t| \). It follows the estimate \( |X_t - Y_t| \leq \frac{3}{2} \Lambda_1(t) + \frac{3}{2} \Lambda_2(t) + \frac{3}{2} \Lambda_3(t) + \frac{3}{2} \Lambda_4(t) \), where

\[
\Lambda_1(t) = \int_0^t \int_{\{|x| > 1\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx),
\]

\[
\Lambda_2(t) = \lambda \int_0^t |u(X_s) - u(Y_s)| ds,
\]

\[
\Lambda_3(t) = \int_0^t \int_{\{|x| \leq 1\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx),
\]

\[
\Lambda_4 = |x - y| + |u(x) - u(y)| \leq \frac{4}{3} |x - y|. \quad \text{Note that, } P \text{-a.s.,}
\]

\[
\sup_{0 \leq s \leq t} |X_s - Y_s|^p \leq C_p |x - y|^p + C_p \sum_{k=1}^{3} \sup_{0 \leq s \leq t} \Lambda_k(s)^p.
\]
We obtain
\[ \sup_{0 \leq s \leq t} \Lambda_2(s)^p \leq c_1(p) t^{p-1} \int_0^t \sup_{0 \leq s \leq r} |X_s - Y_r|^p \, dr. \]

By (4.2) with \( U = \{ x \in \mathbb{R}^d : |x| > 1 \} \) we find
\[ E[\sup_{0 \leq s \leq t} \Lambda_1(s)^p] \]
\[ \leq c(p) E\left[ \left( \int_0^t \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^2 \nu(dx) \right)^{p/2} \right] \]
\[ + c(p) E \int_0^t \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^p \nu(dx). \]

Using \(|u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})| \leq \frac{2}{|x|} |X_{s-} - Y_{s-}|\) and the Hölder inequality, we get
\[ E[\sup_{0 \leq s \leq t} \Lambda_1(s)^p] \leq C_1(p) (1 + t^{p/2-1}). \]

Let us treat \( \Lambda_3(t) \). This requires the condition \( 2\gamma > \alpha \). By using (4.2) with \( U = \{ x \in \mathbb{R}^d : |x| \leq 1, x \neq 0 \} \) and also Lemma 4.1, we get
\[ E[\sup_{0 \leq s \leq t} \Lambda_3(s)^p] \leq c(p) \|u\|_{1+\gamma}^p E\left[ \left( \int_0^t \int_{\{|x|\leq 1\}} |X_s - Y_s|^2 |x|^{2\gamma} \nu(dx) \right)^{p/2} \right] \]
\[ + c(p) \|u\|_{1+\gamma}^p \int_0^t \int_{\{|x|\leq 1\}} |X_s - Y_s|^p |x|^{\gamma} \nu(dx). \]

We obtain
\[ E[\sup_{0 \leq s \leq t} \Lambda_3(s)^p] \leq C_2(p) (1 + t^{p/2-1}) \|u\|^p_{1+\gamma}. \]

Letting
\[ \left( \int_{\{|x| \leq 1\}} |x|^{2\gamma} \nu(dx) \right)^{p/2} + \int_{\{|x| \leq 1\}} |x|^{\gamma} \nu(dx) \]
\[ E[\sup_{0 \leq s \leq r} |X_s - Y_s|^p] \, dr, \]

where \( \int_{\{|x| \leq 1\}} |x|^p \nu(dx) < +\infty \), since \( p \geq 2 \) and \( 2\gamma > \alpha \). Collecting the previous estimates, we arrive at
\[ E[\sup_{0 \leq s \leq t} |X_s - Y_s|^p] \leq C_p |x-y|^p + C_4(p) (1 + t^{p-1}) \int_0^t E[\sup_{0 \leq s \leq r} |X_s - Y_s|^p] \, dr. \]

Applying the Gronwall lemma we obtain (4.8) with \( C(t) = C_p \exp(C_4(p) (1 + t^{p-1})) \). The assertion is proved.

Now we establish the homeomorphism property (ii) (cf. [14, Chapter 3], [1, Chapter 6] and [19, Section V.10]).
First note that, since $\|Du\|_0 < 1/3$, the classical Hadamard theorem (see [19, page 330]) implies that the mapping $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\psi(x) = x + u(x)$, $x \in \mathbb{R}^d$, is a $C^1$-diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Moreover, $D\psi^{-1}$ is bounded on $\mathbb{R}^d$ and $\|D\psi^{-1}\|_0 \leq \frac{1}{1-\delta_x} < \frac{3}{2}$ thanks to
\[
D\psi^{-1}(y) = [I + Du(\psi^{-1}(y))]^{-1} = \sum_{k \geq 0} (-Du(\psi^{-1}(y)))^k, \; y \in \mathbb{R}^d. \tag{4.10}
\]

Let $r \in (0,1)$ and introduce the SDE
\[
Y_t = y + \int_0^t \tilde{b}(Y_s)ds + \int_0^t \int_{\{|z| \leq r\}} g(Y_s-z)\tilde{N}(ds,dz) + \int_0^t \int_{\{|z| > r\}} g(Y_s-z)N(ds,dz), \; t \geq 0,
\]
where $\tilde{b}(y) = \lambda u(\psi^{-1}(y)) - \int_{\{|z| > r\}}[u(\psi^{-1}(y) + z) - u(\psi^{-1}(y))]\nu(dz)$ and $g(y,z) = u(\psi^{-1}(y) + z) + z - u(\psi^{-1}(y))$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^d$.

Note that (4.11) is a SDE of the type considered in [14, Section 3.5]. Due to the Lipschitz condition, there exists a unique solution $Y^y = (Y^y_t)_{t \geq 0}$ to (4.11). Moreover, using (4.4) and the formula
\[
L_t = \int_0^t \int_{\{|z| \leq r\}} x\tilde{N}(ds,dx) + \int_0^t \int_{\{|z| > r\}} xN(ds,dx), \; t \geq 0
\]
(due to the fact that $\nu$ is symmetric) it is not difficult to show that
\[
\psi(X^y_t) = Y^\psi(x), \; x \in \mathbb{R}^d, \; t \geq 0. \tag{4.12}
\]

Thanks to (4.12) to prove our assertion, it is enough to show the homeomorphism property for $Y^y_t$. To this purpose, we will apply [14, Theorem 3.10] to equation (4.11). Let us check its assumptions.

Clearly, $\tilde{b}$ is Lipschitz continuous and bounded. Let us consider [14, condition (3.22)]. For any $y \in \mathbb{R}^d$, $z \in \mathbb{R}^d$, $|g(y,z)| \leq |z|((1 + \|Du\|_0)$ $\leq K(z)$, with $K(z) = \frac{3}{2}|z|$ (recall that $\int_{|z| \leq 1}|z|^2\nu(dz) < \infty$); further by Lemma 4.1 and (4.10) we have, for any $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^d$ with $|z| \leq 1$,
\[
|g(y, z) - g(y', z)| \leq L(z)|y - y'| \; \text{where} \; L(z) = C_1\|u\|_{1+\gamma} |z|^{-\gamma},
\]
with $\int_{|z| \leq 1} L(z)^2\nu(dz) < \infty$, since $2\gamma > \alpha$. Note that we may fix $r > 0$ small enough in (4.11) in order that $K(r) + L(r) < 1$ (according to [14, Section 3.5], this condition is needed to study the homeomorphism property for equation (4.11) without $\int_0^t \int_{\{|z| > r\}} g(Y_s-z)N(ds,dz)$; see also [14, Remark 1, Section 3.4]).

By [14, Theorem 3.10] in order to get the homeomorphism property, it remains to check that, for any $z \in \mathbb{R}^d$, the mapping:
\[
y \mapsto y + g(y, z) \; \text{is a homeomorphism from} \; \mathbb{R}^d \; \text{onto} \; \mathbb{R}^d. \tag{4.13}
\]
Let us fix \( z \). To verify the assertion, we will again apply the Hadamard theorem. We have
\[
D_yg(y, z) = \left[ Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y)) \right] [D\psi^{-1}(y)]
\]
and so by (4.10) (since \( \|Du\|_0 < 1/3 \)) we get \( \|D_yg(\cdot, z)\| \leq \frac{2\lambda}{1-\lambda} < 1 \). We have obtained (4.13). By [14, Theorem 3.10] the homeomorphism property for \( Y_t^y \) follows and this gives the assertion.

Now we show that, for any \( t \geq 0 \), the mapping: \( x \mapsto X_t^x \) is of class \( C^1 \) on \( \mathbb{R}^d \), P-a.s. (see (iii)).

We fix \( t > 0 \) and a unitary vector \( e_k \) of the canonical basis in \( \mathbb{R}^d \). We will show that there exists, P-a.s., the partial derivative \( \lim_{s \to 0} X_t^{s+se_k} - X_t^x \) and, moreover, that the mapping \( x \mapsto D_{e_k}X_t^x \) is continuous on \( \mathbb{R}^d \), P-a.s..

Let us consider the process \( Y^y = (Y_t^y) \) which solves the SDE (4.11). If we prove that the mapping \( y \mapsto Y_t^y \) is of class \( C^1 \) on \( \mathbb{R}^d \), P-a.s., then we have proved the assertion. Indeed, P-a.s.,
\[
D_{e_k}X_t^x = [D\psi^{-1}(Y_t^{\psi(x)}))[D_\psi \psi(x)] D_{e_k} \psi(x), \ x \in \mathbb{R}^d.
\]
We rewrite (4.11) as
\[
Y_t = y + \lambda \int_0^t u(\psi^{-1}(Y_r^y))dr + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(Y_r^y, z)\tilde{N}(dr, dz) + L_t, \quad (4.14)
\]
t \geq 0, \ y \in \mathbb{R}^d, where
\[
h(y, z) = u(\psi^{-1}(y) + z) - u(\psi^{-1}(y)) = g(y, z) - z,
\]
and note that the statement of [14, Theorem 3.4] about the differentiability property holds for SDEs of the form (4.14), provided that the coefficients \( \lambda u \circ \psi^{-1} \) and \( h \) satisfy [14, conditions (3.1), (3.2), (3.8) and (3.9)]. Indeed the presence of \( L_t \) in the equation does not give rise to any difficulty. To check this fact, remark that, for any \( t \geq 0, y \in \mathbb{R}^d, s \neq 0 \), we have the equality
\[
\frac{Y_t^{y+se_k} - Y_t^y}{s} = e_k + \left( \lambda \int_0^t u(\psi^{-1}(Y_r^{y+se_k})) - u(\psi^{-1}(Y_r^y)) \right) dr
\]
\[
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \frac{h(Y_r^{y+se_k}, z) - h(Y_r^y, z)}{s} \tilde{N}(dr, dz),
\]
where \( L_t \) is disappeared. Thus we can apply the same argument which is used to prove [14, Theorem 3.4] (see also the proof of [14, Theorem 3.3]), i.e., we can provide estimates for
\[
E \left[ \sup_{0 \leq t \leq T} \left| \frac{Y_t^{y+se_k} - Y_t^y}{s} \right|^p \right] \text{ and } E \left[ \sup_{0 \leq t \leq T} \left| \frac{Y_t^{y+se_k} - Y_t^y}{s} - \frac{Y_t^{y+s' e_k} - Y_t^{y'}}{s'} \right|^p \right],
\]
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Let us check that \( \lambda u \circ \psi^{-1} \) and \( h \) satisfy the assumptions of \cite[Theorem 3.4]{14} (i.e., respectively, \cite[conditions (3.1), (3.2), (3.8) and (3.9)]{14}). Conditions (3.1) and (3.2) are easy to check. Indeed \( \lambda u(\psi^{-1}(\cdot)) \) is Lipschitz continuous on \( \mathbb{R}^d \) and, moreover, thanks to Lemma 4.1 and to the boundedness of \( D\psi^{-1} \),

\[
|h(y, z) - h(y', z)| \leq C\|u\|_{1+\gamma}(1_{\{|z| \leq 1\}}|z|^\gamma + 1_{\{|z| > 1\}}) \ |y - y'|, \quad z \in \mathbb{R}^d,
\]

\( y, y' \in \mathbb{R}^d \), with \( \int_{\mathbb{R}^d}(1_{\{|z| \leq 1\}}|z|^\gamma + 1_{\{|z| > 1\}})^p \nu(dz) < \infty \), for any \( p \geq 2 \). In addition, \( |h(y, z)| \leq L_0(z), \quad z \in \mathbb{R}^d, \quad y \in \mathbb{R}^d \), where, since \( \|Du\|_0 < 1/3 \),

\[
L_0(z) = \frac{1}{3} 1_{\{|z| \leq 1\}}|z| + 2\|u\|_0 1_{\{|z| > 1\}} \quad \text{with} \quad \int_{\mathbb{R}^d} L_0(z)^p \nu(dz) < \infty, \quad p \geq 2.
\]

Assumptions \cite[(3.8) and (3.9)]{14} are more difficult to check. They require that there exists some \( \delta > 0 \) such that (setting \( l(x) = \lambda u(\psi^{-1}(x)) \))

\[
(1) \sup_{y \in \mathbb{R}^d} |Dl(y)| < \infty; \quad |Dl(y) - Dl(y')| \leq C|y - y'|^{\delta}, \quad y, y' \in \mathbb{R}^d.
\]

\[
(2) |Dg_h(y, z)| \leq K_1(z); \quad |Dg_h(y, z) - Dg_h(y', z)| \leq K_2(z) |y - y'|^{\delta}, \quad (4.15)
\]

for any \( y, y' \in \mathbb{R}^d, \quad z \in \mathbb{R}^d \), with \( \int_{\mathbb{R}^d} K_i(z)^p \nu(dz) < \infty \), for any \( p \geq 2 \), \( i = 1, 2 \). Such estimates are used in \cite{14} in combination with the Kolmogorov continuity theorem to show the differentiability property.

Let us check (1) with \( \delta = \gamma \), i.e., \( Dl \in C^\gamma_0(\mathbb{R}^d, \mathbb{R}^d) \). Since, for any \( y \in \mathbb{R}^d \), \( Dl(y) = \lambda Du(\psi^{-1}(y))D\psi^{-1}(y) \), we find that \( Dl \) is bounded on \( \mathbb{R}^d \). Moreover, thanks to the following estimate (cf. (3.18))

\[
|Dl|_\gamma \leq \lambda \|Du\|_0 |D\psi^{-1}|_\gamma + \lambda \|Du\|_\gamma \|D\psi^{-1}\|_{1+\gamma}^1,
\]

in order to prove the assertion it is enough to show that \( |D\psi^{-1}|_\gamma < \infty \). Recall that for \( d \times d \) real matrices \( A \) and \( B \), we have \( (I + A)^{-1} - (I + B)^{-1} = (I + A)^{-1}(B - A)(I + B)^{-1} \) (if \((I + A)\) and \((I + B)\) are invertible). We obtain, using also that \( D\psi^{-1} \) is bounded, \( D\psi^{-1}(y) - D\psi^{-1}(y') | = | I + Du(\psi^{-1}(y))^{-1} - [I + Du(\psi^{-1}(y'))^{-1}] |

\[
\leq c_1 \|Du\|_\gamma |y - y'|^{\gamma}, \quad y, y' \in \mathbb{R}^d
\]

and the proof of (1) is complete with \( \gamma = \delta \). Let us consider (2). Clearly,

\[
Dg_h(y, z) = [Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y))]D\psi^{-1}(y)
\]

verifies the first part of (2) with \( K_1(z) = c_2 \|Du\|_\gamma (1_{\{|z| \leq 1\}}|z|^{\gamma} + 1_{\{|z| > 1\}}) \).
Let us deal with the second part of (2). We choose \( \gamma' \in (0, \gamma) \) such that \( 2\gamma' > \alpha \) and first show that, for any \( f \in C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d) \), we have
\[
[T_xf]_{\gamma-\gamma'} \leq C[|f|_{\gamma}] |x|^{\gamma'} , \ x \in \mathbb{R}^d ,
\]  
where (as in Lemma 4.1) for any \( x \in \mathbb{R}^d \), we define the mapping \( T_xf : \mathbb{R}^d \to \mathbb{R}^d \) as \( T_xf(u) = f(x + u) - f(u) , u \in \mathbb{R}^d \). Using also (3.14), we get
\[
[T_xf]_{\gamma-\gamma'} \leq N[T_xf]^{\frac{\gamma-\gamma'}{\gamma}} [T_xf]^{1-\frac{\gamma-\gamma'}{\gamma}} \leq cN[|f|_{\gamma}] |x|^{\gamma(1-\frac{\gamma-\gamma'}{\gamma})} \leq cN[|f|_{\gamma}] |x|^{\gamma'} ,
\]  
for any \( x \in \mathbb{R}^d \). By (4.16) we will prove (2) with \( \delta = \gamma - \gamma' > 0 \).

First consider the case when \( |z| \leq 1 \). By (4.16) with \( Du = f \), we get
\[
|D_yh(y, z) - D_yh(y', z)| \leq C_1[Du]_{\gamma}|y - y'|^\delta |z|^{\gamma'},
\]  
for any \( y, y' \in \mathbb{R}^d \). Let now \( |z| > 1 \); we find, for \( y, y' \in \mathbb{R}^d \) with \( |y - y'| \leq 1 \),
\[
|D_yh(y, z) - D_yh(y', z)| \leq C_2|Du|_{\gamma}|y - y'|^\gamma \leq C_2[Du]_{\gamma}|y - y'|^{\gamma-\gamma'}.
\]  
On the other hand, if \( |y - y'| > 1 , |z| > 1 , |D_yh(y, z) - D_yh(y', z)| \leq 4\|Du\|_{0}|y - y'|^{\gamma-\gamma'} \). In conclusion, the second part of (2) is verified with \( \delta = \gamma - \gamma' \) and
\[
K_2(z) = C_3\|Du\|_{\gamma}(1_{\{|z|\leq 1\}}|z|^{\gamma'} + 1_{\{|z|>1\}}).
\]  
(note that \( \int_{\mathbb{R}^d} K_2(z)^p \nu(dz) < \infty \), for any \( p \geq 2 \), since \( 2\gamma' > \alpha \)). Since
\( C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^{\gamma-\gamma'}(\mathbb{R}^d, \mathbb{R}^d) \), we deduce that both (1) and (2) hold with \( \delta = \gamma - \gamma' \).

Arguing as in [14, Theorem 3.4], we get that \( y \mapsto Y_t^y \) is \( C^1 \), \( P \)-a.s., and this proves our assertion. We finally note that [14, Theorem 3.4] also provides a formula for \( H_t^y = DY_t^y \), i.e.,
\[
H_t^y = I + \lambda \int_0^t Du(\psi^{-1}(Y_s^y)) D\psi^{-1}(Y_s^y) H_s^y ds
\]  
\[+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left( D_yh(Y_s^y, z) H_s^{y,z} \right) \tilde{N}(ds, dz) , \ t \geq 0 , y \in \mathbb{R}^d .
\]  
The stochastic integral is meaningful, thanks to (2) in (4.15) and to the estimate \( \sup_{0 \leq t \leq 1} E[|H_t|^p] < \infty \), for any \( t > 0 , p \geq 2 \) (see [14, assertion (3.10)]). The proof is complete.

**Proof of Theorem 1.1.** We may assume that \( 1 - \alpha/2 < \beta < 2 - \alpha \). We will deduce the assertion from Theorem 4.3.

Since \( \alpha \geq 1 \), we can apply Theorem 3.4 and find a solution \( u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d) \) to the resolvent equation (4.3) with \( \gamma = \alpha - 1 + \beta \in (0, 1) \). By the last assertion of Theorem 3.4, we may choose \( \lambda \) sufficiently large in order that \( \|Du\|_0 = \|Du_\lambda\|_0 < 1/3 \). The crucial assumption about \( \gamma \) and \( \alpha \) in Theorem 4.3 is satisfied. Indeed \( 2\gamma = 2\alpha - 2 + 2\beta > \alpha \) since \( \beta > 1 - \alpha/2 \).

By Theorem 4.3 we obtain the result.

\[\square\]
Remark 4.4. Thanks to Theorem 1.1 we may define a stochastic flow associated to (1.1). To this purpose, note that by (ii) we have $X_t^x = \xi_t(x)$, $t \geq 0$, $x \in \mathbb{R}^d$, $P$-a.s., where $\xi_t$ is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let $\xi_t^{-1}$ be the inverse map. As in [14, Section 3.4], we set $\xi_s,t(x) = \xi_t \circ \xi_s^{-1}(x)$, $0 \leq s \leq t$, $x \in \mathbb{R}^d$.

The family $(\xi_{s,t})$ is a stochastic flow since verifies the following properties ($P$-a.s): (i) for any $x \in \mathbb{R}^d$, $(\xi_{s,t}(x))$ is a càdlàg process with respect to $t$ and a càdlàg process with respect $s$; (ii) $\xi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ is an onto homeomorphism, $s \leq t$; (iii) $\xi_{s,t}(x)$ is the unique solution to (1.1) starting from $x$ at time $s$; (iv) we have $\xi_{s,t}(x) = \xi_{u,t}(\xi_{s,u}(x))$, for all $0 \leq s \leq u \leq t$, $x \in \mathbb{R}^d$, and $\xi_{s,s}(x) = x$.

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References


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