A recent theorem of Brattka, Miller and Nies [1] shows that a real number \( r \) in the unit interval is computably random if and only if every nondecreasing computable function from the unit interval to the real numbers is differentiable at \( r \). Here we establish a counterpart result that characterizes normality to a given base in terms of differentiability of functions computable with finite transducers (injective finite state automata).

For a real number \( r \) we consider the unique expansion in base \( b \) of the form
\[
 r = \left\lfloor x \right\rfloor + \sum_{n=1}^{\infty} a_n b^{-n}
\]
where the integers \( 0 \leq a_n < b \), and \( a_n < b - 1 \) infinitely many times. This last condition over \( a_n \) ensures a unique representation of every rational number. Let us recall that Borel’s original definition of normality in [2] is equivalent to the following simpler one [3].

**Definition.** A real number \( r \) is simply normal to a given base \( b \) if each digit in \( \{0, 1, \ldots, (b-1)\} \) occurs with the same limiting frequency \( 1/b \) in the expansion of \( r \) in base \( b \). A number is normal to base \( b \) if it is simply normal to the each base \( b^i \), for very positive integer \( i \).

For a finite set of symbols \( A \) we write \( A^* \) and \( A^\omega \) to denote, respectively, the set of finite and infinite sequences of symbols in \( A \).

**Definition.**

1. A **finite state transducer** is a 4-uple \( C = \langle Q, q_0, \delta, o \rangle \), where \( Q \) is a finite set of states, \( q_0 \in Q \) is the initial state, \( \delta : Q \times A \to Q \) is the transition function and \( o : Q \times A \to A^* \) is the output function. A finite state transducer processes the input symbols according to the current state \( q \). When a symbol \( a \in A \) is read, the automaton moves to state \( \delta(q,a) \) and outputs \( o(q,a) \). The extension of \( \delta \) and \( o \) to process strings are \( \delta^* : Q \times A^* \to Q \) and \( o^* : Q \times A^* \to A^* \) such that, for \( a \in A \), \( s \in A^* \) and \( \lambda \) the empty string, \( \delta^*(q,\lambda) = q \), \( \delta^*(q,as) = \delta^*(\delta(q,a),s) \), and \( o^*(q,\lambda) = \lambda \), \( o^*(q,a) = o(q,a)o^*(\delta(q,a),s) \).

2. The function \( f_C : A^\omega \to A^\omega \) computed by \( C = \langle Q, q_0, \delta, o \rangle \) is \( f_C(x) = o^*(q_0, x) \).

3. A function \( f : A^\omega \to A^\omega \) is computable by a finite state transducer when \( f = f_C \) for some finite state transducer \( C \). A function \( f : A^\omega \to \mathbb{R} \) is computable by a finite state transducer when \( f = \text{conv}(f_C) \) for some finite state transducer \( C \), where \( \text{conv} : A^\omega \to \mathbb{R} \) is the usual map \( \text{conv}(x) = \sum_{i \geq 1} t^{-i} x[i] \), with \( t \) the cardinality of \( A \).

The following example shows that the obvious definition of differentiability is not appropriate for our purposes.
Example. Let $I = \langle q, q, \pi_1, \pi_2 \rangle$ where $\pi_1$ and $\pi_2$ are respectively the projections functions of the first and second argument. So, the function $f_I : \{0, 1\}^\omega \to \mathbb{R}$ is the identify function mapped to the unit interval. The obvious definition of differentiability would yield $\lim_{k \to \infty} 2^{-k}(\text{conv}(\pi^*_x(q, x[1..k-1]1)) - \text{conv}(\pi^*_x(q, x[1..k-1]0))) = 1$. Now, let $C = \langle \{q, r_0, r_1\}, q, \delta, o \rangle$ such that for $a, b \in 2$, $\delta(q, b) = r_b, \delta(r_b, a) = q, o(q, b) = \lambda, o(r_b, a) = ba$. It is easy to check that $f_C : \{0, 1\}^\omega \to \mathbb{R}$ is also the identify function mapped to the unit interval. However, $\lim_{k \to \infty} 2^{-k}(\text{conv}(o^*(q, x[1..k-1]0))) - \text{conv}(o^*(q, x[1..k-1]0)))$ does not exist for any $x$.

Definition. The differential of a non-decreasing function $f : A^\omega \to \mathbb{R}$ at $x$ is $Df(x) = \lim_{k \to \infty} t^{-k}(f(x[1..k]1^\omega) - f(x[1..k]0^\omega)) = \lim_{k \to \infty} \mu(f(T_x[1..k]))/\mu(T_x[1..k])$, where $t$ is the cardinality of $A$, $T_s = \{sx : x \in A^\omega\}$ is the cone defined by the string $s$, and $f(T_s) = \{f(sx) : x \in A^\omega\}$. We say that $f$ is differentiable at $x$ if $Df(x)$ exists.

Now we can formulate the announced result.

Theorem. A real number $r$ is normal to a given base $b$ if, and only if, every real valued non-decreasing function computable by a finite state transducer is differentiable at the expansion of $r$ in base $b$.

The proof relies in the characterization of normal sequences as those incompressible by information lossless finite state compressors, result that follows from [6, 4, 5]. An adaptation is needed to deal with the non-decreasing condition.

References


