The structure of completely positive matrices according to their CP-rank and CP-plus-rank

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Abstract

We study the topological properties of the cp-rank operator cp(A) and the related cp-plus-rank operator cp+(A) (which is introduced in this paper) in the set $S^n$ of symmetric $n \times n$ matrices. For the set of completely positive matrices, $\mathcal{CP}^n$, we show that for any fixed $p$ the set of matrices $A$ satisfying $cp(A) = cp+(A) = p$ is open in $S^n \setminus \text{bd} (\mathcal{CP}^n)$. By making use of the Perron-Frobenius vector we also prove that the set $A^n$ of matrices with $cp(A) = cp^+(A)$ is dense in $S^n$. By applying the theory of semi-algebraic sets we are able to show that membership in $A^n$ is even a generic property. We furthermore answer several questions on the existence of matrices satisfying $cp(A) = cp^+(A)$ or $cp(A) \neq cp^+(A)$, and comment on genericity of having infinitely many minimal cp-decompositions.

1 Introduction

We define a symmetric matrix $A$ to be completely positive if there exists nonnegative vectors $b_1, \ldots, b_p$ such that $A = \sum_{i=1}^p b_i b_i^T$. The set of completely positive matrices forms a proper cone, i.e. a cone which is closed, convex, pointed and full-dimensional. This cone plays an important role in the field of copositive optimisation (see, e.g., [Dir10, Bom12, Bur12]).

In this paper we investigate the cp- and cp-plus-ranks of matrices, which are closely related to complete positivity. These are defined below, where we let $S^n$ be the set of symmetric $n \times n$ matrices, $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{R}_n^+$ be the set of nonnegative real $n$-vectors and $\mathbb{R}_n^{++}$ be the set of strictly positive real $n$-vectors:

Definition 1.1. For $A \in S^n$ we define its cp-rank and its cp-plus-rank respectively as:

$$cp(A) := \min \left\{ p \in \mathbb{N} \mid \exists b_1, \ldots, b_p \in \mathbb{R}_n^+ \text{ s.t. } A = \sum_{i=1}^p b_i b_i^T \right\},$$

$$cp^+(A) := \min \left\{ p \in \mathbb{N} \mid \exists b_1, \ldots, b_p \in \mathbb{R}_n^{++} \text{ s.t. } A = \sum_{i=1}^p b_i b_i^T \right\}.$$

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Note that a matrix $A \in S^n$ is completely positive if and only if $\text{cp}(A) < \infty$.

One motivation for the study of the cp-plus-rank is given by the following theorem, where $\mathcal{CP}^n$ denotes the cone of completely positive matrices of order $n$, $\text{int}(\mathcal{CP}^n)$ denotes its interior, and $\text{rank}(A)$ denotes the standard linear rank of the matrix $A$:

**Theorem 1.2.** For $A \in S^n$ we have

$$A \in \text{int}(\mathcal{CP}^n) \iff \text{cp}^+(A) < \infty \text{ and } \text{rank}(A) = n.$$  

*Proof.* This comes from [Dic10, Theorem 3.8], after noting that for any matrix $B \in \mathbb{R}^{m \times n}$ we have $\text{rank}(B^T B) = \text{rank}(B)$.  

Another point of interest is when $\text{cp}(M) = \text{cp}^+(M)$. We then have the following two properties, where for $\varepsilon > 0$ and $M \in S^n$ we define $N_\varepsilon(M) = \{X \in S^n \mid \|M - X\| \leq \varepsilon\}$, and for a matrix $A = (a_{ij}) \in S^n$, by $\|A\|$ we mean the Frobenius norm, i.e., $\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$:

**Theorem 1.3.** Consider $M \in S^n$ such that $2 \leq \text{cp}(M) = \text{cp}^+(M) < \infty$. Then $M$ has infinitely many minimal cp-decompositions, where a minimal cp-decomposition is a set $\{b_1, \ldots, b_p\} \subseteq \mathbb{R}^n$ such that $p = \text{cp}(M)$ and $M = \sum_{i=1}^p b_i b_i^T$.

*Proof.* This will follow directly from Lemma 2.10.  

**Theorem 1.4.** Consider $M \in \text{int}(\mathcal{CP}^n)$ such that $\text{cp}(M) = \text{cp}^+(M)$. Then there exists $\varepsilon > 0$ such that for all $X \in N_\varepsilon(M)$ we have $\text{cp}(X) = \text{cp}^+(X) = \text{cp}(M)$.

*Proof.* This will be shown in Theorem 2.7.  

The aim of the present paper is to study the topological properties of the functions $\text{cp}(M)$ and $\text{cp}^+(M)$.

In Section 2 we will look at some basic preliminary results on these ranks. In Section 3 we show how orthogonal matrices can be used in considering them. In Section 4 properties of the rank functions are analysed by using Perron-Frobenius vectors. In Section 5 we are interested in properties of the maximum cp- and cp-plus-ranks. Finally, in Section 6 of this paper we shall show that membership in the set $\{M \in S^n \mid \text{cp}(M) = \text{cp}^+(M)\}$ is a generic property. This yields

**Theorem 1.5.** The following properties are generic within the completely positive cone:

1. Having infinitely many minimal cp-decompositions,
2. The cp- and cp-plus- ranks being equal and locally constant.

*Proof.* This will be shown in Corollary 6.8.  

**Notation**

In this paper we shall always consider $n$ to be an integer which is strictly greater than one. In addition to the notation introduced earlier in this section, we shall let $\mathbb{R}^n$ denote the set of real $n$-vectors; $S^n_+$ the set of positive semidefinite matrices of order $n$; $\mathcal{N}^n$ the set of nonnegative symmetric matrices of order $n$; and $\text{bd}(\mathcal{CP}^n)$ the boundary of the set of completely positive matrices. For a vector $a \in \mathbb{R}^n$, whenever we mention a norm we mean the Euclidean norm, i.e., $\|a\| = \sqrt{\sum_i a_i^2}$. 

2
2 Preliminary results

In this section we shall consider some basic results connected to the cp- and cp-plus-ranks. We start with the following three trivial results.

**Lemma 2.1.** For all $M \in S^n$, we have $cp^+(M) \geq cp(M) \geq \text{rank}(M)$.

**Lemma 2.2.** If $M \in S^n \setminus \{0\}$ such that $cp^+(M)$ is finite, then $M \in \text{int}(N^n) \cap CP^n$.

**Lemma 2.3.** For all $A, B \in S^n$ and $\alpha, \beta > 0$ we have

$$cp(\alpha A + \beta B) \leq cp(A) + cp(B).$$

We shall now consider how the cp- and cp-plus-ranks vary in a neighbourhood of a matrix $M \in S^n$.

**Theorem 2.4.** Let $M \in S^n$. Then there exists $\varepsilon > 0$ such that $cp(P) \geq cp(M)$ for all $P \in N_\varepsilon(M)$.

**Proof.** This was shown in [SMBJS13, Proposition 2.4].

A similar result also holds for the cp-plus-rank, although with the inequality reversed.

**Theorem 2.5.** Let $M \in S^n \setminus \text{bd}(CP^n)$. Then there exists $\varepsilon > 0$ such that $cp^+(P) \leq cp^+(M)$ for all $P \in N_\varepsilon(M)$.

**Proof.** If $M \notin CP^n$ then there exists $\varepsilon > 0$ such that for all $P \in N_\varepsilon(M)$ we have $P \notin CP^n$, and thus $cp^+(P) = \infty = cp^+(M)$.

If $M \in \text{int}(CP^n)$ then the result comes directly from considering the proof of [DS08, Theorem 2.3].

**Remark 2.6.** The result of the previous theorem does not in general hold when $M \in \text{bd}(CP^n)$. For example, if $M \in \text{bd}(CP^n)$ such that $cp^+(M) \neq \infty$, then for all $\varepsilon > 0$ there exists $P \in N_\varepsilon(M) \setminus CP^n$ and thus $cp^+(P) = \infty > cp^+(M)$.

Combining Lemma 2.1 and Theorems 2.4 and 2.5, we get the following result.

**Theorem 2.7.** Let $M \in S^n \setminus \text{bd}(CP^n)$ such that $cp(M) = cp^+(M) = p$. Then there exists $\varepsilon > 0$ such that $cp^+(P) = cp(P) = p$ for all $P \in N_\varepsilon(M)$.

**Corollary 2.8.** The following sets are open for all $p \in \mathbb{N}$:

$$\{ M \in S^n \setminus \text{bd}(CP^n) \mid cp(M) = cp^+(M) = p \},$$

$$\{ M \in S^n \setminus \text{bd}(CP^n) \mid cp(M) = cp^+(M) \}.$$

We finish this section by considering some equivalent definitions of the cp- and cp-plus-ranks, which will be used regularly throughout the paper.

We begin with the following trivial result:

**Lemma 2.9.** For all $A \in S^n \setminus \{0\}$ we have

$$cp(A) = \min \left\{ p \in \mathbb{N} \mid \exists B \in \mathbb{R}^{p \times n}_+ \text{ s.t. } A = B^T B \right\},$$

$$cp^+(A) = \min \left\{ p \in \mathbb{N} \mid \exists B \in \mathbb{R}^{p \times n}_{++} \text{ s.t. } A = B^T B \right\}.$$
Proof. This comes from noting that if we have a matrix $B \in \mathbb{R}^{p \times n}$ whose rows are given by $b^T_1, \ldots, b^T_p$ then $B^T B = \sum_{i=1}^p b_i b_i^T$.

We now consider another equivalent definition of the cp-plus-rank which is less trivial. These results come from [Dic13, Lemma 7.13] and its proof.

**Lemma 2.10.** Consider $a, b \in \mathbb{R}^n$, and for all $\theta \in \mathbb{R}$ let $c_\theta = a \sin \theta + b \cos \theta$ and $d_\theta = a \cos \theta - b \sin \theta$. Then we have:

1. $aa^T + bb^T = c_\theta c_\theta^T + d_\theta d_\theta^T$ for all $\theta \in \mathbb{R}$, and
2. if $a \in \mathbb{R}^+_n$ and $b \in \mathbb{R}^+_n$ then there exists $\Theta > 0$ such that $c_\theta, d_\theta \in \mathbb{R}^+_n$ for all $\theta \in (0, \Theta]$.

**Corollary 2.11.** For $A \in S^n \setminus \{0\}$, we have

$$\text{cp}^+(A) = \min \{ p \in \mathbb{N} \mid \exists b_1, \ldots, b_p \in \mathbb{R}^n_+ \text{ s.t. } b_1, \ldots, b_p \in \mathbb{R}^+_n \text{ and } A = \sum_{i=1}^p b_i b_i^T \}.$$ 

This result leads to an inequality for the function cp-plus-rank similar to Lemma 2.3, but note that here we have a mixture of cp- and cp-plus-ranks.

**Corollary 2.12.** For all $A, B \in S^n \setminus \{0\}$ and $\alpha, \beta > 0$ we have

$$\text{cp}^+(\alpha A + \beta B) \leq \text{cp}^+(A) + \text{cp}(B).$$

### 3 Orthogonal matrices

The concept of cp-plus-rank connects to orthogonal matrices through the following lemma.

**Lemma 3.1.** Let $A, B \in \mathbb{R}^{p \times n}$. Then $A^T A = B^T B$ if and only if there exists an orthogonal matrix $Q \in \mathbb{R}^{p \times p}$ such that $A = QB$.

**Proof.** The reverse implication (which we will need below) is trivial. The forward implication is a well known result in linear algebra, and a sketch of the proof is presented in [Xu04, Lemma 1].

In the paper [SSMS13] the authors considered matrices $B \in \mathbb{R}^{p \times n}_+$ and defined such matrices to be nearly positive if there exist orthogonal matrices $\{Q_i \mid i \in \mathbb{N}\}$ such that $Q_i B > 0$ for all $i$ and $\lim_{i \to \infty} Q_i = I$ (where $I$ is the identity matrix). Using the lemma above we then get the following sufficient condition for when the cp-rank of a matrix is equal to its cp-plus-rank.

**Corollary 3.2.** Let $X \in \mathcal{CP}^n$ with $\text{cp}(X) = p$, and let $B \in \mathbb{R}^{p \times n}_+$ such that $X = B^T B$. If $B$ is a nearly positive matrix, then $\text{cp}^+(X) = p$.

In [SSMS13, Example 7.4] it was shown that the reverse implication to this does not hold. In that example, for $n \geq 4$, they considered a family of $M = A^T A \in \text{int} (\mathcal{CP}^n)$, with $A \in \mathbb{R}^{n \times n}_+$ not being a nearly positive matrix, but $\text{cp}^+(M) = \text{cp}(M) = \text{rank}(M) = n$.

In [SSMS13] the authors looked at many interesting results on nearly positive matrices, including the following:

**Theorem 3.3.** Let $X \in \mathcal{CP}^n \cap \text{int} (\mathcal{N}^n)$ and let $B \in \mathbb{R}^{p \times n}_+$ such that $X = B^T B$. If either $n \leq 3$ or $p \leq 2$ (or both) then $B$ is nearly positive.

Translating this result for the cp-plus-rank we get the following corollary.

**Corollary 3.4.** Let $X \in \mathcal{CP}^n \cap \text{int} (\mathcal{N}^n)$. If $n \leq 3$ or $\text{cp}(X) \leq 2$ (or both) then $\text{cp}(X) = \text{cp}^+(X)$. 

4
4 Perron-Frobenius Vectors

In this section we shall analyse the cp- and cp-plus-ranks using the theory of Perron-Frobenius vectors. We begin by recalling some basic definitions and results on Perron-Frobenius vectors, applied to matrices in \( N^n \setminus \{0\} \).

**Theorem 4.1.** Let \( M \in N^n \setminus \{0\} \). Then there exists \( \lambda \in \mathbb{R}_{++} \) such that:

1. \( \lambda \) is an eigenvalue of \( M \),
2. the absolute values of all eigenvalues of \( M \) are less than or equal to \( \lambda \),
3. there is an eigenvector \( x \in \mathbb{R}^n_+ \), with \( \|x\| = 1 \), corresponding to the eigenvalue \( \lambda \).

We refer to this as a Perron-Frobenius (P-F) vector of \( M \).

Furthermore, if \( M \in \text{int}(N^n) \), then for \( \lambda \) and \( x \) given above we have:

4. the absolute values of all eigenvalues of \( M \), excluding \( \lambda \), are strictly less than \( \lambda \),
5. \( x \in \mathbb{R}^n_+ \), and \( x \) is the unique eigenvector of \( M \) corresponding to \( \lambda \), up to multiplication by a scalar (i.e. the eigenvalue \( \lambda \) has multiplicity one). We shall denote this eigenvector by \( x_M \).

**Remark 4.2.** Note that any matrix \( M \in \text{int}(CP^n) \) satisfies \( M \in \text{int}(N^n) \). Also note that if in the theorem above we have \( M \notin \text{int}(N^n) \), then we do not necessarily have a unique P-F vector. For example, consider \( M \) being equal to the identity matrix.

We now recall the following well known lemma on eigenvectors and eigenvalues.

**Lemma 4.3.** Consider a matrix \( A \in S^n \) with eigenvectors \( x_1, x_2 \in \mathbb{R}^n \), whose corresponding eigenvalues are \( \lambda_1, \lambda_2 \in \mathbb{R} \). If \( \lambda_1 \neq \lambda_2 \) then \( x_1^T x_2 = 0 \).

**Proof.** This comes from noting that \( \lambda_1 x_1^T x_2 = x_1^T A x_2 = \lambda_2 x_1^T x_2 \). \( \square \)

From this we then get the following result on P-F vectors.

**Lemma 4.4.** Consider \( M \in N^n \setminus \{0\} \) and let \( x \in \mathbb{R}^n_+ \) be an eigenvector of \( M \) such that \( \|x\| = 1 \). Then \( x \) is a P-F vector of \( M \).

**Proof.** Assume for the sake of contradiction that the eigenvector \( x \) with corresponding eigenvalue \( \mu \) is not a P-F vector. Then there exists a P-F vector \( y \in \mathbb{R}^n_+ \setminus \{0\} \) with eigenvalue \( \lambda > \mu \). By Lemma 1.3 it would follow \( y^T x = 0 \), a contradiction to \( 0 \neq y \in \mathbb{R}^n_+ \), \( x \in \mathbb{R}^n_+ \). \( \square \)

Another well known lemma on eigenvectors is the following.

**Lemma 4.5.** Consider \( P, Q \in S^n \) and \( x \in \mathbb{R}^n \setminus \{0\} \) such that \( P = Q + \mu xx^T \) for some \( \mu \in \mathbb{R} \). Then \( x \) is an eigenvector of \( P \) if and only if it is an eigenvector of \( Q \).

**Proof.** Without loss of generality let \( \|x\| = 1 \). Then we have

\[
Qx = \lambda x \iff Px = (\lambda + \mu)x.
\]

Now combining Lemmas 4.4 and 4.5 we get the following result:
Lemma 4.6. Consider $P, Q \in \mathcal{N}^n \setminus \{0\}$ and $x \in \mathbb{R}_{++}^n$ such that $P = Q + \mu xx^T$ for some $\mu \in \mathbb{R}$. Then $x$ is a P-F vector of $P$ if and only if it is a P-F vector of $Q$.

We will now look at what P-F vectors can tell us about the cp- and cp-plus-ranks. In order to do this for $M \in \text{int} \{(N^n)\}$ and $\mu \in \mathbb{R}$ we let

$$M(\mu) := M + \mu xx_M^T.$$ 

Note from Theorem 4.4 that this is well defined. Furthermore, from the definition, we have that $|M - M(\mu)| = |\mu|$ and thus $M(\mu) \in N_{|\mu|}(M)$. We also note the following basic result.

Lemma 4.7. Let $M, P \in \text{int} \{(N^n)\}$ and $\mu \in \mathbb{R}$ such that $P = M(\mu)$. Then we have $M = P(-\mu)$.

Proof. This comes directly from Theorem 4.4 and Lemma 4.6.

We are now ready to present the main results of this section.

Theorem 4.8. For all $M \in \text{int} \{(N^n)\}$ and all $\mu > 0$ we have $\text{cp}(M) \geq \text{cp}^+(M(\mu))$.

Proof. This proof is an adaptation of one from [SMBJ13].

In this proof we will in fact prove the more general result that considers $M \in \mathcal{N}^n \setminus \{0\}$ with a P-F vector $x \in \mathbb{R}_{++}^n$. Under these circumstances, we have

$$\text{cp}(M) \geq \text{cp}^+(M + \mu xx^T) \quad \text{for all } \mu > 0.$$ 

Indeed, if $M \notin \mathcal{CP}^n$ then we have $\text{cp}(M) = \infty$ and the result is trivial. From now on we assume $M \in \mathcal{CP}^n \setminus \{0\}$ and consider an arbitrary $\mu > 0$.

Letting $p = \text{cp}(M) \in (0, \infty)$, there exists $V \in \mathbb{R}_{++}^{p \times n}$ such that $M = V^TV$. All rows of $V$ are nonzero, otherwise $\text{cp}(M) < p$. Therefore, letting $y = Vx$, we have $y \in \mathbb{R}_{++}^n$.

As $x$ is a P-F vector of $M$, there exists $\lambda > 0$ such that $\lambda x = Mx = V^TVx = V^Ty$. We thus have that $y^T y = x^T V^TV y = \lambda x^T x = \lambda$.

The proof is now completed by letting $\nu = \sqrt{1 + (\mu/\lambda)} - 1 > 0$, noting that we have $(V + \nu yy^T) \in \mathbb{R}_{++}^{p \times n}$ and considering the following:

$$(V + \nu yy^T)^T (V + \nu yy^T) = V^TV + \nu (V^Tyy^T + xy^TV) + \nu^2 yx^T yx^T$$

$$= M + \nu\lambda(2 + \nu) xx^T$$

$$= M + \mu xx^T. \quad \Box$$

Theorem 4.9. Consider $M \in \text{int} \{\mathcal{CP}^n\}$ with $p := \text{cp}(M) \leq \text{cp}^+(M(\varepsilon)) =: q$. Then there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon}]$ we have

$$\text{cp}(M(\varepsilon)) = \text{cp}^+(M(\varepsilon)) = p \quad \text{and} \quad \text{cp}(M(-\varepsilon)) = \text{cp}^+(M(-\varepsilon)) = q.$$ 

Proof. From Theorem 2.4 there exists $\varepsilon_+ > 0$ such that for all $\varepsilon \in (0, \varepsilon_+]$ we have $p \leq \text{cp}(M(\varepsilon))$, and from Theorem 4.8 we have $\text{cp}^+(M(\varepsilon)) \leq p$. We now note from Lemma 2.1 that $\text{cp}(M(\varepsilon)) \leq \text{cp}^+(M(\varepsilon))$, and combining these three inequalities together we get $\text{cp}(M(\varepsilon)) = \text{cp}^+(M(\varepsilon)) = p$ for all $\varepsilon \in (0, \varepsilon_+]$.

Similarly, from Theorem 2.5 there exists $\varepsilon_- > 0$ such that for all $\varepsilon \in (0, \varepsilon_-]$ we have $q \geq \text{cp}^+(M(-\varepsilon))$. For such $\varepsilon$, as the cp-plus-rank is finite, we have $M(-\varepsilon) \in \text{int} \{(N^n)\}$, and thus from Lemma 4.7 and Theorem 4.8 we have $\text{cp}(M(-\varepsilon)) \geq q$. We now note from Lemma 2.1 that $\text{cp}^+(M(-\varepsilon)) \geq \text{cp}^+(M(-\varepsilon))$, and combining these three inequalities together we get $\text{cp}(M(-\varepsilon)) = \text{cp}^+(M(-\varepsilon)) = q$ for all $\varepsilon \in (0, \varepsilon_-]$.

Now letting $\hat{\varepsilon} = \min\{\varepsilon_-, \varepsilon_+\}$, this completes the proof. \[\Box\]
5 Maximum \( \text{cp-} \) and \( \text{cp-plus-ranks} \)

Let us define the numbers

\[
p_n := \max\{\text{cp}(M) \mid M \in \mathcal{CP}^n\}, \\
p_n^+ := \max\{\text{cp}^+(M) \mid \text{cp}^+(M) < \infty\}.
\]

In the following theorem we collect some known results on these numbers, along with a couple of new ones.

**Theorem 5.1.** We have that

\[
p_n = \max\{\text{cp}(M) \mid M \in \text{bd}(\mathcal{CP}^n)\}, \\
= \max\{\text{cp}(M) \mid M \in \text{int}(\mathcal{CP}^n)\} \\
= \max\{\text{cp}^+(M) \mid M \in \text{int}(\mathcal{CP}^n)\}
\]

(1)

\[
p_n \leq p_n^+ \leq p_n + 1
\]

(2)

\[
p_n = n \quad \text{for all } n = 2, 3, 4,
\]

(3)

\[
p_n \geq \left\lfloor \frac{n^2}{4} \right\rfloor > n \quad \text{for all } n \geq 5
\]

(4)

\[
p_n \geq \frac{1}{2} n(n+1) - 4 - \sqrt{2n^2 + \frac{9}{2} n} > \left\lfloor \frac{n^2}{4} \right\rfloor \quad \text{for all } n \geq 5
\]

(5)

\[
p_n \leq \frac{1}{2} n(n+1) - 4 \quad \text{for all } n \geq 5
\]

(6)

\[
\text{for all } k \in \{1, \ldots, n-1\} \exists M \in \mathcal{CP}^n \text{ s.t. } \text{cp}(M) = \text{cp}^+(M) = k
\]

(7)

\[
\text{and we have } M \in \text{bd}(\mathcal{CP}^n),
\]

(8)

\[
\text{for all } k \in \{n+1, \ldots, p_n\} \exists M \in \text{int}(\mathcal{CP}^n) \text{ s.t. } \text{cp}(M) = \text{cp}^+(M) = k
\]

(9)

\[
\text{for all } k \in \{n+1, \ldots, p_n\} \exists M \in \text{int}(\mathcal{CP}^n) \text{ s.t. } k - 1 = \text{cp}(M) \neq \text{cp}^+(M) = k
\]

(10)

\[
\text{for all } k \in \{n+1, \ldots, p_n\} \exists M \in \text{int}(\mathcal{CP}^n) \text{ s.t. } k - 1 = \text{cp}(M) \neq \text{cp}^+(M) = k
\]

(11)

**Proof.** (1) and (2) were proven in \[SMBJS13\], and (3) follows directly from (2) and Theorem 4.9. The leftmost inequality in (4) follows from (3). To prove the rightmost inequality in (4), we consider an arbitrary \( M \in \mathcal{S}^n \setminus \{0\} \) such that \( \text{cp}^+(M) < \infty \). From the definitions, there exists \( v \in \mathbb{R}^n_+ \) such that \( M - vv^T \in \mathcal{CP}^n \setminus \{0\} \) and from Corollary 2.12 we have \( \text{cp}^+(M) \leq \text{cp}^+(vv^T) + \text{cp}(M - vv^T) \leq 1 + p_n \).

While (5) and (6) are well known since long, see for example \[BSM03\], the bounds in (7) and (8) were established quite recently, namely in \[SMBB+13\] and in \[BSU14b\]. For \( n = 5 \) we have \( p_n = \lfloor n^2/4 \rfloor \) \[SMBJST13\]. It was conjectured in \[DJL94\] that this equality holds for all \( n \geq 5 \), however counter examples to this conjecture for \( n = 7, \ldots, 11 \) were recently presented in \[BSU14a\]. For \( n \geq 15 \) this conjecture is refuted by (7), and for \( n = 12, 13, 14 \) tighter lower bounds also refute it \[BSU14b\].

We shall now prove (9), (10) and (11). From Theorem 1.2 and Lemma 2.1 if \( \text{cp}(M) < n \) then \( M \in \text{bd}(\mathcal{CP}^n) \). From (2), (5), (6) and Theorem 4.9 there exists \( M \in \text{int}(\mathcal{CP}^n) \) such that \( \text{cp}(M) = \text{cp}^+(M) = p_n \geq n \), and thus statement (10) holds for \( k = p_n \). From Theorem 1.2 and using that \( \text{rank}(M) = \text{rank}(B) \) holds for \( M = B^TB \), there exists \( b_1, \ldots, b_{p_n} \in \mathbb{R}^n_+ \) such that \( \text{span}\{b_1, \ldots, b_{p_n}\} = \mathbb{R}^n \) and \( M = \sum_{i=1}^{p_n} b_i b_i^T \). For all \( k \in \{1, \ldots, p_n\}, \theta \in [0,1] \) we let \( M_k(\theta) := \sum_{i=1}^{k-1} b_i b_i^T + \theta b_k b_k^T \). From Theorem 1.2 we have

\[
M_k(\theta) \in \text{int}(\mathcal{CP}^n) \quad \text{for all } k \in \{n, \ldots, p_n\}, \theta \in [0,1],
\]

\[
M_k(\theta) \in \text{bd}(\mathcal{CP}^n) \quad \text{for all } k \in \{1, \ldots, n-1\}, \theta \in [0,1].
\]
Furthermore, for all \( k \in \{1, \ldots, n\} \), \( \theta \in [0, 1] \) we have \( M = M_k(\theta) + (1-\theta)b_kb_k^T + \sum_{i=k+1}^{p_n} b_i b_i^T \), and thus by Lemma \( \text{2.3} \) we have
\[
p_n = \text{cp}(M) \leq \text{cp}(M_k(\theta)) + \text{cp} \left( (1-\theta)b_kb_k^T + \sum_{i=k+1}^{p_n} b_i b_i^T \right) \leq \text{cp}(M_k(\theta)) + 1 + p_n - k.
\]
It is also trivial to see from the definitions that \( \text{cp}(M_k(\theta)) \leq \text{cp}^+(M_k(\theta)) \leq k \). Combining these inequalities together, we get
\[
k - 1 \leq \text{cp}(M_k(\theta)) \leq \text{cp}^+(M_k(\theta)) \leq k \quad \text{for all} \ k \in \{1, \ldots, n\}, \ \theta \in [0, 1].
\]
For all \( k \in \{1, \ldots, p_n - 1\} \) we have \( M_{k+1}(0) = M_k(1) \) and thus using the above we get \( \text{cp}(M_k(1)) = \text{cp}^+(M_k(1)) = k \), which completes the proof for statements (9) and (10). Similar arguments can also be found in [SMBB13 Prop.4.1, Thm.4.1]

For an arbitrary \( k \in \{n+1, \ldots, p_n\} \), we now let \( \vartheta_k = \sup_{\theta \in [0,1]} \{ \theta \mid \text{cp}(M_k(\theta)) = k-1 \} \) and note by Corollary \( \text{2.8} \) that \( 0 < \vartheta_k < 1 \). For all \( \theta \in (\vartheta_k, 1] \) we have \( k = \text{cp}(M_k(\theta)) = \text{cp}^+(M_k(\theta)). \)

Therefore, by Theorem \( \text{2.5} \), we have \( k \leq \text{cp}^+(M_k(\vartheta_k)) \), and thus \( \text{cp}^+(M_k(\vartheta_k)) = k \).

Additionally, for all \( \varepsilon > 0 \) there exists \( \theta \in \left[ \vartheta_k - \varepsilon, \vartheta_k \right] \) such that \( k - 1 = \text{cp}(M_k(\theta)) \). Therefore, by Theorem \( \text{2.4} \) we have \( k - 1 \geq \text{cp}(M_k(\vartheta_k)) \), and thus \( \text{cp}(M_k(\vartheta_k)) = k - 1 \), which completes the proof.

From the following lemma we get \( p_n = p_n^* \) for \( n = 2, 3, 4 \). It is an open question whether this equality continues to hold for \( n \geq 5 \).

**Lemma 5.2.** For \( n = 2, 3, 4 \) let \( M \in \mathcal{CP}^n \cap \text{int} \left( \mathcal{N}^n \right) \). Then \( \text{cp}^+(M) \leq p_n = n \).

**Proof.** From Corollary \( \text{3.4} \) for \( n = 2, 3 \) we already have \( \text{cp}(M) = \text{cp}^+(M) \). However, the following proof will be for a general \( n = 2, 3, 4 \), as nothing is lost in doing this.

We begin by recalling that for \( n = 2, 3, 4 \) we have \( \mathcal{CP}^n = \mathcal{S}_+^n \cap \mathcal{N}^n \) and thus \( \mathcal{CP}^n \cap \text{int} \left( \mathcal{N}^n \right) = \mathcal{S}_+^n \cap \text{int} \left( \mathcal{N}^n \right) \), see [MM62].

Let \( M \in \mathcal{CP}^n \cap \text{int} \left( \mathcal{N}^n \right) \), with P-F vector \( x \). For \( \varepsilon > 0 \) small enough we have \( P = (M - \varepsilon xx^T) \in \text{int} \left( \mathcal{N}^n \right) \) and thus from Lemma \( \text{4.6} \) \( x \) is also the P-F vector of \( P \). When going from \( M \) to \( P \), the only eigenvalue that we are affecting is the eigenvalue corresponding to \( x \), which remains strictly positive. Therefore we have \( P \in \mathcal{S}_+^n \). This implies that \( P \in \mathcal{CP}^n \), and thus \( \text{cp}(P) \leq p_n \). Finally, from Theorem \( \text{4.8} \) we have \( \text{cp}^+(M) \leq \text{cp}(P) \leq p_n \), completing the proof.

For \( n \geq 5 \) this lemma no longer holds, consider for example the following:

**Example.** In [DS08 Example 2.2], the authors showed that the following matrix is on the boundary of the completely positive cone:

\[
B = \begin{pmatrix}
8 & 5 & 1 & 1 & 5 \\
5 & 8 & 5 & 1 & 1 \\
1 & 5 & 8 & 5 & 1 \\
1 & 1 & 5 & 8 & 5 \\
5 & 1 & 1 & 5 & 8
\end{pmatrix}.
\]

We have \( \text{rank}(B) = 5 \), but \( B \notin \text{int} \left( \mathcal{CP}^5 \right) \), and thus from Theorem \( \text{1.2} \) we have \( \text{cp}^+(B) = \infty \).
6 Genericity of the property $\text{cp}^+(M) = \text{cp}(M)$

6.1 Genericity vs. open and dense

In this section we consider the topological properties of the following set:

$$\mathcal{A}^n := \{ M \in S^n \mid \text{cp}(M) = \text{cp}^+(M) \}. \quad (12)$$

As usual, we say that a set $\mathcal{A} \subseteq S^n$ is dense if for all $X \in S^n$ and $\varepsilon > 0$ we have $N_\varepsilon(X) \cap \mathcal{A} \neq \emptyset$. From the results so far it follows that the set $\mathcal{A}^n$ contains an open and dense subset of $S^n$.

**Theorem 6.1.** The set $\mathcal{A}^n := \mathcal{A}^n \setminus \text{bd} (\mathcal{CP}^n)$ is open and dense in $S^n$.

**Proof.** For an arbitrary $M \in S^n$, we consider the following cases, which will complete the proof:

1. $M \notin \mathcal{CP}^n$: We have $M \in \mathcal{A}^n$, and as the set of completely positive matrices is closed, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq S^n \setminus \mathcal{CP}^n \subseteq \mathcal{A}^n$.

2. $M \in \text{bd} (\mathcal{CP}^n)$: We have $M \notin \mathcal{A}^n$, and for all $\varepsilon > 0$ there exists $M_\varepsilon \in N_\varepsilon(M)$ such that $M_\varepsilon \in S^n \setminus \mathcal{CP}^n \subseteq \mathcal{A}^n$.

3. $M \in \text{int} (\mathcal{CP}^n) \cap \mathcal{A}^n$: From Theorem 2.7, there exists $\varepsilon > 0$ such that $N_\varepsilon(M) \subseteq \mathcal{A}^n$.

4. $M \in \text{int} (\mathcal{CP}^n) \setminus \mathcal{A}^n$: From Theorem 4.9, for all $\varepsilon > 0$ we have $N_\varepsilon(M) \cap \mathcal{A}^n \neq \emptyset$.

By this theorem we know that the set $\mathcal{A}^n$ contains an open and dense set. But it is well known that for a set $\mathcal{A} \subseteq S^n$, being dense and open does not necessarily imply that the Lebesgue measure of $S^n \setminus \mathcal{A}$, denoted $\mu_L(S^n \setminus \mathcal{A})$, is equal to zero. Indeed, the set of rational numbers, $\mathbb{Q} \subseteq \mathbb{R}$, is a well-known dense set with $\mu_L(\mathbb{Q}) = 0$ [Bea04, p.133]. Considering approximations of measurable sets [Bea04, p.139], for all $\varepsilon > 0$ there exists an open set $\mathcal{A}_\varepsilon$ such that $\mathbb{Q} \subseteq \mathcal{A}_\varepsilon \subseteq \mathbb{R}$ and $\mu_L(\mathcal{A}_\varepsilon) \leq \mu_L(\mathbb{Q}) + \varepsilon = \varepsilon$. We then have that $\mathcal{A}_\varepsilon$ is an open and dense set in $\mathbb{R}$ with $\mu_L(\mathbb{R} \setminus \mathcal{A}_\varepsilon) = \infty \neq 0$.

In what follows we wish to strengthen the statement of Theorem 6.1, and we will show that the membership in $\mathcal{A}^n$ is a generic property.

Recall that in topology for a subset $\mathcal{A} \subseteq \mathbb{R}^N$, we say that membership in $\mathcal{A}$ is generic in $\mathbb{R}^N$ if $\mathcal{A}$ contains a set $\mathcal{A}_0$ such that the following two statements hold:

1. the set $\mathcal{A}_0$ is open, and
2. the Lebesgue measure of $\mathbb{R}^N \setminus \mathcal{A}_0$ is equal to zero.

Statement [1] means that membership in $\mathcal{A}_0$ is stable for small variations. Statement [2] means that ‘almost all’ elements of $\mathbb{R}^N$ are in $\mathcal{A}_0$ (and thus also in $\mathcal{A}$).

In the next subsection we prove that indeed membership in $\mathcal{A}^n$ is a generic property.
6.2 Semi-algebraic sets

In order to show that membership in the set $A^n$ is generic we make use of the theory of semi-algebraic sets and only need the density part of Theorem 6.1.

We prove that $A^n$ is a semi-algebraic set and apply the fact that for a semi-algebraic set, being dense is a sufficient condition for membership in the set being generic. We note that similar arguments have been used recently to obtain genericity results in cone programming [BDL11].

The results on semi-algebraic sets will be stated for the space $\mathbb{R}^N$. The results can then be trivially applied to the space $S^n \equiv \mathbb{R}^{(n+1)n/2}$.

We begin by recalling some preliminary definitions and results on semi-algebraic sets (see [BR90, GWdL76]).

Definition 6.2. A set $A \subset \mathbb{R}^N$ is called semi-algebraic if it is given by a finite union of sets of the form
\[
\{ x \in \mathbb{R}^N \mid p_i(x) = 0 \text{ for all } i = 1, \ldots, k, \quad q_j(x) > 0 \text{ for all } j = 1, \ldots, s \}
\]
with $k, s \in \mathbb{N}$ and polynomial functions $p_i, q_j \in \mathbb{R}[x]$.

Remark 6.3. Since $\{ x \mid p(x) \geq 0 \} = \{ x \mid p(x) = 0 \} \cup \{ x \mid p(x) > 0 \}$, also sets defined by polynomial inequalities $p(x) \geq 0$ are semi-algebraic.

The following theorem states some well-known facts on semi-algebraic sets.

Theorem 6.4. For $N, M \in \mathbb{N}$, consider semi-algebraic sets $A, B \subset \mathbb{R}^N$ and a polynomial function $h : \mathbb{R}^N \to \mathbb{R}^M$ (e.g. a projection operator). Then the following sets are also semi-algebraic:
\[
A \cup B, \quad A \cap B, \quad A \setminus B, \quad h(A).
\]

Proof. For a proof we refer to [BR90, Section 2.1–2.3].

We shall now show that the set $A^n$ from (12) is a semi-algebraic set.

Lemma 6.5. The set $A^n$ from (12) is semi-algebraic.

Proof. Recalling from the definition that the union of finitely many semi-algebraic sets is also semi-algebraic and recalling from Theorem 5.1 that we have $p_n < \frac{1}{4}n(n + 1)$, it is sufficient to show that the following set is semi-algebraic for all $p \in \mathbb{N} \cup \{ \infty \}$:
\[
A^n_p := \{ A \in S^n \mid \text{cp}(A) = \text{cp}^+(A) = p \}.
\]

For all $p \in \mathbb{N}$ we have that the following sets are trivially semi-algebraic:
\[
E := \{ (X, V) \in S^n \times \mathbb{R}^{p \times n} \mid v_{ij} \geq 0 \text{ for all } i, j, \quad X = V^T V \},
\]
\[
F := \{ (X, V) \in S^n \times \mathbb{R}^{p \times n} \mid v_{ij} > 0 \text{ for all } i, j, \quad X = V^T V \}.
\]

From Theorem 6.4 the projections $\text{proj}_X(E)$ and $\text{proj}_X(F)$ are also semi-algebraic for all $p \in \mathbb{N}$; but obviously
\[
\text{proj}_X(E) = \{ X \in S^n \mid \text{cp}(X) \leq p \} \quad \text{and} \quad \text{proj}_X(F) = \{ X \in S^n \mid \text{cp}^+(X) \leq p \}.
\]
Therefore again considering Theorem 5.1 and Theorem 6.4 the following sets are semi-algebraic for all $p \in \mathbb{N}$:

\[
\{X \in \mathcal{S}^n \mid \text{cp}(X) = \infty\} = \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \text{cp}(A) \leq p_n\}, \\
\{X \in \mathcal{S}^n \mid \text{cp}^+(X) = \infty\} = \mathcal{S}^n \setminus \{X \in \mathcal{S}^n \mid \text{cp}^+(A) \leq p_n^+\}, \quad \text{as well as} \\
\{X \in \mathcal{S}^n \mid \text{cp}(X) = p\} = \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p\} \setminus \{X \in \mathcal{S}^n \mid \text{cp}(X) \leq p-1\}, \\
\{X \in \mathcal{S}^n \mid \text{cp}^+(X) = p\} = \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p\} \setminus \{X \in \mathcal{S}^n \mid \text{cp}^+(X) \leq p-1\}.
\]

Since

\[
\mathcal{A}_p = \{X \in \mathcal{S}^n \mid \text{cp}(X) = p\} \cap \{X \in \mathcal{S}^n \mid \text{cp}^+(X) = p\},
\]

this finally implies that also $\mathcal{A}_p$ is semi-algebraic for all $p \in \mathbb{N} \cup \{\infty\}$. \qed

We can also combine Theorem 6.4 with other well-known results to obtain the following which may be of general interest in algebraic geometry:

**Theorem 6.6.** Let $\mathcal{A} \subseteq \mathbb{R}^N$ be a semi-algebraic set. Then the membership in $\mathcal{A}$ is generic if and only if $\mathcal{A}$ is dense in $\mathbb{R}^N$.

**Proof.** The forward implication is trivial. To prove the reverse implication we make use of the following facts on semi-algebraic sets.

From [GWdL76, 2.7], we have that any semi-algebraic set $\mathcal{A} \subseteq \mathbb{R}^N$ admits a (stratification) partition $\mathcal{A} = \bigcup_{i=0}^d \mathcal{S}_i$ with some $d \in \mathbb{N}$ such that

1. $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for $i \neq j$ and
2. the sets $\mathcal{S}_i$ are smooth manifolds of $\mathbb{R}^N$ of dimension $i$ (or are empty).

It is a well-known result, see for example [GP74], that the manifolds of dimension $N$ in $\mathbb{R}^N$ are precisely the open sets in $\mathbb{R}^N$. Furthermore, manifolds of dimension $k < N$ in $\mathbb{R}^N$ have Lebesgue measure zero (cf., e.g., [GP74, p.45]).

We first consider the set $\mathcal{M} = \mathbb{R}^N \setminus \mathcal{A}$, and note from Theorem 6.4 that this set is semi-algebraic. So $\mathcal{M}$ allows a stratification $\mathcal{M} = \bigcup_{i=0}^d \mathcal{S}_i$ with some $d \in \mathbb{N}$. As $\mathcal{A}$ is dense, $\mathcal{M}$ cannot contain any open sets. This implies that for all $0 \leq i \leq d$ we have $\dim \mathcal{S}_i < N$ and thus $\mu_L(\mathcal{S}_i) = 0$, implying:

\[
0 \leq \mu_L(\mathcal{M}) = \mu_L\left(\bigcup_{i=0}^d \mathcal{S}_i\right) \leq \sum_{i=0}^d \mu_L(\mathcal{S}_i) = 0.
\]

Now we take the semi-algebraic set $\mathcal{A}$ and a stratification $\mathcal{A} = \bigcup_{i=0}^q \hat{\mathcal{S}}_i$ with some $q \in \mathbb{N}$.

We claim that the manifold $\hat{\mathcal{S}}_q$ with (highest) dimension $q$ must be of dimension $q = N$. So by the remark above, $\hat{\mathcal{S}}_q$ must be an open set. Indeed, the condition $\dim \hat{\mathcal{S}}_q < N$ would also imply $\mu_L(\hat{\mathcal{S}}_q) = 0$ and then

\[
\mu_L(\mathbb{R}^N) = \mu_L\left(\left(\mathbb{R}^N \setminus \mathcal{A}\right) \cup \mathcal{A}\right) \leq \mu_L\left(\left(\mathbb{R}^N \setminus \mathcal{A}\right)\right) + \mu_L(\mathcal{A}) = 0,
\]

a contradiction. Altogether we have shown that the set $\mathcal{A}$ contains the open set $\mathcal{A}_0 := \hat{\mathcal{S}}_N$ with complement

\[
\mathbb{R}^N \setminus \mathcal{A}_0 = (\mathcal{A} \setminus \hat{\mathcal{S}}_N) \cup (\mathbb{R}^N \setminus \mathcal{A}) = (\bigcup_{i=0}^{N-1} \hat{\mathcal{S}}_i) \cup (\mathbb{R}^N \setminus \mathcal{A})
\]
of Lebesgue measure

$$\mu_L(\mathbb{R}^N \setminus A_0) \leq \sum_{i=0}^{N-1} \mu_L(\tilde{S}_i) + \mu_L(\mathbb{R}^N \setminus A) = 0.$$ 

So membership in the set $A$ is a generic property.

We are now ready to present the main result of this section.

**Theorem 6.7.** Membership in the set $A^n$ from (12) is generic in $S^n$.

**Proof.** By Theorem 6.1, the set $A^n$ is dense in $S^n$. The result then follows by Lemma 6.5 and Theorem 6.6.

**Corollary 6.8.** The following properties are generic within the completely positive cone:

1. Having infinitely many minimal cp-decompositions,
2. The cp- and cp-plus-ranks being equal and locally constant.

**Proof.** From Theorems 1.2 to 1.4 and Lemma 2.1, it is sufficient to show that membership of the open set $A^n \cap \text{int}(CP^n)$ is generic in $CP^n$. Since $CP^n$ is convex we have $\mu_L(bd(CP^n)) = 0$ (see e.g. [Lan86]), and from Theorem 6.7 we have $\mu_L(S^n \setminus A^n) = 0$. The proof is then completed by noting the following:

$$\mu_L(CP^n \setminus (A^n \cap \text{int}(CP^n))) = \mu_L((CP^n \setminus A^n) \cup \text{bd}(CP^n))$$

$$\leq \mu_L(S^n \setminus A^n) + \mu_L(\text{bd}(CP^n)) = 0.$$ 

7 **Concluding Remarks**

In this paper we studied the distribution of completely positive matrices according to their cp- and cp-plus-ranks. One interesting result found was that whereas it was previously known that in a sufficiently small neighbourhood of a matrix $M \in S^n$ the cp-rank cannot go down, we have shown that in a sufficiently small neighbourhood of a matrix $M \in S^n \setminus \text{bd}(CP^n)$ the cp-plus-rank can not go up. As the cp-plus-rank of a matrix is an upper bound on its cp-rank, this means that for a matrix $M \in S^n \setminus \text{bd}(CP^n)$ with its cp-rank equal to its cp-plus rank, in a sufficiently small neighbourhood of the matrix, neither the cp-rank nor the cp-plus-rank will change.

Motivated by this result we considered the open sets

$$\{M \in S^n \setminus \text{bd}(CP^n) | \text{cp}(M) = \text{cp}^+(M) = p\},$$

which were shown to be nonempty for all $p \in \{n, \ldots, p_n, \infty\}$. An interesting open question is whether these are also connected sets.

We have also established that the sets

$$\{M \in \text{int}(CP^n) | k - 1 = \text{cp}(M) < \text{cp}^+(M) = k\}$$

are nonempty for all $k \in \{n + 1, \ldots, p_n\}$. 

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Considering the set \( \mathcal{A}_n = \{ M \in S^n \mid \text{cp}(M) = \text{cp}^+(M) \} \), we have shown that this is dense in \( S^n \) and open in \( S^n \setminus \text{bd}(CP^n) \). By applying the theory of semi-algebraic sets we in addition established that membership in \( \mathcal{A}_n \) is a generic property in \( S^n \).

Some interesting questions are still open: For example, is the set \( \mathcal{A}_n \setminus \{0\} \) open in \( S^n \)? Note that around the zero matrix the set of matrices \( M \) satisfying \( \text{cp}(M) = \text{cp}^+(M) \) is not open. Indeed, take any matrix \( B \in CP^n \) with \( \text{cp}(B) \neq \text{cp}^+(B) \). For all \( \lambda > 0 \) we have \( \text{cp}(\lambda B) = \text{cp}(B) \neq \text{cp}^+(B) = \text{cp}^+(\lambda B) \), but for \( A = 0 \) we have \( A = \lim_{\lambda \to 0} \lambda B \) and \( \text{cp}(A) = \text{cp}^+(A) = 0 \).

On the other hand in contrast to the behaviour on \( S^n \) the set of matrices \( \mathcal{A}_n \) is not dense on \( \text{bd}(CP^n) \). Indeed there exist matrices \( A \in \text{bd}(CP^n) \) and \( \varepsilon > 0 \), such that for all \( M \in \text{bd}(CP^n) \cap N_\varepsilon(A) \) we have \( \text{cp}(M) \neq \text{cp}^+(M) = \infty \). Take for example the identity matrix \( I \in \text{bd}(CP^n) \). Since \( I \) has full rank \( n \), by Theorem 1.2 we must have \( \text{cp}^+(M) = \infty \) (otherwise \( I \) would be in the interior of \( CP^n \)) and this argument holds for all \( M \in \text{bd}(CP^n) \cap N_\varepsilon(I) \) for some \( \varepsilon > 0 \).

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