The higher sharp I

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Abstract

We establish the descriptive set theoretic representation of the mouse $M_n^\#$, which is called $0^{(n+1)\#}$. This part deals with the case $n \leq 3$.

1 Introduction

The collection of projective subsets of $\mathbb{R}$ is the minimum one which contains all the Borel sets and is closed under both complements and continuous images. Despite its natural-looking definition, many fundamental problems about projective sets are undecidable in ZFC, for instance, if all projective sets are Lebesgue measurable. The axiom of Projective Determinacy (PD) is the most satisfactory axiom that settles these problems by producing a rich structural theory of the projective sets. PD implies certain regularity properties of projective sets: all projects of reals are Lebesgue measurable (Mycielski, Swierczkowski), have the Baire property (Banach, Mazur) and are either countable or have a perfect subset (Davis) (cf. [36]). The structural theory of the projective sets are centered at good Suslin representations of projective sets. Moschovakis [36] shows that PD implies the scale property of the pointclasses $\Pi^1_{2n+1}$ and $\Sigma^1_{2n+2}$. It follows that there is a nicely behaved tree $T_{2n+1}$ that projects to the good universal $\Sigma^1_{2n+2}$ set. So the analysis of $\Sigma^1_{2n+2}$ sets is reduced to that of the tree $T_{2n+1}$, the canonical model $L[T_{2n+1}]$.
and its relativizations. The canonicality of $\mathcal{L}[T_{2n+1}]$ is justified by Becker-Kechris [3] in the sense that $\mathcal{L}[T_{2n+1}]$ does not depend on the choice of $T_{2n+1}$. The model $\mathcal{L}[T_{2n+1}]$ turns out to have many analogies with $\mathcal{L} = \mathcal{L}[T]$. These analogies support the generalizations of classical results on $\Sigma^1_2$ sets to $\Sigma^1_{2n+2}$ sets.

The validity of PD is further justified by Martin-Steel [34]. They show that PD is a consequence of large cardinals: if there are $n$ Woodin cardinals below a measurable cardinal, then $\Pi^1_{n+1}$ sets are determined. Inner model theory essentially developed into the region of Woodin cardinals, turns out to have its particular meaning in descriptive set theory. Martin [33] (for $\Sigma^1_1$), Neeman [37, 38] (for $n \geq 1$) show that $M^#$ is many-one equivalent to the good universal $\Sigma^{n+1}(<\omega^2,\Pi^1_1)$ real. Steel [49] shows that $\mathcal{L}[T_{2n+1}] = \mathcal{L}[M^#_{2\omega+\infty}\delta^1_{2n+1}]$ where $M^#_{2\omega+\infty}$ is the direct limit of all the countable iterates of $M^#_{2n}$, and that $\delta^1_{2n+1}$ is the least cardinal that is strong up to the least Woodin of $M^#_{2\omega+\infty}$. This precisely explains the analogy between $\mathcal{L}[T_{2n+1}]$ and $\mathcal{L}$. The mechanism of inner model theory is therefore applicable towards understanding the structure $\mathcal{L}[T_{2n+1}]$.

In this paper and its sequel, we generalize the Silver indiscernibles for $\Pi^1_n$ to the level-$(2n+1)$ indiscernibles of $\mathcal{L}[T_{2n+1}]$. The theory of $\mathcal{L}[T_{2n+1}]$ with the level-$(2n+1)$ indiscernibles will be called $0^{(2n+1)}#$, which is many-one equivalent to $M^#_{2n}$. At the level of mice with an odd number of Woodins, $M^#_{2n-1}$ is the optimal real with the basis result for $\Sigma^1_{2n+1}$ sets (cf. [51, Section 7.2]): Every nonempty $\Sigma^1_{2n+1}$ set has a member recursive in $M^#_{2n-1}$. The basis result for $\Sigma^1_{2n+1}$ was originally investigated in [27], with the intention of generalizing Kleene’s basis theorem: Every nonempty $\Sigma^1_1$ set of real has a member recursive in Kleene’s $\mathcal{O}$. The real $y_{2n+1}$, defined [27], turns out $\Delta^1_{2n+1}$ equivalent to $M^#_{2n-1}$. In this paper and its sequel, we define the canonical tree $T_{2n}$ that projects to a good universal $\Pi^1_{2n}$ set. It is the natural generalization of the Martin-Solovay tree $T_2$ that projects to a good universal $\Pi^1_2$ set. We show that $\mathcal{L}_{n2n+1}[T_{2n}]$, the minimum admissible set over $T_{2n}$, shares most of the standard properties of $L_{\omega^CK}$, in particular, the higher level analog of the Kechris-Martin theorem [21, 23]. We define $0^{(2n)}#$ as the set of truth values in $\mathcal{L}_{n2n+1}[T_{2n}]$ for formulas of complexity slightly higher than $\Sigma_1$. $0^{(2n)}#$ is many-one equivalent to both $M^#_{2n+1}$ and $y_{2n+1}$. Summing up, we have

$$0^{(n+1)}# \equiv_m M^#_n.$$

We start to give a detailed explanation of the influence of the higher sharp in the structural theory of projective sets and in inner model theory. The set theoretic structures tied to $\Pi^1_1$ sets are $L_{\omega^CK}$ and its relativizations. The classical results on $\Pi^1_1$ sets and $L_{\omega^CK}$ include:
1. (Model theoretic representation of $\Pi^1_1$) $A \subseteq \mathbb{R}$ is $\Pi^1_1$ iff there is a $\Sigma^1_1$ formula $\varphi$ such that $x \in A \leftrightarrow L_{\omega_1}[x] \models \varphi(x)$.

2. (Mouse set) $x \in \mathbb{R} \cap L_{\omega_1^CK}$ if $x$ is $\Delta^1_1$ iff $x$ is $\Delta^1_1$ in a countable ordinal.

3. (The transcendental real over $L_{\omega_1^CK}$) $\mathcal{O}$ is the $\Sigma^1_1$-theory of $L_{\omega_1^CK}$.

4. (\Pi^1_1-coding of ordinals below $\omega_1$) $x \in WO$ if $x$ codes a wellordering of a subset of $\omega$. Every ordinal below $\omega_1$ is coded by a member of $WO$. $WO$ is $\Pi^1_1$.

$\Sigma^1_2$ sets are $\omega_1$-Suslin via the Shoenfield tree $T_1$. The complexity of $T_1$ is essentially that of $WO$, or $\Pi^1_1$. The set-theoretic structures in our attention are $L = L[T_1]$ and its relativizations. Assuming every real has a sharp, the classical results related to $L$ include:

1. (Model theoretic representation of $\Sigma^1_2$) $A \subseteq \mathbb{R}$ is $\Sigma^1_2$ iff there is a $\Sigma^1_1$ formula $\varphi$ such that $x \in A \leftrightarrow L[x] \models \varphi(x)$.

2. (Mouse set) $x \in \mathbb{R} \cap L$ if $x$ is $\Delta^1_2$ in a countable ordinal.

3. (The transcendental real over $L$) $0^\#$ is the theory of $L$ with Silver indiscernibles, or equivalently, the least active sound mouse projecting to $\omega$.

4. (\Delta^1_3-coding of ordinals below $u_\omega$) $WO_{u_\omega}$ is the set of sharp codes. Every ordinal $\alpha < u_\omega$ has a sharp code $\langle \tau^1, x^\# \rangle$ so that $\alpha = \tau^L[x](x, u_1, \ldots, u_k)$. The comparison of sharp codes is $\Delta^1_3$.

Inner model theory start to participate at this level. Based on the theory of sharps for reals, the Martin-Solovay tree $T_2$ is defined. $T_2$ is essentially a tree on $u_\omega$. The complexity of $T_2$ is $\Delta^1_3$ via the sharp coding of ordinals.

$\Sigma^1_3$ sets are $u_\omega$-Suslin via the Martin-Solovay tree $T_2$. The structures tied to $\Pi^1_3$ sets are $L_{\kappa_3}[T_2]$ and its relativizations. The theory at this level is in parallel to $\Pi^1_1$ sets and $L_{\omega_1^CK}$:

1. (Model theoretic representation of $\Pi^1_3$, $[21, 23]$) $A \subseteq \mathbb{R}$ is $\Pi^1_3$ iff there is a $\Sigma^1_1$ formula $\varphi$ such that $x \in A \leftrightarrow L_{\kappa_3}[T_2, x] \models \varphi(T_2, x)$.

2. (Mouse set, $[21, 23, 27, 47]$) $x \in \mathbb{R} \cap L_{\kappa_3}[T_2]$ if $x$ is $\Delta^1_3$ in a countable ordinal iff $x \in \mathbb{R} \cap M_{1^\#}$.

3. (The transcendental real over $L_{\kappa_3}[T_2]$, Theorem $4.8$) $M_{1^\#} \equiv_m 0^{2\#}$. 

3
4. \((\Pi_3^1\text{-coding of ordinals below } \delta_3^1, \text{ essentially by Kunen in } \square_{46}) \text{WO}^{(3)} \text{ is the set of reals that naturally code a wellordering of } u_\omega. \text{WO}^{(3)} \text{ is } \Pi_3^1.\)

In general, if \(\Gamma\) is a pointclass, \(\alpha\) is an ordinal, and \(f : \mathbb{R} \to \alpha\) is a surjection, then Code(\(f\)) = \(\{(x, y) : f(x) \leq f(y)\}\) and \(f\) is in \(\Gamma\) iff Code(\(f\)) is in \(\Gamma\); \(\alpha\) is \(\Gamma\)-wellordered cardinal iff there is a surjection \(f : \mathbb{R} \to \alpha\) such that \(f\) is in \(\Gamma\) but there is no \(\beta < \alpha\) and surjections \(g : \mathbb{R} \to \beta, h : \beta \to \alpha\) such that both \(g\) and \(\{(x, y) : f(x) = h \circ g(y)\}\) is in \(\Gamma\). The above list can be continued:

5. The uncountable \(\Delta_3^1\) wellordered cardinals are \((u_k : 1 \leq k \leq \omega).\)

The heart of the new knowledge is the equality of pointclass in Theorem 4.5
\(\bigodot^2(<\omega^2\Pi_1^1) =< u_\omega\Pi_3^1.\) Philosophically speaking, as \(\bigodot^2\Pi_1^1 = \Pi_3^1\), this equality reduces the “non-linear” part \(\bigodot^2\) to the “linear” part \(< u_\omega.\) Based on this equality, \(0^{\#}\) is defined to be the set of truth of \(L_{\kappa_3}[T_2]\) for formulas of complexity slightly larger than \(\Sigma_1,\) cf. Definitions 4.6-4.7. \(0^{\#}\) is essentially \(y_3,\) defined in \(\square_{27}\). It is a good universal \(< u_\omega\Pi_3^1\) subset of \(\omega.\) The many-one equivalence \(M_1^{\#} \equiv_m 0^{\#}\) is thus obtained using Neeman \(\square_{37}, \square_{38}\). Under \(AD,\) we have \(u_k = \aleph_k,\) and \(\square_{25}\) summarizes the further structural theory at this level. The expression of \(0^{\#}\) opens the possibility of running recursion-theoretic arguments in \(L_{\kappa_3}[T_2]\) that generalize those in \(L_{\omega_1^{CK}}.\)

The Moschovakis tree \(T_{2n+1}\) projects to the good universal \(\Sigma_{2n+2}\) set. The structures tied to \(\Sigma_{2n+2}^1\) sets are \(L[T_{2n+1}]\) and its relativizations. \(L[T_{2n+1}]\) is the higher level analog of \(L:\)

1. (Model theoretic representation of \(\Sigma_{2n+2}^1\) A \(\subseteq \mathbb{R}\) is \(\Sigma_{2n+2}^1\) iff there is a \(\Sigma_1\) formula \(\varphi\) such that \(x \in A \leftrightarrow L[T_{2n+1}, x] \models \varphi(T_{2n+1}, x).\)

2. (Mouse set) \(\square_{47}\) \(x \in \mathbb{R} \cap L[T_{2n+2}]\) iff \(x\) is \(\Delta_2^{1n+2}\) in a countable ordinal iff \(x \in \mathbb{R} \cap M_{2n}^{\#}.\)

3. (The transcendental real over \(L[T_{2n+2}].\) Theorem \(\square_{5.13}\) for \(n = 1) M_{2n}^{\#} \equiv_m 0^{(2n+1)\#}.\)

4. \((\Delta_{2n+3}^1\text{-coding of ordinals below } u_{E(2n+1)}^{(2n+1)} \text{WO}^{(2n+1)} E_{(2n+1)}\) is the set of level \((2n + 1)\) sharp codes for ordinals in \(u_{E(2n+1)}^{(2n+1)}\). The comparison of level \((2n + 1)\) sharp codes is \(\Delta_{2n+3}^1.\)

\(0^{(2n+1)\#}\) is the theory of \(L[T_{2n+1}]\) with level \((2n + 1)\) indiscernibles. The structure of the level \((2n + 1)\) indiscernibles is more complicated than their order, as opposed to the order indiscernibles for \(L.\) The level \((2n + 1)\) indiscernibles form a tree structure, and the type realized in \(L[T_{2n+1}]\) by finitely many of them depends only on the finite tree structure that relates them. This tree
structure resembles the structure of measures (under AD) witnessing the homogeneity of $S_{2n+1}$, a tree on $\omega \times \delta_3$ that projects to the good universal $\Pi_{2n+1}^1$ set. We give a purely syntactical definition of $0^{(2n+1)\#}$ as the unique iterable, remarkable, level $\leq 2n$ correct level-(2n + 1) EM blueprint. This is the higher level analog of $0^\#$ as the unique wellfounded remarkable EM blueprint. The “iterability” part takes the form $\forall \exists^R (\Pi_{2n+1}^1 \to \Pi_{2n+1}^1)$, making the complexity of the whole definition $\Pi_{2n+2}^1$. The ordinal $u_{E(2n+1)}^{(2n+1)}$ is a level-(2n + 1) uniform indiscernible. It will be discussed in the next paragraph. When $n = 0$, $u_{E(1)}^{(1)} = u_\omega$.

The structure tied to arbitrary $\Pi_{2n+1}^1$ sets are defined. By induction, we have level-(2n – 1) indiscernibles for $L^{\delta_{2n-1}}[T_{2n-1}]$ and the real $0^{(2n-1)\#}$. Based on the EM blueprint formulation of $0^{(2n-1)\#}$, we define the level-2n Martin-Solovay tree $T_{2n}$. It is the higher level analog of $T_2$. This is the most canonical tree that enables the correct generalization of the structural theory related to $\Pi_{2n+1}^1$ sets. The structures in our attention are $L_{\kappa_{2n+1}}[T_{2n}]$, the least admissible set over $T_{2n}$, and its relativizations:

1. (Model theoretic representation of $\Pi_{2n+1}^1$, Theorem 7.4 for $n = 2$) $A \subseteq \mathbb{R}$ is $\Pi_{2n+1}^1$ iff there is a $\Sigma_1$ formula $\varphi$ such that $x \in A \leftrightarrow L^{\kappa_{2n+1}}[T_{2n}, x] \models \varphi(T_{2n}, x)$.

2. (Mouse set, [47]) $x \in \mathbb{R} \cap L^{\kappa_{2n+1}}[T_{2n+1}]$ iff $x$ is $\Delta_{2n+1}^1$ in a countable ordinal iff $x \in M_{2n-1}^\#$.

3. (The transcendental real over $L^{\kappa_{2n+1}}[T_{2n}, x]$, Theorem 7.10 for $n = 2$) $M_{2n-1}^\# = m(0^{(2n)\#})$.

4. ($\Pi_{2n+1}^1$-coding of ordinals below $\delta_{2n+1}^1$) $WO^{(2n+1)}$ is the set of reals that naturally code a wellordering of $u_{E(2n-1)}^{(2n-1)}$. $WO^{(2n+1)}$ is $\Pi_{2n+1}^1$.

5. The uncountable $\Delta_{2n+1}^1$ wellordered cardinals are $(u_k : 1 \leq k \leq \omega)$, $(u_\xi^{(3)} : 1 \leq \xi \leq E(3))$, ..., $(u_\xi^{(2n-1)} : 1 \leq \xi \leq E(2n - 1))$, where $E(0) = 1$, $E(i+1) = \omega^{E(i)}$ via ordinal exponentiation.

The equivalence $M_{2n-1}^\# = m(0^{(2n)\#})$ is based on the equality of pointclasses (Theorem 7.7 for $n = 2$): $\exists^{2n}(<\omega^2-\Pi_1^1) = <u_{E(2n-1)}^{(2n-1)}\Pi_{2n+1}^1$. $\{u_\xi^{(2n-1)} : 1 \leq \xi \leq E(2n - 1)\}$ is the set of level-(2n – 1) uniform indiscernibles. It is the higher level analog of the first $\omega + 1$ uniform indiscernibles $\{u_n : 1 \leq n \leq \omega\}$. Under full AD, the uncountable $\Delta_{2n+1}^1$ wellordered cardinals enumerate all the uncountable cardinals below $\delta_{2n+1}^1$: $u_k = \aleph_k$ for $1 \leq k < \omega$, $u_\xi^{(2n+1)} = \aleph_{E(2i-1) + \xi}$ for $1 \leq \xi \leq E(2i + 1)$. Assume AD for the moment. The equation
\( \delta_{2n+1} = \mathcal{N}_{E(2n-1)+1} \) is originally proved by Jackson in [12][15]. Jackson shows that every successor cardinal in the interval \([\delta_{2n-1}, \mathcal{N}_{E(2n-1)}]\) is the image of \( \delta_{2n-1} \) via an ultrapower map induced by a measure on \( \delta_{2n-1} \). [16] goes on to show that for a certain collection of measures \( \mu \) on \( \delta_{2n-1} \), every description leads to a canonical function representing a cardinal modulo \( \mu \). [16][17] compute the cofinality of the cardinals below \( \delta_{2n} \). In this paper and its sequel, we demonstrate the greater importance of the set theoretic structures tied to these cardinals over their order type. It is the inner model \( L[T_{2n-1}] \) and its images via different ultrapower maps that give birth to the uncanny order type \( E(2n-1)+1 \). The level-\((2n-1)\) uniform indiscernibles \( (u_{(2n-1)}^{(2n-1)} : 1 \leq \xi \leq E(2n-1)) \) are defined under this circumstance. Recall that the first \( \omega \) uniform indiscernibles can be generated by \( j^\omega (L_{\omega_1}) = L_{u_{n+1}} \), where \( \mu^n \) is the \( n \)-fold product of the club measure on \( \omega_1 \); if \( 1 \leq i \leq n+1 \), then \( u_i \) is represented modulo \( \mu^n \) by a projection map; every ordinal below \( u_{n+1} \) is in the Skolem hull of \( \{x, u_1, \ldots, u_n\} \) over \( L[x] \) for some \( x \in \mathbb{R} \). This scenario is generalized by the level-\((2n-1)\) uniform indiscernibles. As a by-product, we simplify the arguments in [12][15][17], show in full generality that any description represents a cardinal modulo any measure on \( \delta_{2n-1} \), and establish the effective version of the cofinality computations.

The whole argument is inductive. Assume AD for simplicity. In the computation of \( \delta_{2n+1} \) in [12][15], the strong partition property of \( \delta_{2n+1} \) is proved and used inductively in the process. Our argument reproves the strong partition property of \( \delta_{2n+1} \) using the EM blueprint formulation of \( 0^{(2n+1)}# \). The definition of \( 0^{(2n+1)}# \) is based on the analysis of level-\((2n+1)\) indiscernibles, whose existence depend on the homogeneous Suslin representations of \( \Pi^1_{2n} \) sets, which in turn follow from the strong partition property of \( \delta_{2n-1} \). Just as the main ideas of the computation of \( \delta_{2n+1} \) boil down to that of \( \delta_5 \), this paper defines \( 0^2#, 0^3#, 0^4# \), which contains all the key ideas in a general inductive step. The sequel to this paper will deal with the general inductive step. It will be merely a technical manifestation.

A deeper insight into the interaction between inner model theory and Jackson’s computation of projective ordinals in [12][15] is the concrete information on the direct system of countable iterates of \( M_{2n}^\# \). Put \( n = 1 \) and assume AD for simplicity sake. Put \( M_{2,\infty}^- = L_{\delta_3^1}[T_3] \). We define \( (\xi^{(3)}_\xi : \xi < \delta_3^1) \), a continuous sequence in \( \delta_3^1 \) that generates the set of level-3 indiscernibles for \( M_{2,\infty}^- \). Each \( M_{2,\infty}^-|\xi^{(3)}_\xi \) is the direct limit of \( \Pi^1_3 \)-iterable mice whose Dodd-Jensen order is \( \xi^{(3)}_\xi \). We define an alternative direct limit system indexed by ordinals in \( u_\omega \), which is dense in the system leading to \( M_{2,\infty}^-|\xi^{(3)}_\xi \). The advantage of this dense subsystem is that it leads to a good coding of \( M_{2,\infty}^-|\xi^{(3)}_\xi \) by
a subset of $u_\omega$. The indexing ordinals are represented by wellorderings on $\omega_1$ of order type $\omega_1 + 1$ modulo measures on $\omega_1$ arising from the strong partition property on $\omega_1$. Any order-preserving injection between two such wellorderings corresponds to an elementary embedding between models of this new direct limit. This injection is an isomorphism just in case its corresponding elementary embedding is essentially an iteration map, i.e., commutes with the comparison maps. The new direct system is then guided by isomorphisms between wellorderings on $\omega_1$ of order type $\omega_1 + 1$. In this regard, the Dodd-Jensen property of mice corresponds to the simple fact that if $f$ is an order preserving map between ordinals, then $\alpha \leq f(\alpha)$ pointwise. This observation is not surprising at all, as the Dodd-Jensen property on iterates of $0^#$ is originated from this simple fact. This viewpoint might be a prelude to understanding the combinatorial nature of iteration trees on mice with finitely many Woodin cardinals.

A key step in computing the upper bound of $\delta_3^1$ in [12] is the (level-3) Martin tree. For the reader familiar with the Martin tree and the purely descriptive set theoretical proof of the Kechris-Martin theorem in [13, Section 4.4], the level-1 version of the Martin tree is essentially an analysis of partially iterable sharps. The level-3 Martin tree is therefore replaced by an analysis of partially iterable level-3 sharps in this paper. The aforementioned new direct limit system indexed by ordinals in $u_\omega$ applies to any partially iterable mouse, so that its possibly illfounded direct limit is naturally coded by a subset of $u_\omega$. This is yet another incidence that descriptive set theory and inner model theory are two sides of the same coin.

Apart from inner model theory, the pure computational component in [12, 15] has a major simplification. Under AD, a successor cardinal in the interval $[\delta_3^1, \kappa_{\omega_1}]$ is represented by a measure $\mu$ on $\delta_3^1$ and a description. The original definition of description involves a finite iteration of ultrapowers on $u_\omega$. The “finite iteration of ultrapowers” part is now simplified to a single ultrapower, due to Lemma 4.53.

As $L[\kappa_{2n+1}, T_{2n}]$ is the correct structure tied to $\Pi^1_{2n+1}$ sets, it is natural to investigate its intrinsic structure. However, little is known at this very step. The closest result is on the full model $L[T_{2n}]$. The uniqueness of $L[T_{2n}]$ is proved by Hjorth [9] for $n = 1$ and Atmai [2] for general $n$. Here, uniqueness means that if $T'$ is the tree of another $\Delta^1_{2n+1}$-scale on a good universal $\Pi^1_{2n}$ set, then $L[T_{2n}] = L[T']$. Atmai-Sargsyan [2] goes on to show that the full model $L[T_{2n}]$ is just $L(M^\#_{2n-1, \infty})$, where $M^\#_{2n-1, \infty}$ is the direct limit of all the countable iterates of $M^\#_{2n-1}$. A test question that separates $L[\kappa_3, T_2]$ from $L[T_2]$ is the inner model theoretic characterization of $C_3$, the largest countable $\Pi^1_3$ set: if $x \in C_3$, must $x$ be $\Delta^1_3$-equivalent to a master code.
in $M_2$? (cf. [50, p.13]) Section 4.5 sets up a good preparation for tackling this problem.

Looking higher up, the technique in this paper and its sequel should generalize to arbitrary projective-like pointclasses in $L(\mathbb{R})$ and beyond. The descriptive set theory counterpart of larger mice should enhance our understanding of large cardinals. Typical open questions in the higher level include:

1. (cf. [1, Problem 19]) Assume AD. Let $\Gamma$ be a $\Pi^1_1$-like scaled pointclass (i.e., closed under $\forall^\infty_\infty$, continuous preimages and non-self-dual) and Let $\Delta = \Gamma \cap \Gamma^\perp$, $\delta = \sup\{|<| : <$ is a prewellordering in $\Delta\}$. Is $\Gamma$ closed under unions of length $< \delta$?

2. Assume AD. Let $\Gamma$, $\delta$ be as in 1. Must $\delta$ have the strong partition property?

3. Assume AD. If $\kappa \leq \lambda$ are cardinals, must $\text{cf}(\kappa^{++}) \leq \text{cf}(\lambda^{++})$?

We now switch to some immediate applications on the theory of higher level indiscernibles. Our belief is that any result in set theory that involves sharp and Silver indiscernibles should generalize to arbitrary projective levels.

Woodin [43] proves that boldface $\Pi^1_{2n+1}$-determinacy is equivalent to “for any real $x$, there is an $(\omega,\omega_1)$-iterable $M^\#_n(x)$”. The lightface scenario is tricky however. Neeman [37][38] proves that the existence of an $\omega_1$-iterable $M^\#_n$ implies boldface $\Pi^1_n$-determinacy and lightface $\Pi^1_{n+1}$-determinacy.

**Question 1.1** (cf. [4, #9]). Assume $\Pi^1_n$-determinacy and $\Pi^1_{n+1}$-determinacy. Must there exist an $\omega_1$-iterable $M^\#_n$?

Note that the assumption of boldface $\Pi^1_n$-determinacy in Question 1.1 is necessary, as $\Delta^1_2$-determinacy alone is enough to imply that there is a model of OD-determinacy (Kechris-Solovay [28]). The cases $n \in \{0, 1\}$ in Question 1.1 are solved positively by Harrington in [8] and by Woodin in [48]. The proof of the $n = 1$ case heavily relies on the theory of Silver indiscernibles for $L$. The theory of level-3 indiscernibles for $L_{\delta_3_3}[T_3]$ is thus involved in proving the general case when $n$ is odd.

**Theorem 1.2.** Assume $\Pi^1_{2n+1}$-determinacy and $\Pi^1_{2n+2}$-determinacy. Then there exists an $(\omega,\omega_1)$-iterable $M^\#_{2n+1}$.

The proof of Theorem 1.2 will appear in further publications. The case $n \geq 2$ even in Question 1.1 remains open.

Another application is the $\delta$-ordinal of intermediate pointclasses between $\Pi^1_m$ and $\Delta^1_{m+1}$. If $\Gamma$ is a pointclass, $\delta(\Gamma)$ is the supremum of the lengths of
Γ-prewellorderings on $\mathbb{R}$. A $\subseteq \mathbb{R}$ is $\Gamma_{m,n}(z)$ iff for some formula $\psi$ we have $x \in A \iff M_{m-1}[x,z] \models \psi(x,z,N_1,\ldots,N_n)$. A is $\Gamma_{m,n}$ iff $A$ is $\Gamma_{m,n}(z)$ for some real $z$. Hjorth [11] proves that $\delta(\Gamma_{1,n}) = u_{n+2}$ under $\Delta^1_2$-determinacy. Sargsyan [41] proves that under $\text{AD}$, $\sup_{n<\omega} \delta(\Gamma_{2k+1,n})$ is the cardinal predecessor of $\delta^{2k+3}$. The exact value of $\delta(\Gamma_{2k+1,n})$ remains unknown. Based on the theory of higher level indiscernibles, we can define the pointclasses $\Lambda_{2k+1,\xi}$ for $0 < \xi \leq E(2k + 1)$. For the moment we need the notations in this paper. $A \subseteq \mathbb{R}$ is $\Lambda_{3,\xi}(z)$ iff for some level-$3$ tree $R$ such that $\langle \emptyset \rangle_R = \hat{\xi}$, for some $L^z$-formula $\psi$ we have $x \in A \iff \langle \gamma \rangle \in (x,z)^{3#}(R)$. When $\xi$ is a limit, $\Lambda_{3,\xi}(z) = \bigcup_{\eta<\xi} \Lambda_{3,\eta}(z)$. A is $\Lambda_{3,\xi}$ iff $A$ is $\Lambda_{3,\xi}(z)$ for some real $z$.

**Theorem 1.3.** Assume $\Delta^1_4$-determinacy and $0 < \xi < \omega^{\omega\omega}$. If $\xi$ is a successor ordinal, then $\delta(\Lambda_{3,\xi}) = u^{(3)}_{\xi+1}$. If $\xi$ is a limit ordinal, then $\delta(\Lambda_{3,\xi}) = u^{(3)}_{\xi}$.

The proof of Theorem 1.3 and its higher level analog will appear in further publications. The question on the value of $\delta(\Gamma_{3,n})$ is then reduced to the relative position of $\Gamma_{3,n}$ in the hierarchy $(\Lambda_{3,\xi} : 0 < \xi < \omega^{\omega\omega})$. The results of this paper combined with Neeman [37,38] yields the following estimate:

$$\Lambda_{3,\omega^{\omega\omega}n} \subseteq \Gamma_{3,n} \subseteq \Lambda_{3,\omega^{\omega\omega}n+1+1}.$$ 

We conjecture that $\Lambda_{3,\omega^{\omega\omega}n+1} \not\subseteq \Gamma_{3,n} \not\subseteq \Lambda_{3,\omega^{\omega\omega}n+1+1}$ and $\delta(\Gamma_{3,n}) = u^{(3)}_{\omega^{\omega\omega}n+1+1}$. We try to make this paper as self-contained as possible. The reader is assumed to have some minimum background knowledge in descriptive set theory and inner model theory. On the descriptive set theory side, we assume basic knowledge of determinacy, scale and its tree representation, homogeneous tree and its ultrapower representation, and at least the results of Moschovakis periodicity theorems. We will briefly recall them in Section 2. Theorem 2.1 by Becker-Kechris [3] and Kechris-Martin [21,23] will basically be treated as a black box. Knowing its proof would help, though not necessary. On the inner model theory side, we assume basic knowledge of mice and iteration trees in the region of finitely many Woodin cardinals, especially Theorem 6.10 in [51]. The level-wise projective complexity associated to mice will be recalled in Section 2.5. Theorem 2.18 by Steel [49] will be treated as a black box. In particular, we require absolutely no knowledge of Jackson’s analysis in [12,15].

This paper is structured as follows. Section 2 fixes notations and briefly reviews the background knowledge. Section 3 is basically a review of sharps and the Martin-Solovay tree, expressed in a form that is easy to generalize. Section 4 proves the many-one equivalence of $0^{2#}$ and $M^{\#}_1$, generalizes Jackson’s level-2 and level-3 analysis, and establishes useful properties of the
coding system for ordinals in $\delta_3^1$. Built on these results, Section 5 defines the level-3 indiscernibles for $L_{\delta_1^1}[T_3]$, proves the many-one equivalence of $0^#^3$ and $M_2^*$, and gives a $\Pi^1_1$-axiomatization of the real $0^#$. Section 6 defines the uniform level-3 indiscernibles and the level-4 Martin-Solovay tree. Section 7 proves the level-4 Kechris-Martin theorem and the many-one equivalence of $0^4#$ and $M_3^*(x)$, which prepares for the induction into the next level.

2 Backgrounds and preliminaries

Following the usual treatment in descriptive set theory, $\mathbb{R} = \omega^\omega$ is the Baire space, which is homeomorphic to the irrationals of the real line. If $A \subseteq \mathbb{R} \times X$, then $y \in \exists^R A$ iff $\exists x \in \mathbb{R} \ (x, y) \in A$, $y \in \forall^R A$ iff $\forall x \in \mathbb{R} \ (x, y) \in A$, $y \in \exists^R A$ iff Player I has a winning strategy in the game with output $A_y = \text{DEF} \ \{x : (x, y) \in A\}$. $\exists^R A = \exists(D^n(A))$ when $A$ is a subset of an appropriate product space. A pointclass is a collection of subsets of Polish spaces (typically finite products of $\omega$ and $\mathbb{R}$). If $\Gamma$ is a pointclass, then $\exists^R \Gamma = \{\exists^R A : A \in \Gamma\}$, and similarly for $\forall^R \Gamma$, $\forall \Gamma$, $\exists^R \Gamma$. $\Sigma^1_0 = \Sigma^1_0$ is the pointclass of open sets. $\Sigma^1_0 = \Sigma^1_0$ is the pointclass of effectively open sets. $\Pi^1_{n+1} = \forall^R \Sigma^1_n$, $\Sigma^1_{n+1} = \exists^R \Pi^1_n$.

If $\alpha$ is an ordinal and $A \subseteq \alpha \times X$, then

$$x \in \text{Diff} A \iff \exists i < \alpha \ (\alpha \text{ is odd } \land \forall j < i (j, x) \in A) \land (i, x) \notin A.$$ 

If $\alpha < \omega^\omega^{CK}$ then $A \subseteq X$ is $\alpha$-$\Pi^1_1$ iff $A = \text{Diff} B$ for some $\Pi^1_1 B \subseteq \alpha \times X$. $A$ is $\alpha$-$\Pi^1_1$ iff $A$ is $\beta$-$\Pi^1_1$ for some $\beta < \alpha$. Martin [33] proves that $\Pi^1_1$-determinacy implies $<\omega^2$-$\Pi^1_1$-determinacy.

A tree on $X$ is a subset of $X^{<\omega}$ closed under initial segments. If $T$ is a tree on $X$, $[T]$ is the set of infinite branches of $T$, i.e., $x \in T$ iff $\forall n \ (x \upharpoonright n) \in T$. If $T$ is a tree on $\lambda$, $\lambda$ is an ordinal, $[T] \neq \emptyset$, the leftmost branch is $x \in [T]$ such that for any $y \in [T]$, $(x(0), x(1), \ldots)$ is lexicographically smaller than or equal to $(y(0), y(1), \ldots)$. In addition, if $x \in [T]$ and for any $y \in [T]$ we have $\forall n \ x(n) \leq y(n)$, then $x$ is the honest leftmost branch of $T$. A tree $T$ on $\omega \times X$ is identified with a subset of $\omega^{<\omega} \times X^{<\omega}$ consisting of $(s, t)$ so that $\text{lh}(s) = \text{lh}(t)$ and $((s(i), t(i)))_{i<\text{lh}(s)} \in T$. If $T$ is a tree on $\omega \times X$, $[T] \subseteq \omega^{<\omega} \times X^{<\omega}$ is the set of infinite branches of $T$. $p[T] = \{x : \exists y \ (x, y) \in [T]\}$ is the projection of $T$. If $T$ is a tree on $\omega \times \lambda$ and $p[T] \neq \emptyset$, then $x$ is the leftmost real of $T$ iff $\exists \alpha \ (x, \alpha)$ is the leftmost branch of $T$.

Suppose $A \subseteq \mathbb{R}$. A norm on $A$ is a function $\varphi : A \rightarrow \text{Ord}$. $\varphi$ is regular iff $\text{ran}(\varphi)$ is an ordinal. A scale on $A$ is a sequence of norms $\vec{\varphi} = (\varphi_n)_{n<\omega}$ on $A$ such that if $(x_i)_{i<\omega} \subseteq A$, $x_i \rightarrow x(i \rightarrow \infty)$ in the Baire topology, and for all $n$, $\varphi_n(x_i) \rightarrow \lambda_n(i \rightarrow \infty)$ in the discrete topology, then $x \in A$ and $\forall n \ \varphi_n(x) \leq \lambda_n$. 

10
$\varphi$ is regular iff each $\varphi_n$ is regular. If $A = p[T]$, $T$ is a tree on $\omega \times \lambda$, the $\lambda$-scale associated to $T$ is $(\varphi_n)_{n<\omega}$ where $\varphi_n(x) = \langle \alpha_n^0, \ldots, \alpha_n^n \rangle$, $(\alpha_n^n)_{n<\omega}$ is the leftmost branch of $T_x$ for all $x \in [T]$, $(\ldots) : \lambda^{n+1} \rightarrow \text{Ord}$ is order preserving with respect to the lexicographic order and is onto an ordinal. Suppose $\Gamma$ is a pointclass. If $\varphi$ is a norm on $A$, then $\varphi$ is a $\Gamma$-norm iff the relations

$$x \leq_{\varphi} y \iff x \in A \land (y \in A \rightarrow \varphi(x) \leq \varphi(y)),$$

$$x <_{\varphi} y \iff x \in A \land (y \in A \rightarrow \varphi(x) < \varphi(y)).$$

are both in $\Gamma$. $\varphi = (\varphi_n)_{n<\omega}$ is a $\Gamma$-scale iff the relations $x \leq_{\varphi_n} y$ and $x <_{\varphi_n} y$ in $(x, y, n)$ are both in $\Gamma$. $\Gamma$ has the prewellordering property iff every set in $\Gamma$ has a $\Gamma$-norm. $\Gamma$ has the scale property iff every set in $\Gamma$ has a $\Gamma$-scale. Assuming PD, Moschovakis [36] shows that the pointclasses $\Pi^0_2$, $\Pi^0_3$, $\Sigma^1_2$, $\Sigma^1_3$ are both in $\Gamma$.

In Sections 2.1-2.3, we assume $\Delta^1_2$-determinacy.

The set $\{\langle x^\# \rangle : x \in \mathbb{R}\}$ is $\Pi^1_3$. $\text{WO} = \text{WO}_1$ is the set of codes for countable ordinals. For $1 \leq m < \omega$, $\text{WO}_{m+1}$ is the set of $\langle \tau, x^\# \rangle$ where $\tau$ is an $(m+1)$-ary Skolem term for an ordinal in the language of set theory and $x \in \mathbb{R}$. The
ordinal coded by $w = \langle \tau^\gamma, x^\# \rangle \in \text{WO}_{m+1}$ is

$$|w| = \tau^{L[x]}(x, u_1, \ldots, u_m).$$

Every ordinal in $u_{m+1}$ is of the form $|w|$ for some $w \in \text{WO}_{m+1}$. For each $1 \leq m < \omega$,

$$\{ \tau^{L[x]}(x, u_m) : \langle \tau^\gamma, x^\# \rangle \in \text{WO}_2 \}$$

is a cofinal subset of $u_{m+1}$. $\text{WO}_\omega = \bigcup_{1 \leq m < \omega} \text{WO}_m$. $\text{WO}$ is $\Pi^1_1$, and $\text{WO}_{m+1}$ is $\Pi^1_2$ for $1 \leq m < \omega$.

If $X$ is a Polish space, $A \subseteq X \times u_{\omega}$ and $\Gamma$ is a pointclass, say that $A$ is in $\Gamma$ iff

$$A^* = \{(x, w) : x \in \text{WO}_\omega \land (x, |w|) \in A\}$$

is in $\Gamma$. $\Gamma$ acting on product spaces are similarly defined.

$T_2$ refers to the Martin-Solovay tree on $\omega \times u_{\omega}$ that projects to $\{x^\# : x \in R\}$, giving the scale

$$\varphi_{\tau^\gamma}(x^\#) = \tau^{L[x]}(x, u_1, \ldots, u_{k_\tau}),$$

where $\tau^\gamma$ is the Gödel number of $\tau$, $\tau$ is $k_\tau + 1$-ary. Details can be found in [31] or [323], or in Section 3.2 of this paper. $T_2$ is a $\Delta^1_3$ subset of $(\omega \times u_{\omega})^{<\omega}$.

From $T_2$ one can compute a tree $\hat{T}_2$ on $\omega \times u_{\omega}$ that projects to a good universal $\Pi^1_2$ set. The definition of $\hat{T}_2$ will be recalled in Section 3.2.

For $x \in R$, $L_{\kappa_3}[T_2, x]$ is the minimum admissible set containing $(T_2, x)$.

The fact that the $\hat{T}_2$ projects to a good universal $\Pi^1_2$ set implies for every $\Pi^1_3$ set of reals $A$, there is a $\Sigma_1$-formula $\varphi$ such that $x \in A$ iff $L_{\kappa_3}[T_2, x] \models \varphi(T_2, x)$: $\varphi$ can be effectively computed from the definition of $A$. Becker-Kechris in [3] strengthens this fact by allowing a parameter in $u_{\omega}$. The converse direction is shown by Kechris-Martin in [21, 23]. The back-and-forth conversion is concluded in [3].

**Theorem 2.1** (Becker-Kechris, Kechris-Martin). Assume $\Delta^1_2$-determinacy. Then for each $A \subseteq u_{\omega} \times R$, the following are equivalent.

1. $A$ is $\Pi^1_3$.

2. There is a $\Sigma_1$ formula $\varphi$ such that $(\alpha, x) \in A$ iff $L_{\kappa_3}[T_2, x] \models \varphi(T_2, \alpha, x)$.

The conversion between the $\Pi^1_3$ definition of $A$ and the $\Sigma_1$-formula $\varphi$ are effective. The original proof of $2 \Rightarrow 1$ in Theorem 2.1 is based on Theorem 2.2 and Corollary 2.3.

**Theorem 2.2** (Kechris-Martin, [21, 23]). Assume $\Delta^1_2$-determinacy. Let $x \in R$. If $A$ is a nonempty $\Pi^1_3(x)$ subset of $u_{\omega}$, then $\exists w \in \Delta^1_3(x) \cap \text{WO}_\omega(|w| \in A)$. 

12
Corollary 2.3 (Kechris-Martin, [21, 23]). Assume $\Delta^1_2$-determinacy. Then $\Pi^1_3$ is closed under quantifications over $u_\omega$, i.e., if $A \subseteq (u_\omega)^2 \times \mathbb{R}$ is $\Pi^1_3$, then so are

$$B = \{ (\alpha, x) : \exists \beta < u_\omega \ (\beta, \alpha, x) \in A \},$$

$$C = \{ (\alpha, x) : \forall \beta < u_\omega \ (\beta, \alpha, x) \in A \}.$$

Suppose $\mathcal{X}$ is a Polish space. For $x \in \mathbb{R}$ and $\alpha < u_\omega$, $A \subseteq \mathcal{X}$ is $\Sigma^1_3(x, \alpha)$ iff there is a $\Sigma^1_3(x, \alpha)$ set $B \subseteq u_\omega \times \mathcal{X}$ such that $y \in A$ iff $(\alpha, y) \in B$. Or equivalently, $A$ is $\Sigma^1_3(x, \alpha)$ iff there is a $\Sigma^1_3(x)$ set $B \subseteq \mathbb{R} \times \mathcal{X}$ such that $y \in A$ iff $\exists w \in \text{WO}_\omega \ (|w| = \alpha \land (w, \alpha) \in B)$. $A$ is $\Pi^1_3(x, \alpha)$ iff $\mathcal{X} \setminus A$ is $\Sigma^1_3(x, \alpha)$. $A$ is $\Delta^1_3(x, \alpha)$ iff $A$ is both $\Sigma^1_3(x, \alpha)$ and $\Pi^1_3(x, \alpha)$. $\Sigma^1_3(x, <\beta)$ means $\Sigma^1_3(x, \alpha)$ for some $\alpha < \beta$. Similarly define $\Pi^1_3(x, <\beta)$ and $\Delta^1_3(x, <\beta)$.

In the proof of Theorem 2.1 the prewellordering property for $\Pi^1_1$ subsets of $\omega \times u_\omega$, originally proved by Solovay, is used.

Theorem 2.4 (Solovay, [24, Theorem 3.1]). Assume $\Delta^1_2$-determinacy. Suppose $A \subseteq u_\omega \times \mathbb{R}$ is $\Pi^1_3(x, \alpha)$, where $x \in \mathbb{R}$, $\alpha < u_\omega$. Then there is a $\Pi^1_3(x, \alpha)$ norm $\varphi : A \rightarrow \text{Ord}$, i.e., the relations

$$(\beta, y) \leq^* \varphi (\gamma, z) \iff (\beta, y) \in A \land ((\gamma, z) \in A \rightarrow \varphi(\beta, y) \leq \varphi(\gamma, z))$$

$$(\beta, y) <^* \varphi (\gamma, z) \iff (\beta, y) \in A \land ((\gamma, z) \in A \rightarrow \varphi(\beta, y) < \varphi(\gamma, z))$$

are $\Pi^1_3(x, \alpha)$.

Corollary 2.5 (Reduction). Assume $\Delta^1_2$-determinacy. Suppose $A, B \subseteq u_\omega \times \mathbb{R}$ are both $\Pi^1_3(x, \alpha)$, where $x \in \mathbb{R}$, $\alpha < u_\omega$. Then there exist $\Pi^1_3(x, \alpha)$ sets $A', B' \subseteq u_\omega \times \mathbb{R}$ such that $A' \subseteq A$, $B' \subseteq B$, $A \cup B = A' \cup B'$ and $A' \cap B' = \emptyset$.

Corollary 2.6 (Easy uniformization). Assume $\Delta^1_2$-determinacy. Suppose $A \subseteq (u_\omega \times \mathbb{R}) \times u_\omega$ is $\Pi^1_3(x, \alpha)$, where $x \in \mathbb{R}$, $\alpha < u_\omega$. Then $A$ can be uniformized by a $\Pi^1_3(x, \alpha)$ function, i.e., there is a $\Pi^1_3(x, \alpha)$ function $f$ such that $\text{dom}(f) = \{ (\beta, y) : \exists \gamma ((\beta, y), \gamma) \in A \}$ and that $((\beta, y), f(\beta, y)) \in A$ for all $(\beta, y) \in \text{dom}(f)$.

The $\Pi^1_1$ coding system for $\Delta^1_2$ sets (e.g., [7, Theorem 3.3.1]) applies to the larger pointclass $\Delta^1_3(<u_\omega)$. The proof is similar.

Corollary 2.7 ($\Pi^1_1$-codes for $\Delta^1_3(<u_\omega)$). Assume $\Delta^1_2$-determinacy. Then there is a $\Pi^1_3$ set $C \subseteq u_\omega$ and sets $P, S \subseteq u_\omega \times \mathbb{R}$ in $\Pi^1_3, \Sigma^1_3$ respectively such that for any $\alpha \in C$,

$$P_\alpha = S_\alpha = \text{DEF} \ D_\alpha$$

and

$$\{ D_\alpha : \alpha \in C \} = \{ A \subseteq \mathbb{R} : A \text{ is } \Delta^1_3(<u_\omega) \}.$$
Proof. Let $U \subseteq \omega \times \mathbb{R}^2$ be a good universal $\Pi^1_3$ set. Define
\[
((n, \alpha), (m, \beta), x) \in A \leftrightarrow \forall w \in \text{WO}_\omega \ (|w| = \alpha \rightarrow (n, w, x) \in U) \\
((n, \alpha), (m, \beta), x) \in B \leftrightarrow \forall w \in \text{WO}_\omega \ (|w| = \beta \rightarrow (m, w, x) \in U)
\]
Then $A, B$ are $\Pi^1_3$ subsets of $(\omega \times u_\omega)^2$. Reduce them to $A', B'$ according to Corollary 2.5. Define
\[
((n, \alpha), (m, \beta)) \in C \leftrightarrow (A')_{(n,\alpha),(m,\beta)} \cup (B')_{(n,\alpha),(m,\beta)} = \mathbb{R}
\]
$C$ is a $\Pi^1_3$ subset of $(\omega \times u_\omega)^2$. Let $P = A'$, $S = (\omega \times u_\omega)^2 \times \mathbb{R} \setminus B'$. Identifying $(\omega \times u_\omega)^2$ with $u_\omega$ with the Gödel pairing function, $C, P, S$ are as desired. \qed

Theorem 2.1 provides a model-theoretic view of $Q$-theory [27] at the level of $Q_3$-degrees. We give an exposition of these results, probably with simple strengthenings thereof.

The higher level analog of the hyperarithmetic reducibility on reals is $Q_3$ reducibility. $Q_3$-degrees are coarser than $\Delta^1_3$-degrees. $y \in Q_3(x)$ iff $y$ is $\Delta^1_3(x)$ in a countable ordinal, i.e., there is $\alpha < \omega_1$ such that $\forall w \in \text{WO}(|w| = \alpha \rightarrow y \in \Delta^1_3(x))$. $x$ is $\Delta^1_3(x)$ in an ordinal $< u_\omega$ iff there is $\alpha < u_\omega$ such that $\forall w \in \text{WO}_\omega(|w| = \alpha \rightarrow y \in \Delta^1_3(x))$. $y \leq \Delta^1_3 x$ iff $y \in \Delta^1_3(x)$. $y \equiv_{\Delta^1_3} x$ iff $y \leq \Delta^1_3 x \leq \Delta^1_3 y$. $y \leq \Delta^1_3 x$ iff $y \in Q_3(x)$. $y \equiv_{\Delta^1_3} x$ iff $y \leq \Delta^1_3 x \leq \Delta^1_3 y$.

**Proposition 2.8** ( [20][21][23][27][47]).

1. Let $x, y \in \mathbb{R}$. Then $y \in L_{\kappa^*_3}[T_2, x]$ iff $y \in M^*_1[x]$ iff $y$ is $\Delta^1_3(x)$ in a countable ordinal iff $y$ is $\Delta^1_3(x)$ in an ordinal $< u_\omega$.

2. The relation $y \in L_{\kappa^*_3}[T_2, x]$ is $\Pi^1_3$, where $x, y$ ranges over $\mathbb{R}$.

3. The relation $y \in \Delta^1_3(x)$ is $\Pi^1_3$, where $x, y$ ranges over $\mathbb{R}$.

$\kappa^*_3$ is the higher level analog of $\omega^*_3$, the least $x$-admissible. It is defined in a different way in [27] Section 14]. As in [23][27], we define
\[
\lambda^*_3 = \sup\{|W| : W \text{ is a } \Delta^1_3(x) \text{ prewellordering on } \mathbb{R}\} \\
= \sup\{\xi < \kappa^*_3 : \xi \text{ is } \Delta^1_3 \text{-definable over } L_{\kappa^*_3}[T_2, x] \text{ from } \{T_2, x\}\}.
\]
The equivalence of these two definitions of $\kappa^*_3$ is proved in [23]:
\[
\kappa^*_3 = \sup\{\omega.t.(W) : W \text{ is a } \Delta^1_3(x, < u_\omega) \text{ wellordering on } \mathbb{R}\} \\
= \sup\{\lambda^*_3 x : M^*_1(T_2, x) \not\subseteq \Delta^1_3 (x, y)\}.
\]
Moreover,
\[
\forall \alpha < u_\omega \exists w \in \text{WO}_\omega \ (|w| = \alpha \land \lambda^*_3 w < \kappa^*_3).
\]
Note that \( \kappa_3^2 < \lambda_3^{\omega} < \delta_3^1 \), as proved in [27, Lemma 14.2].

The Kunen-Martin theorem implies that \( \kappa_3^2 \) is a bound on the rank of any \( \Sigma^3_2(x, \prec u_\omega) \) wellfounded relation.

**Theorem 2.9** (Kunen-Martin, [36, 2G.2]). Suppose \( W \) is a wellfounded relation on \( \mathbb{R} \). Suppose \( \gamma \) is an ordinal and \( T \) is a tree on \((\omega \times \omega) \times \gamma\) such that \( W = p[T] \). Let \( L_\kappa[T] \) be the least admissible set containing \( T \) as an element. Then the rank of \( W \) is smaller than

\[
\sup\{ \xi < \kappa : \xi \text{ is } \Delta_1 \text{-definable over } L_\kappa[T] \text{ from } \{ T \} \}.
\]

**Corollary 2.10.** Suppose \( W \) is a \( \Sigma^3_2(x, \prec u_\omega) \) wellfounded relation on \( \mathbb{R} \). Then the rank of \( W \) is smaller than \( \kappa_3^2 \).

### 2.2 A \( \Delta^1_3 \) coding of subsets of \( u_\omega \) in \( \mathbb{L}_{\delta_3^1}[T_2] \)

As a corollary to Theorem 2.1, every subset of \( u_\omega \) in \( \mathbb{L}_{\delta_3^1}[T_2] \) is \( \Delta^1_3 \). The proof of Theorem 2.1 gives a better definability estimate of \( \mathcal{P}(u_\omega) \cap \mathbb{L}_{\delta_3^1}[T_2] \).

For \( x \in \mathbb{R} \), a putative \( x \)-sharp is a remarkable EM blueprint over \( x \). Suppose \( x^* \) is a putative \( x \)-sharp. For any ordinal \( \alpha \), \( \mathcal{M}_{x^*, \alpha} \) is the EM model built from \( x^* \) and indiscernibles of order type \( \alpha \). The wellfounded part of \( \mathcal{M}_{x^*, \alpha} \) is transitive. For any limit ordinal \( \alpha < \beta \), \( \mathcal{M}_{x^*, \alpha} \) is a rank initial segment of \( \mathcal{M}_{x^*, \beta} \). Say that \( x^* \) is \( \alpha \)-wellfounded iff \( \alpha \in \text{wfp}(\mathcal{M}_{x^*, \alpha}) \). A putative sharp code for an increasing function is \( w = \langle \langle \tau \rangle, x^* \rangle \) such that \( x^* \) is a putative \( x \)-sharp, \( \tau \) is a \( \{ \subseteq, x \} \)-ary Skolem term for an ordinal and

\[
\forall v, v'(v, v' \in \text{Ord} \land v < v') \rightarrow (\tau(v) \in \text{Ord} \land \tau(v) < \tau(v'))
\]

is a true formula in \( x^* \). The statement \( \langle \langle \tau \rangle, x^* \rangle \) is a putative sharp code for an increasing function, \( x^* \) is \( \alpha \)-wellfounded, \( r \) codes the order type of \( \tau^{\mathcal{M}_{x^*, \alpha}}(\alpha) \) about \( \langle \langle \tau \rangle, x^* \rangle, r \rangle \) is \( \Sigma^1_1 \) in the code of \( \alpha \). In addition, when \( x^* = x^\# \), \( \langle \langle \tau \rangle, x^* \rangle \) is called a (true) sharp code for an increasing function.

A subset \( A \subseteq u_n \) is coded by \( \text{Code}_n(A) = \{ w \in \text{WO}_n : |w| \in A \} \).

**Lemma 2.11.** Assume \( \Delta^1_3 \)-determinacy. Suppose \( n < \omega \), \( A \subseteq u_n \) and \( A \in \mathbb{L}_{\delta_3^1}[T_2] \). Then \( \text{Code}_n(A) \) is in \( \mathcal{D}(\omega(n + 1) - \Pi^1_1) \).

**Proof.** By Kechris-Woodin [29, \( \mathcal{D}(<\omega^2 - \Pi^1_1) \) sets are determined. We prove by induction on \( n \) the following claim:

Suppose \( A \subseteq u_n \). Suppose \( B, C \subseteq \mathbb{R} \) are \( \Pi^1_2 \) subsets of \( \mathbb{R}^2 \) such that \( \langle w \in \text{WO}_n \land |w| \in A \rangle \) iff \( \exists z((w, z) \in B) \) iff \( \neg \exists z((w, z) \in C) \). Then there is \( x \in \mathbb{R} \) such that

\[
\forall v \in \text{WO}_n \exists w \in \text{WO}_n \cap L[v, x] (|v| = |w| \land L[v, x] \models \exists z((w, z) \in B \cup C)) \).
\]
By Shoenfield absoluteness, this claim gives a uniform definition of the relation $|v| \in A$ over $L[v, x]$ from parameters in $\{u_1, \ldots, u_n\}$. In the definition, the parameters $u_1, \ldots, u_n$ are used to decide whether or not $|v| = |w|$ for $v, w \in \text{WO}_\eta$. Combined with the fact from Theorem 2.1 that every subset of $u_\omega$ in $\mathcal{L}_{\omega_1}[T_2]$ is $\Delta^1_3$, the lemma will follow from our claim.

We start the induction with $n = 1$. Consider the game $G(B, C, 0)$, where I produces $v$, II produces $(w, y)$. II wins iff either $v \notin \text{WO}$ or $v, w \in \text{WO}$ and $|v| < |w|$ and $\forall \alpha < |v| \exists (\bar{w}, z) \in L[v, y] (\bar{w} \in \text{WO} \land |\bar{w}| = \alpha \land (\bar{w}, z) \in B \cup C)$.

This game is $\Pi^1_2$ for Player II, hence determined. I does not have a winning strategy by $\Sigma^1_1$-boundedness. So II has a winning strategy $g$. $g$ plays the role of $x$ in the claim, verifying the $n = 1$ case.

Suppose the claim holds for $n$ and we want to prove for $n + 1$. Consider the game $G(B, C, n + 1)$, where I produces $\langle \tau, a^* \rangle$, II produces $\langle \sigma, b^* \rangle, y)$. II wins iff

1. If $\langle \tau, a^* \rangle$ is a putative sharp code for an increasing function, then so is $\langle \sigma, b^* \rangle$. Moreover, for any $\eta < \omega_1$, if

$$a^* \text{ is } \eta\text{-wellfounded } \land \tau^{M_{a^*}, \eta}(\eta) \in \text{wfp}(M_{a^*}, \eta)$$

then

$$b^* \text{ is } \eta\text{-wellfounded } \land \sigma^{M_{b^*}, \eta}(\eta) \in \text{wfp}(M_{b^*}, \eta)$$

$$\land \tau^{M_{a^*}, \eta}(\eta) < \sigma^{M_{b^*}, \eta}(\eta).$$

2. If $\langle \tau, a^* \rangle$ is a true sharp code for an increasing function, $a^* = a^\#$, then

$$\forall v \in \text{WO}_{n+1}(|v| < \tau^{L[a]}(u_n) \rightarrow \exists (\bar{w}, z) \in L[v, y]$$

$$\land \bar{w} \in \text{WO}_{n+1} \land |\bar{w}| = |v| \land (\bar{w}, z) \in B \cup C).$$

This game is $\psi(\omega(n + 1) \cdot \Pi^1_1)$, hence determined. If Player I has a winning strategy $f$, then for each $\eta$, let $X_\eta$ be the set of $r \in \mathbb{R}$ such that there are putative sharp codes for increasing functions on ordinals $\langle \tau, a^* \rangle, \langle \sigma, b^* \rangle$ and an ordinal $\beta \leq \eta$ such that

1. $\langle \tau, a^* \rangle = f \ast \langle \sigma, b^* \rangle;$
2. for any $\beta < \beta$, $b^*$ is $\beta$-wellfounded, $\sigma^{M_{b^*}}(\beta) \in \text{wfp}(M_{b^*}, \eta)$, $\sigma^{M_{b^*}}(\beta) \leq \eta$;

3. $a^*$ is $\beta$-wellfounded, $\tau^{M_{a^*}}_n(\beta)$ has order typed in $r$.

$X_\eta$ is a $\Sigma^1_1$ set in the code of $\eta$. Since $f$ is a winning strategy for I, $X_\eta \subseteq \text{WO}$. Let $C$ be the set of countable $f$-admissibles and their limits. By $\Sigma^1_1$-boundedness, if $\langle \sigma^\gamma, b^\# \rangle$ is a true sharp code for an increasing function, such that $\forall \beta < \omega_1 \sigma^L[\beta] \in C$, then $\langle \tau^\gamma, a^\# \rangle =_{\text{DEF}} f \ast \langle \sigma^\gamma, b^\# \rangle$ is a true sharp code for an increasing function, and for any $\eta \in C$ such that $\forall \beta < \eta \sigma^L[\beta] < \eta$, $\tau^L[\eta](\eta) < \min(C \setminus \eta + 1)$, and in particular, $\tau^L[\eta](u_n) < \text{the least } f\text{-admissible above } u_n$. Let $\xi$ be the least $f$-admissible above $u_n$. In $L[f]$, there is a bijection $\pi : u_n \rightarrow \xi$, definable from $\{u_n\}$. $A \cap \beta$ is thus identified with $(\pi^{-1})'(A \cap \beta)$, as a subset of $u_n$. $\pi$ induces a $\Delta^1_3(f)$ map $\pi_*$ such that for any $v \in \text{WO}_n$, $\pi_*(v) \in \text{WO}_{n+1}$ and $|\pi_*(v)| = \pi(|v|)$. Let $\pi(v) = w$ iff $\exists z(v, w, z) \in D$, where $D$ is $\Pi^1_2(f)$.

Let $(v, z) \in B' \text{ iff } v \in \text{WO}_n$, $(v, (z)_0, (z)_1) \in D$, and $((z)_0, (z)_2) \in B$. Similarly define $C'$. Then $|v| \in (\pi^{-1})'(A \cap \beta)$ iff $\exists z \in B'$ then $\exists z (v, z) \in C'$. $B', C'$ are $\Pi^1_2$. By induction hypothesis, there is a real $x^*$ such that

$$\forall v \in \text{WO}_n \exists w \in \text{WO}_n \cap L[v, x^*](|v| = |w| \wedge L[v, x^*] \models \exists z ((w, z) \in B' \cup C'))$$

In $G(B, C, n+1)$, II defeats $f$ by playing $((\tau^\gamma, f^\#), f \oplus x^*)$, where $(\sigma^*)^L[\beta]$ is the $\beta$-th $f$-admissible. This is a contradiction.

Thus, II has a winning strategy $g$ in $G(B, C, n+1)$. $g$ plays the role of $x$ in the claim, verifying the inductive case.

As a corollary to Lemma 2.11, we obtain a $\Delta^1_3$ coding of subsets of $u_\omega$ that lie in $L_{\delta^1_3}[T_2]$. The $\Delta^1_3$ coding was first established by Kunen under $\text{AD}$ in a less effective way in [16].

**Corollary 2.12.** Assume $\Delta^1_3$-determinacy. There is a $\Delta^1_3$ set $X \subseteq \mathbb{R} \times u_\omega$ such that $\{X_v : v \in \mathbb{R}\} = \mathcal{P}(u_\omega) \cap L_{\delta^1_3}[T_2]$. Here $X_v = \{\alpha < u_\omega : (x, \alpha) \in X\}$.

**Proof.** If $v = \langle k, \langle \varphi \rangle, z \rangle$, $k < \omega$, $z \in \mathbb{R}$, $(\varphi, z)$ defines a $\Pi^1_1(z)$ subset $A_\varphi$ of $\omega k \times \mathbb{R}^2$, put $X_{v, k} = \exists \text{Diff } A_\varphi$. Put $X_v = \bigcup_{k<\omega} X_{(v)_k:k}$. $X = \{(v, \beta) : \beta \in X_v\}$. $X$ is clearly $\Delta^1_3$. The map $v \mapsto X_v$ is onto $\mathcal{P}(u_\omega) \cap L_{\delta^1_3}[T_2]$ by Lemma 2.11.

As a corollary, assuming $\Delta^1_3$-determinacy, if $A \subseteq \omega_1$, then $A \in L$ iff $A \in L_{\delta^1_3}[T_2]$.

17
2.3 Silver’s dichotomy on $\Pi_3^1$ equivalence relations

Harrington’s proof [22], [18, Chapter 32] of Silver’s dichotomy [44] on $\Pi_3^1$ equivalence relations generalizes to $\Pi_3^1$ in a straightforward fashion. This folklore generalization is stated in [9,10] in a slightly weaker form.

An equivalence relation $E$ on $\mathbb{R}$ is thin iff there is no perfect set $P$ such that $\forall x, y \in P \ (xEy \rightarrow x = y)$. If $\Gamma$ is a pointclass, for equivalence relations $E,F$ (possibly on different spaces of the form $\mathbb{R}^m \times (\omega^m)^n$), $E$ is $\Gamma$-reducible to $F$ iff there is a function $\pi$ in $\Gamma$ such that $xEy \Leftrightarrow \pi(x)F\pi(y)$.

**Theorem 2.13** (Folklore). Assume $\Delta_2^1$-determinacy. Let $x \in \mathbb{R}$. If $E$ is a thin $\Pi_3^1(x)$ equivalence relation on $\mathbb{R}$, then $E$ is $\Delta_3^1(x)$ reducible to a $\Pi_3^1(x)$ equivalence relation on a $\Pi_3^1(x)$ subset of $\omega_\omega$.

**Proof.** For simplicity, let $x = 0$. The generalization of Harrington’s proof of Silver’s dichotomy shows that for every $y \in \mathbb{R}$, there is a $\Delta_3^1(<\omega_\omega)$ set $A$ such that $y \in A \subseteq [y]_E$.

Let $C,P,S,(D_\alpha)_{\alpha \in C}$ be the $\Pi_3^1$ coding system for $\Delta_3^1(<\omega_\omega)$ subsets of $\mathbb{R}$, given by Corollary 2.7. Let $\alpha \in C'$ iff $\alpha \in C$ and $\forall y \in D_\alpha \forall z \in D_\alpha (yEy)$. $C'$ is $\Pi_3^1$. The set

$$A = \{(y,\alpha) : \alpha \in C' \land y \in D_\alpha\}$$

is $\Pi_3^1$. By Corollary 2.6 $A$ can be uniformized by a $\Pi_3^1$ function $\pi$. Let $\alpha F \beta$ iff $\alpha \in C'$, $\beta \in C'$, and $\forall y \in D_\alpha \forall z \in D_\beta (yEy)$. $F$ is a $\Pi_3^1$ equivalence relation on $C'$. $\pi$ is a reduction from $E$ to $F$. To see that $\pi$ is also $\Sigma_3^1$, apply Corollary 2.3 and use the fact that $\pi$ is a total function taking values in $\omega_\omega$.

The reduction $\pi$ and the target equivalence relation $F$ in Theorem 2.13 are uniformly definable from the $\Pi_3^1(x)$ definition of $E$, independent of $x$. A similar uniformity applies to the following corollary.

**Corollary 2.14.** Assume $\Delta_3^1$-determinacy. Let $x \in \mathbb{R}$. If $E$ is a thin $\Delta_3^1(x)$ equivalence relation on $\mathbb{R}$, then $E$ is $\Delta_3^1(x)$ reducible to $=_{\omega_\omega}$. Here $\alpha =_{\omega_\omega} \beta$ iff $\alpha = \beta < \omega_\omega$.

**Proof.** Assume $x = 0$. Proceed as in the proof of Theorem 2.13 until we reach the set $A$. We now show that $A$ can be uniformized by a $\Pi_3^1$ function $\pi$ such that $yEy$ iff $\pi(y) = \pi(z)$. Indeed, let $\varphi$ be a $\Pi_3^1$-norm on $A$, given by Theorem 2.4 and let $\pi(y) = \alpha$ iff $(y,\alpha) \in A$ and $(\varphi(y,\alpha))$ is lexicographically minimal among the set $\{((\varphi(z,\beta)), \beta) : zEy \land (z,\beta) \in A\}$. Similarly to the proof of Corollary 2.6, $\pi$ is $\Pi_3^1$ (we use $E \in \Delta_3^1$ here). Again, $\pi$ is $\Sigma_3^1$, $\pi$ is the desired reduction from $E$ to $=_{\omega_\omega}$. □
It should be possible to give an alternative proof of Corollary 2.14 using
the forceless proof of the dichotomy of chromatic numbers of graphs in [35],
but the author has not checked the details.

**Corollary 2.15.** Assume $\Delta^1_2$-determinacy. Let $x \in \mathbb{R}$. If $\leq^*$ is a $\Delta^1_3(x)$
prewellordering on $\mathbb{R}$ and $A$ is a $\Sigma^1_3(x)$ subset of $\mathbb{R}$, then $|\leq^*|$ and $\{|x|_\leq^* : x \in A\}$
are both in $L_{\kappa^+_1(x)}[T_2, M_1^\#(x)]$ and $\Delta_1$-definable over $L_{\kappa^+_3(x)}[T_2, M_1^\#(x)]$
from parameters in $\{T_2, M_1^\#(x)\}$.

**Proof.** The equivalence relation $a \equiv^* b \iff a \leq^* b \leq^* a$ is thin. By Corollary 2.14 we get a $\Delta^1_4(x)$-function $\pi : \mathbb{R} \to u_\omega$ such that $a \equiv^* b$ iff $\pi(a) = \pi(b)$. $\pi$ induces a wellordering $<^{**}$ on $\text{ran}(\pi)$ where $\pi(a) <^{**} \pi(b)$ iff $a <^* b$.
$|\leq^*|$ is then the order type of $<^{**}$. $\text{ran}(\pi)$ and $<^{**}$ are $\Sigma^3_3$, hence $\Pi_1$-
definable over $L_{\kappa^+_3}[T_2, x]$ from $\{T_2, x\}$ by Theorem 2.1. Put $w = M_1^\#(x)$. By [27] Lemma 14.2, $\kappa^+_3 < \kappa^+_5$. So $\text{ran}(\pi)$ and $<^{**}$ are $\Delta_1$-definable over $L_{\kappa^+_5}[T_2, w]$ from $\{T_2, w\}$. By admissibility, $|\leq^*|$ is $\Delta_1$-definable in $L_{\kappa^+_5}[T_2, w]$ from $\{T_2, w\}$. The part concerning $\{|x|_\leq^* : x \in A\}$ is similar.

**Remark 2.16.** We do not know if $M_1^\#(x)$ can be replaced by $x$ in the conclusion of Corollary 2.15.

### 2.4 $N$-homogeneous trees

As this paper deals with restricted ultrapowers and “restricted homogeneous
trees” over and over again, it is convenient to abstract the relevant properties.

A transitive set or class $N$ is **admissibly closed** iff

$$\forall M \in N \exists M' \in N (M' \text{ is admissible } \land M \subseteq M')$$

Suppose $N$ is admissibly closed and $X \in N$. $\nu$ is an $N$-filter on $X$ iff there is
a filter $\nu^*$ on $X$ such that $\nu = \nu^* \cap N$. An $N$-filter $\nu$ is an $N$-measure
on $X$ iff $\nu$ is countably complete and for any $A \in \mathcal{P}(X) \cap N$, either $A \subseteq \nu$
or $X \setminus A \in \nu$. If $\nu$ is an $N$-measure on $X$, then $\text{Ult}(N, \nu)$ is the ultrapower
consisting of equivalence classes of functions $f : X \to N$ that lie in $N$. Denote
by $j_N^\nu : N \to \text{Ult}(N, \nu)$ the ultrapower map and $[f]_N^\nu$ the $\nu$-equivalence class
of $f$ in $\text{Ult}(N, \nu)$. The ultrapower is well-defined by admissible closedness of $N$,
and is wellfounded by countable completeness of $\nu$. The usual Loś proof shows for any transitive $M \in N$ containing $\{X\}$, for any first order formula $\varphi$, for any $f_i : X \to M$ that belongs to $N$, $1 \leq i \leq n$,

$$j_N^\nu(M) \models \varphi([f_1]_N^\nu, \ldots, [f_n]_N^\nu)$$
if
for \( \nu \)-a.e. \( a \in X \), \( M \models \varphi(f_1(a), \ldots, f_n(a)) \).

Suppose \( \nu \) is an \( N \)-measure on \( X^n \) and \( \mu \) is an \( \bar{N} \)-measure on \( X^m \), \( m \leq n \). \( \nu \) projects to \( \mu \) iff for all \( A \subseteq X^m \), \( A \in \mu \) iff \( \{ \bar{a} : \bar{a} \mid m \in A \} \in \nu \). \( \bar{\nu} = (\nu_n)_{n<\omega} \) is a tower of \( N \)-measures on \( X \) iff for each \( n \), \( \nu_n \) is an \( N \)-measure on \( X^n \) and \( \nu_n \) projects to \( \nu_m \) for all \( m < n \).

Suppose \( N \) is admissibly closed, \( X \in N \), and \( \bar{\nu} = (\nu_n)_{n<\omega} \) is a tower of \( N \)-measures on \( X \). This naturally induces factor maps \( \jmath_N^{\nu_m,\nu_n} \) from \( \text{Ult}(N, \nu_m) \) to \( \text{Ult}(N, \nu_n) \). We say \( \bar{\nu} \) is close to \( N \) iff whenever \( (A_n)_{n<\omega} \) is a sequence such that \( A_n \in \nu_n \cap N \) for all \( n \), there exists \( (B_n)_{n<\omega} \in N \) such that \( B_n \subseteq A_n \) and \( B_n \in \nu_n \) for all \( n \). If \( \bar{\nu} \) is close to \( N \), we say \( \bar{\nu} \) is \( N \)-countably complete iff whenever \( (A_n)_{n<\omega} \) is a sequence such that \( A_n \in \nu_n \cap N \) for all \( n \), there exists \( (a_n)_{n<\omega} \) such that \( (a_1, \ldots, a_n) \in A_n \) for all \( n \). The usual homogeneous tree argument shows:

**Proposition 2.17.** Suppose \( \bar{\nu} = (\nu_n)_{n<\omega} \) is close to \( N \). Then \( \bar{\nu} \) is \( N \)-countably complete iff the direct limit of \( (\jmath_N^{\nu_m,\nu_n})_{m<n<\omega} \) is wellfounded.

**Proof.** The new part is to show \( N \)-countable completeness of \( \bar{\nu} \) from wellfoundedness of the direct limit of \( (\jmath_N^{\nu_m,\nu_n})_{m<n<\omega} \). Given \( (A_n)_{n<\omega} \) such that \( A_n \in \nu_n \cap N \) for all \( n \), suppose towards contradiction that there does not exist \( (a_n)_{n<\omega} \) such that \( (a_1, \ldots, a_n) \in A_n \) for all \( n \). By closedness of \( \bar{\nu} \) to \( N \), let \( (B_n)_{n<\omega} \in N \) such that \( B_n \subseteq A_n \) and \( B_n \in \nu_n \) for all \( n \). The tree \( T \) consisting of \( (a_1, \ldots, a_n) \) such that \( a_i \in B_i \) for all \( i \) is wellfounded. The ranking function \( f \) of \( T \) belongs to \( N \) by admissible closedness. From \( f \) we can construct \( f_n : X^n \to N \) so that \( f_n \in N \) and \( [f_n]_{\nu_n} > [f_{n+1}]_{\nu_{n+1}} \) as usual, contradicting to wellfoundedness of \( (\jmath_N^{\nu_m,\nu_n})_{m<n<\omega} \). \( \square \)

An \( N \)-homogeneous system is a sequence \( (\nu_s)_{s<\omega} \) such that for any \( x \in \mathbb{R} \), \( \nu_x =_{\text{DEF}} (\nu_{\bar{x}s})_{n<\omega} \) is a tower of \( N \)-measures which is close to \( N \). For \( X \in N \), a tree \( T \) on \( \omega \times X \) is \( N \)-homogeneous iff there is an \( N \)-homogeneous system \( (\nu_s)_{s<\omega} \) such that \( T_s \in \nu_s \) for all \( s \in \omega^{<\omega} \) and for all \( x \in p[T] \), \( \nu_x \) is \( N \)-countably complete. If \( T \) is \( N \)-homogeneous, by Proposition 2.17 and standard arguments, \( x \in p[T] \) iff the direct limit of \( (\jmath_N^{\nu_m,\nu_n})_{m<n<\omega} \) is wellfounded.

### 2.5 \( L[T_3] \) as a mouse

The notations concerning inner model theory follow [51]. If \( \mathcal{M} \) is a premouse, \( o(\mathcal{M}) \) denotes \( \text{Ord} \cap \mathcal{M} \). In Steel [47], the level-wise projective complexity associated to mice is discussed in detail. In this paper, we find it more convenient to work with \( \Pi^1_{n+1} \)-iterability rather than \( \Pi^1_n^{HC} \)-iterability in [47].

20
A countable normal iteration tree $T$ on a countable premouse is $\Pi_1^1$-guided iff for any limit $\lambda \leq \text{lh}(T)$, there is $\xi \leq o(M^T_\lambda)$ such that $M^T_\xi\models\delta(T \upharpoonright \alpha)$] and $J_{\xi+1}[M(T \upharpoonright \alpha)] \models \text{"}\delta(T \upharpoonright \alpha)\text{ is not Woodin"}\text{.}$ A countable stack of countable normal iteration trees $\vec{T}$ is $\Pi_1^1$-guided iff every normal component of $\vec{T}$ is $\Pi_1^1$-guided.

$x \in \mathbb{R}$ codes a $\Pi_3^1$-iterable mouse iff $x$ codes a 1-small premouse $P_x$ such that for any $\Pi_1^1$-guided normal iteration trees $\vec{T} \in HC$ on $P_x$, either $T$ has a last wellfounded model or $\text{lh}(T)$ is a limit ordinal and for any $\xi \geq o(M(T))$, if $J_{\xi}[M(T)] \models \text{"}\delta(T \upharpoonright \alpha)\text{ is Woodin"}$, then there is a cofinal branch $b$ through $T$ such that either $J_{\xi}[M(T)] \leq M^T_b$ or $M^T_b \leq J_{\xi}[M(T)]$.

$\Pi_2^1$-iterability is enough to compare countable 1-small premice that project to $\omega$. A countable normal iteration tree $T$ on a countable premouse is $\Pi_2^1$-guided iff for any limit $\lambda \leq \text{lh}(T)$, there is $\xi \leq o(M^T_\lambda)$ such that $M^T_\xi\models\delta(T \upharpoonright \lambda)$ and $\text{rud}(M(T \upharpoonright \alpha)) \models \text{"}\delta(T \upharpoonright \alpha)\text{ is not Woodin"}$]. A countable stack of countable normal iteration trees $\vec{T}$ is $\Pi_2^1$-guided iff every normal component of $\vec{T}$ is $\Pi_2^1$-guided.

Assume $\Delta^1_2$-determinacy. $x \in \mathbb{R}$ codes a $\Pi_3^1$-iterable mouse iff $x$ codes a countable 2-small premouse $P_x$ such that for any $v \in \mathbb{R}$ coding $\Pi_2^1$-guided stack of normal iteration trees $\vec{T} = (T_i)_{i<\alpha}$ on $P_x$, either

1. $M^T_\alpha$ exists, either as the last model of $T_{\alpha-1}$ when $\alpha$ is a successor or as the direct limit of $(M^T_i : i < \alpha)$ when $\alpha$ is a limit, and there is $Q \succ M^T_\alpha$ such that $\alpha \models o(M^T_\alpha)$, $\text{rud}(Q) \models \text{"}there is no Woodin cardinal \leq o(M^T_\alpha)\text{"}$, or

2. $\alpha$ is a successor cardinal and there is $b \in M^T_\alpha(x,v)$ such that $b$ is a maximal branch through $T_{\alpha-1}$, and there is $Q \succ M^T_{\alpha-1}$ such that $\alpha \models o(M^T_{\alpha-1})$, $\text{rud}(Q) \models \text{"}there is no Woodin cardinal \leq o(M^T_{\alpha-1})\text{"}$.

$\Pi_3^1$-iterability is a $\Pi_3^1$ property by Spector-Gandy. “countable” and “2-small” are usually omitted for prefixing “$\Pi_3^1$-iterable mouse”. $P$ is a $\Pi_3^1$-iterable mouse iff there is $x \in \mathbb{R}$ that codes a $\Pi_3^1$-iterable mouse $P = P_x$. Note that $\Pi_3^1$-iterable mice are genuinely $(\omega_1, \omega_1)$-iterable. $\leq_{DJ}$ is the Dodd-Jensen prewellordering on $\Pi_3^1$-iterable mice. $M \leq_{DJ} N$ iff $M, N$ are $\Pi_3^1$-iterable mice and in the comparison between $M$ and $N$, the main branch on the $M$-side does not drop. $M \sim_{DJ} N$ iff $M \leq_{DJ} N \leq_{DJ} M$. $M \leq_{DJ} N$ iff $M \leq_{DJ} N \not\leq_{DJ} M$. The norm $x \mapsto \|P_x\|_{<DJ}$ for $x$ coding a $\Pi_3^1$-iterable mouse $P_x$ is $\Pi_3^1$. For instance, $(P_x$ is a $\Pi_3^1$-iterable mouse $\land \langle y \rangle$ is a $\Pi_3^1$-iterable mouse $\rightarrow P_x \leq_{DJ} P_y$) iff $P_x$ is a $\Pi_3^1$-iterable mouse and for any $\Pi_3^1$-guided normal iteration trees $\vec{T}, \vec{U}$ on $P_x, P_y$ respectively, if $\vec{T}, \vec{U}$ have the common
last model \( Q \) and the main branch of \( T \) drops, then the main branch of \( U \) also drops.

If \( N \) is a \( \Pi^1_3 \)-iterable mouse, then \( I_N \) is the direct system consisting of countable nondropping iterates of \( N \), and \( N_{\infty} \) is the direct limit of \( I_N \), \( \pi_{N_{\infty}} : N \rightarrow N_{\infty} \) is the direct limit map. \( o(N_{\infty}) < \delta^1_3 \) as it is the length of a \( \Delta^1_3 \)-prewellordering.

For a real \( z \), all the iterability notions relativize to \( z \)-mice. \( <_{DJ(z)} \) is the Dodd-Jensen prewellordering on \( \Pi^1_3 \)-iterable \( z \)-mice.

Theorem 2.18 (Steel [49]). Assume \( \Pi^1_3 \)-determinacy. Then for any real \( z \),

1. \( \delta^1_3 \) is the least \( < \delta^2_{2,\infty} \)-strong cardinal of \( M^\#_{2,\infty}(z) \), where \( \delta^2_{2,\infty} \) is the least Woodin cardinal of \( M_{2,\infty}(z) \).
2. \( M^\sim_{2,\infty}(z) = L[T_3, z] \).

3 The level-1 sharp

The level-1 sharp is the usual sharp, originally published in [45]. We present the usual arguments of Martin’s proof of \( \Pi^1_4 \)-determinacy and the Martin-Solovay tree on a \( \Pi^1_3 \)-complete set in a form that conveniently generalizes to higher levels.

3.1 The tree \( S_1 \), level-1 description analysis

We are working under ZF + DC.

The technical definition of tree of uniform cofinalities is extracted from [20], defined in [14], and redefined in our paper in a more convenient way. A tree of uniform cofinality pinpoints a particular measure that appears in a homogeneity system for a projective set. A level-1 tree of uniform cofinalities, or a level-1 tree, is a set \( P \subseteq \omega^{<\omega} \) such that:

1. \( \emptyset \notin A \).
2. If \( (i_1, \ldots, i_{k+1}) \in T, k \geq 1 \), then \( (i_1, \ldots, i_k) \in T \) and for every \( j < i_{k+1}, (i_1, \ldots, i_k, j) \in T \).
Any countable linear ordering is isomorphic to $<_{BK}\mid P$ for some level-1 tree $P$. If $P, P'$ are finite level-1 trees, $s \notin P, P' = P \cup \{s\}$, then the $<_{BK}\mid P'$-predecessor of $s$ is $s$. Level-1 trees are just convenient representations of countable linear orderings and their extensions.

A level-1 tree $P$ is said to be regular iff $(1) \notin P$. In other words, when $P$ is regular and $P \neq \emptyset$, $(0)$ must be the $<_{BK}$-maximal node of $P$.

The ordinal representation of $P$ is

$$\text{rep}(P) = \{(p) : p \in P\} \cup \{(p, n) : p \in P, n < \omega\}.$$ 

rep($P$) is endowed with the ordering

$$<^P = <_{BK}\mid \text{rep}(P).$$

Thus, for $p \in P$, $(p)$ is the $<^P$-supremum of $(p, n)$ for $n < \omega$. If $B \subseteq \omega_1$ is in $\mathbb{L}$, let $B^{P_1}$ the set of functions $f : \text{rep}(P) \to B$ which are continuous, order preserving (with respect to $<^P$ and $<$) and belong to $\mathbb{L}$. If $f \in \omega_1^{P_1}$, let

$$[f]^P = ([f]_p^P)_{p \in P},$$

where $[f]_p^P = f((p))$ for $p \in P$. Let $[B]^{P_1} = \{[f]^P : f \in B^{P_1}\}$. $P$ is said to be $\Pi_1^1$-wellfounded iff $P \cup \{\emptyset\}$ is a wellfounded tree, or equivalently, $<^P$ is a wellordering. $\Pi_1^1$-wellfoundedness of a level-1 tree is a property in the real coding the tree. A tuple $\vec{\alpha} = (\alpha_p)_{p \in P}$ is said to respect $P$ iff $\vec{\alpha} \in [\omega_1]^{P_1}$. In other words, each $\alpha_p$ is a countable limit ordinal, and the map $p \mapsto \alpha_p$ is an isomorphism between $(P; <_{BK}\mid P)$ and $(\{\alpha_p : p \in P\}; <)$. In particular, when $P$ is regular, $P \neq \emptyset$ and $\vec{\alpha}$ respects $P$, then $\alpha_{\langle 0 \rangle} > \alpha_p$ whenever $p \in P \setminus \{(0)\}$.

A finite level-1 tower is a tuple $(P_i)_{i \leq n}$ such that $n < \omega$, $P_i$ is a level-1 tree of cardinality $i$ for any $i$, and $i \leq j \Rightarrow P_i \subseteq P_j$. An infinite level-1 tower is $(P_i)_{i \leq \omega}$ such that $(P_i)_{i \leq n}$ is a finite level-1 tower for any $n < \omega$. A level-1 system is a sequence $\vec{P} = (P_s)_{s \in \omega^\omega}$ such that for each $s \in \omega^\omega$, $(P_s)_{i < \text{lh}(s)}$ is a finite level-1 tower. $\vec{P}$ is regular iff each $P_s$ is regular. Associated to a $\Pi_1^1$ set $A$ we can assign a regular level-1 system $(P_s)_{s \in \omega^\omega}$ so that $x \in A$ iff the infinite regular level-1 tree $P_x = \text{DEF } \cup_{n < \omega} P_{x|n}$ is $\Pi_1^1$-wellfounded. If $A$ is lightface $\Pi_1^1$, then $(P_s)_{s \in \omega^\omega}$ can be picked effective.

**Definition 3.1.** $S_1$ is the tree on $V_\omega \times \omega_1$ such that $(\emptyset, \emptyset) \in S_1$ and a nonempty node

$$(\vec{P}, \vec{\alpha}) = ((P_i)_{i \leq n}, (\alpha_i)_{i \leq n}) \in S_1$$

iff $(P_i)_{i \leq n}$ is a finite regular level-1 tower and putting $p_i \in P_{i+1} \setminus P_i$, $\beta_{p_i} = \alpha_i$, then $(\beta_p)_{p \in P_n}$ respects $P_n$. 

23
Since every tree occurring in $S_1$ is regular, for a nonempty node $(\vec{P}, \vec{a}) \in S_1$, we must have $a_0 > \max(\alpha_1, \ldots, \alpha_n)$.

$S_1$ projects to the universal $\Pi_1^1$ set:

$$p[S_1] = \{ \vec{P} : \vec{P} \text{ is a } \Pi_1^1\text{-wellfounded regular level-1 tower}\}.$$ 

The (non-regular) $\omega_1$-scale associated to $S_1$ is $\Pi_1^1$.

**Definition 3.2.**

1. Suppose $P$ is a level-1 tree. The set of $P$-descriptions is $\text{desc}(P) = \text{DEF} P \cup \{\emptyset\}$. The constant $P$-description is $\emptyset$.

2. $p < p'$ iff $p, p' \in \text{desc}(P)$ and $p <_{\text{BK}} p'$.

3. Suppose $P, W$ are level-1 trees. A function $\sigma : P \cup \{\emptyset\} \rightarrow W \cup \{\emptyset\}$ is said to factor $(P, W)$ iff $\sigma(\emptyset) = \emptyset$ and $\sigma$ preserves the $<_{\text{BK}}$-order. ($\sigma$ does not necessarily preserve the tree order.)

4. Suppose $P$ is a level-1 tree. $\sigma$ factors $(P, *)$ iff $\sigma$ factors $(P, W)$ for some level-1 tree $W$.

Suppose $P, W$ are $\Pi_1^1$-wellfounded. Then o.t.($<P$) $\leq$ o.t.($<W$) is equivalent to “$\exists \sigma \ (\sigma$ factors $(P, W)$)”. o.t.($<P$) $< \text{o.t.}(<W)$ is equivalent to “$\exists \sigma \exists w \in W \ (\sigma$ factors $(P, W) \land \forall p \in P \sigma(p) \prec_{w} w)$”. The higher level analog of this simple fact will be established in Section 4.9 which will be an ingredient in the axiomatization of $0^\#_3$ in Section 5.

### 3.2 Homogeneity properties of $S_1$

From now on, we assume $\Pi_1^1$-determinacy. This is equivalent to $\forall x \in \mathbb{R}(x^\# \text{ exists})$ by Martin [32] and Harrington [8].

The first $\omega$ uniform indiscernibles $(u_n)_{n<\omega}$ can be generated by restricted ultrapowers of $L$. Recall that $L = \bigcup_{x \in \mathbb{R}} L[x]$, which is admissibly closed. Then for every subset $A \subseteq \omega_1$ in $L$, there is a real $x$ such that $A$ is $\Sigma_1$-definable over $(L_{\omega_1}[x] ; \in, x)$. Let

$$\mu_L$$

be the $L$-club measure on $\omega_1$, i.e., $A \in \mu_L$ iff $A \in L$ and $\exists C \in L \ (C \subseteq A \land C$ is a club in $\omega_1)$. When $P$ is a finite level-1 tree, $\mu^P$ is the $L$-measure on $\text{card}(P)$-tuples in $\omega_1$ given by: $A \in \mu^P$ iff there is $C \in \mu_L$ such that $[C]^P \subseteq A$. So $\mu^P$ is essentially a variant of the $\text{card}(P)$-fold product of $\mu_L$, concentrating on tuples whose ordinals are ordered according to the $<_{\text{BK}}$-order of $P$. In particular, $\mu^\emptyset$ is the principal ultrafilter concentrating on
\[ j^P = j_{\mu^P}^P, \quad [f]_{\mu^P}^P = [f]_{\mu^P}^L \quad \text{for} \quad f \in \mathbb{L}. \]

Standard arguments show that \( \text{Ult}(\mathbb{L}, \mu^P) = \mathbb{L} \), and \( j^P(\omega_1) = u_{\text{card}(P)+1} \). For any real \( x \), \( j^P \restriction L[x] \) is elementary from \( L[x] \) to \( \mathbb{L} \).

The set of uncountable \( \mathbb{L} \)-regular cardinals below \( u_\omega \) is \( \{ u_n : 1 \leq n < \omega \} \).

For any real \( x \), \( j^P \restriction L[x] \) is elementary from \( L[x] \) to \( L[x] \).

The relation \( \beta = \text{cf}^\mathbb{L}(\alpha) \) is \( \Delta^3_1 \) (in the sharp codes). Suppose \( P \) is a finite level-1 tree, \( p \in \text{desc}(P) \).

Then \( \text{seed}_P^P \in \mathbb{L} \)

is the element represented modulo \( \mu^P \) by the projection map sending \( \vec{\alpha} = (\alpha_{p'})_{p' \in P} \) to \( \alpha_p \) if \( p \in P \), by the constant function with value \( \omega_1 \) if \( p = \emptyset \). We have \( \text{seed}_P^P = u_{\|p\|_{\prec P} + 1} \), where \( \|p\|_{\prec P} \) is the \( \prec P \)-rank of \( p \). In particular, \( \text{seed}_P^0 = u_{\text{card}(P)+1} = j^P(\omega_1) \). For each \( p \in P \), \( \mu^P \) projects to \( \mu^L \) via the map \( \vec{\alpha} \mapsto \alpha_p \).

\( p^P : \mathbb{L} \to \mathbb{L} \)

is the induced factoring map that sends \( j_{\mu^L}(h)(\omega_1) \) to \( j^P(h)(\text{seed}_P^P) \). Thus, \( p^P \) is the unique map such that for any \( z \in \mathbb{R} \), \( p^P \) is elementary from \( L[z] \) to \( L[z] \) and \( p^P \circ j_{\mu^L} = j^P, \quad p^P(\omega_1) = \text{seed}_P^P \). If \( p \) is the \( \prec P \)-predecessor of \( p' \), then \( (p^P)^\tau u_2 \) is a cofinal subset of \( \text{seed}_{P'}^P \). Put

\[ \text{seed}_P^P = (\text{seed}_P^P)_{p \in \text{desc}(P)}, \]

So \( p \prec^P p' \) iff \( \text{seed}_P^P < \text{seed}_{P'}^P \). Every element in \( \mathbb{L} \) is expressible in the form \( j^P(h)(\text{seed}_P^P) \) for some \( h \in \mathbb{L} \).

If \( P, P' \) are finite level-1 trees, \( P \) is a subtree of \( P' \), then \( \mu^P \) projects to \( \mu^P \) in the language of Section 2.4 i.e., the identity map factors \((P, P')\). Let

\[ j^{P,P'} : \mathbb{L} \to \mathbb{L} \]

be the factor map given by Section 2.4. Thus, for any real \( x \),

\[ j^{P,P'} \restriction L[x] : L[x] \to L[x] \]

is elementary and

\[ j^{P,P'}(\tau^L[x](\text{seed}_{P_1}^P, \ldots, \text{seed}_{P_n}^P)) = \tau^L[x](\text{seed}_{P_1}^{P'}, \ldots, \text{seed}_{P_n}^{P'}) \]

for \( p_1, \ldots, p_n \in P \). If \( (P_n)_{n<\omega} \) is an infinite level-1 tower, the associated measure tower \((\mu^P_n)_{n<\omega} \) is easily seen close to \( \mathbb{L} \).

The proof of \( \Pi^1_1 \)-determinacy \([32]\) shows:

**Theorem 3.3** (Martin). Assume \( \Pi^1_1 \)-determinacy. Let \( (P_n)_{n<\omega} \) be an infinite level-1 tower. The following are equivalent.
1. \((P_n)_{n<\omega}\) is \(\Pi^1_1\)-wellfounded.

2. \([\omega_1] \cup (P_{n:n<\omega})^\uparrow \neq \emptyset\).

3. \((\mu^{P_n})_{n<\omega}\) is \(L\)-countably complete.

4. The direct limit of \((j^{P_m}P_n)_{m<n<\omega}\) is wellfounded.

If \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n'\}\) is order preserving, let

\[ j^\sigma : L \to L \]

where \(j^\sigma(\tau^{L|x}(u_1, \ldots, u_n)) = \tau^{L|x}(u_{\sigma(1)}, \ldots, u_{\sigma(n)})\). Let

\[ j^\sigma_{sup} : u_{n+1} \to u_{n'+1} \]

where \(j^\sigma_{sup}(\beta) = \sup(j^\sigma)^\sigma(\beta)\). So \(j^\sigma\) is continuous at \(\beta\) iff \(j^\sigma(\beta) = j^\sigma_{sup}(\beta)\). The continuity points of \(j^\sigma\) are characterized by their \(L\)-cofinalities:

**Lemma 3.4.** Suppose \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n'\}\) is order preserving, \(\beta < u_{n+1}\). Put \(\sigma(0) = 0\). Then \(j^\sigma(\beta) \neq j^\sigma_{sup}(\beta)\) iff for some \(k\), \(cf^L(\beta) = u_k\) and \(\sigma(k) > \sigma(k-1) + 1\). If \(cf^L(\beta) = u_k\) and \(\sigma(k) > \sigma(k-1) + 1\), then \(j^\sigma_{sup}(\beta) = j^\sigma_k \circ j^\tau_{sup,k}(\beta)\), where \(\sigma = \sigma_k \circ \tau_k\), \(\sigma_k(i) = \sigma(i)\) for \(1 \leq i < k\), \(\sigma_k(k) = \sigma(k-1) + 1\), \(\sigma_k(i) = \sigma(i-1)\) for \(k < i \leq n+1\).

The second half of this lemma states that \(j^\sigma_{sup}\) acting on points of \(L\)-cofinality \(u_k\) is factored into the “continuous part” \(j^\sigma_k\) and the “discontinuous part” \(j^\tau_{sup,k}\). This simple fact about factoring \(j^\sigma_{sup}\) is essentially part of effectivized Kunen’s analysis on \(u_\omega\) in [46].

A **partial level \(\leq 1\) tree** is a pair \((P, t)\) such that \(P\) is a finite regular level-1 tree, and either

1. \(t \notin P \land P \cup \{t\}\) is a regular level-1 tree, or

2. \(P \neq \emptyset, t = -1\).

\(-1\) is regarded as the “level-0” component, hence the name “level \(\leq 1\)”. \((P, t)\) is of degree 0 if \(t = -1\), of degree 1 otherwise. We put \(dom(P, t) = P \cup \{t\}\).

The **uniform cofinality** of \((P, t)\) is

\[ ucf(P, t) = \begin{cases} -1 & \text{if } t = -1, \\ t^- & \text{if } t \neq -1. \end{cases} \]

\(\vec{\alpha} = (\alpha_s)_{s \in P \cup \{t\}}\) respects \((P, t)\) iff \(\vec{\alpha} \upharpoonright P\) respects \(P\) and \(t = -1 \to \alpha_t < \omega\), \(t \neq -1 \to \alpha_t < \alpha_{t^-}\). The **cardinality** of \((P, t)\) is \(\text{card}(P, t) = \text{card}(P) + 1\).
The unique partial level \( \leq 1 \) tree of cardinality 1 is \((\emptyset, (0))\). If \((P, t)\) is of degree 1, its completion is \( P \cup \{t\} \). \((P, -1)\) has no completion. \((P, t)\) is a partial subtree of \( P' \) iff the completion of \((P, t)\) exists and is a subtree of \( P' \).

A partial level \( \leq 1 \) tower of discontinuous type is a nonempty finite sequence \((\vec{P}, \vec{p}) = (P, p_i)_{i \leq k}\) such that \( \text{card}(P_0, p_0) = 1 \), each \((P_i, p_i)\) is a partial level \( \leq 1 \) tree, and \( P_{i+1} \) is the completion of \((P_i, p_i)\). Its signature is \((p_i)_{i < k}\). Its uniform cofinality is \( ucf(P_k, p_k) \). A partial level \( \leq 1 \) tower of continuous type is \((P_i, p_i)_{i < k} \subseteq (P, P)\) such that either \( k = 0 \land P = \emptyset \) or \((P_i, p_i)_{i < k}\) is a partial level \( \leq 1 \) tower of discontinuous type \( \wedge P = \text{the completion of} \ (P_{k-1}, p_{k-1}) \). Its signature is \((p_i)_{i < k}\). When \( k > 0 \), its uniform cofinality is \( p_{k-1} \). For notational convenience, the information of a partial level \( \leq 1 \) tower is compressed into a potential partial level \( \leq 1 \) tower. We say a potential partial level \( \leq 1 \) tower is \((P_*, \vec{p}) = (P_*, (p_i)_{i < \text{lh}(\vec{p})})\) such that for some level-1 tower \( \vec{P} = (P_i)_{i \leq k}\), either \( P_* = P_k \wedge (\vec{P}, \vec{p}) \) is a partial level \( \leq 1 \) tower of discontinuous type or \((\vec{P}, \vec{p}) \subset (P_*)\) is a partial level \( \leq 1 \) tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of \((P_*, \vec{p})\) are defined according to the partial level \( \leq 1 \) tree generating \((P_*, \vec{p})\).

\[ ucf(P_*, \vec{p}) \]

denotes the uniform cofinality of \((P_*, \vec{p})\). If \((P_*, (p_i)_{i < k})\) is a potential partial level \( \leq 1 \) tower of discontinuous type, its completion is the completion of \((P, p_k)\).

Clearly, a potential partial level \( \leq 1 \) tower \((P_*, \vec{p})\) of discontinuous type iff \( \text{card}(P_*) = \text{lh}(\vec{p}) \), of discontinuous type iff \( \text{card}(P_*) = \text{lh}(\vec{p}) - 1 \).

Suppose \( P, W \) are level-1 trees, \( \sigma \) factors \((P, W)\). Given a tuple \( \vec{\alpha} = (\alpha_w)_{w \in W} \in [\omega_1]^W \), define

\[ \vec{\alpha}_\sigma = (\alpha_{\sigma P})_{P \in P} \in [\omega_1]^P \]

where \( \alpha_{\sigma P} = \alpha_{\sigma (P)} \). If \( W \) is finite, put \( \text{seed}_W = (\text{seed}_{\sigma (P)})_{P \in P} \), i.e., \( \text{seed}_{\sigma} \) is represented modulo \( \mu_W \) by the function \( \vec{\alpha} \mapsto \vec{\alpha}_\sigma \).

If \( P, W \) are finite, \( \sigma \) factors \((P, W)\), then for any \( A \in \mu^P \), for \( \mu^W \)-a.e. \( \vec{\alpha} \), \( \vec{\alpha} \in A \). Thus, for any \( A \in \mu^P \), \( \text{seed}_W \in j^W(A) \). Thus, we can unambiguously define

\[ \sigma^W : \mathbb{L} \rightarrow \mathbb{L} \]

by \( \sigma^W(j^P(h)(\text{seed}_P)) = j^W(h)(\text{seed}_W) \). \( \sigma^W \) is the unique map such that for any \( z \in \mathbb{R} \), \( \sigma^W \) is elementary from \( L_z \) to \( L_z \), \( \sigma^W \circ j^P = j^W \), and for any \( p \in P \), \( \sigma^W \circ p^P = \sigma(p)^W \). Define \( \sigma^W_{\sup}(\beta) = \sup(\sigma^W)^\nu(\beta) \).

The next two lemmas compute the “effective uniform cofinality” of the image of certain ordinals under level-1 tree factoring maps. They will be useful in the level-2 description analysis in Section 4.4.
Lemma 3.5. Suppose \((P^-, p)\) is a partial level \(\leq 1\) tree whose completion is \(P\). \(\sigma, \sigma'\) both factor \((P, W)\). \(\sigma\) and \(\sigma'\) agree on \(P^-\), \(\sigma'(p)\) is the \(\prec^W\)-predecessor of \(\sigma(p)\). Then for any \(\beta < j^{P^-}(\omega_1)\) such that \(\text{cf}^Z(\beta) = \text{seed}^{P^-}_p\),

\[
\sigma^W \circ j^{P^-}_{\text{sup}}(\beta) = (\sigma')^W_{\text{sup}} \circ j^{P^-}_{\text{sup}}(\beta).
\]

Proof. Note that \(\text{cf}^Z(j^{P^-}(\beta)) = \text{seed}^{P^-}_p\). As in Lemma 3.4, \((\sigma')^W_{\text{sup}}\) acting on points of \(\mathbb{L}\)-cofinality \(\text{seed}^{P^-}_p\) is decomposed into the discontinuous part \(j^{P,P+}_{\text{sup}}\) and the continuous part \((\sigma^+)^W\), where \(P^+\) is the completion of the partial level \(\leq 1\) tree \((P, p^+)\), \((p^+) = p^-, \sigma^+\) factors \((P^+, W)\), \(\sigma'\) and \(\sigma^+\) agree on \(P\), \(\sigma^+(p^+) = \sigma(p)\). Let \(\nu\) factor \((P, P^+)\) where \(\nu \upharpoonright P^- = \text{id}, \nu(p) = p^+\). So \(\sigma^+ \circ \nu = \sigma\). By considering the seed\(_{p^-}\)-cofinal sequence in \(\beta\), it is not hard to show that \(j^{P,P+}_{\text{sup}} \circ j^{P^-}_{\text{sup}}(\beta) = \nu^P \circ j^{P^-}_{\text{sup}}(\beta)\). Hence,

\[
(\sigma')^W_{\text{sup}} \circ j^{P^-}_{\text{sup}}(\beta) = (\sigma^+)^W_{\text{sup}} \circ j^{P,P+}_{\text{sup}} \circ j^{P^-}_{\text{sup}}(\beta)
\]

\[
= (\sigma^+)^W_{\text{sup}} \circ \nu^P \circ j^{P^-}_{\text{sup}}(\beta)
\]

\[
= \sigma^W \circ j^{P^-}_{\text{sup}}(\beta).
\]

Lemma 3.6. Suppose \((P, p)\) is a partial level \(\leq 1\) tree, \(\sigma\) factors \((P, W)\). Suppose \(\beta < j^{P^-}(\omega_1)\) and either

1. \(p = -1, P^+ = \mathbb{P}, \sigma' = \sigma, \text{cf}^Z(\beta) = \omega, \) or

2. \(p \neq -1, P^+\) is the completion of \((P, p)\), \(\sigma'\) factors \((P^+, W)\), \(\sigma = \sigma' \upharpoonright P\), \(\sigma'(p)\) is the \(\prec^W\)-predecessor of \(\sigma(p)\), \(\text{cf}^Z(\beta) = \text{seed}^{P^-}_p\).

Then

\[
\sigma^W(\beta) = (\sigma')^W_{\text{sup}} \circ j^{P,P+}(\beta).
\]

Proof. By commutativity of factoring maps, \(\sigma^W(\beta) = (\sigma')^W_{\text{sup}} \circ j^{P,P+}(\beta)\). Note that \(\text{cf}^Z(j^{P,P+}(\beta)) = \text{seed}^{P^-}_p\) when \(p \neq -1\), \(\text{cf}^Z(j^{P,P+}(\beta)) = \omega\) when \(p = -1\). In either case, by Lemma 3.4, \((\sigma')^W_{\text{sup}}\) is continuous at \(j^{P,P+}(\beta)\).

To conclude this section, we define the Martin-Solovay tree \(T_2\) projecting to \(\{x^\#: x \in \mathbb{R}\}\) and its variant \(\hat{T}_2\) projecting to a good universal \(\Pi^1_2\) set. This formulation of \(T_2\) and \(\hat{T}_2\) will generalize to the higher level in Section 6.3. Let \(T \subseteq 2^{\omega}\) be a recursive tree such that \([T]\) is the set of remarkable EM blueprints over some real. Here we have fixed in advance an effective Gödel coding of first order formulas in the language \(\{\in, x, c_n : n < \omega\}\), so that an infinite string \(x \in 2^\omega\) represents the theory \(\{\varphi : x_\varphi = 0\}\). Fix an effective list
of Skolem terms \((\tau_k)_{k \in \omega}\) in the language of set theory, where \(\tau_k\) is \(f(k) + 1\)-ary, \(f\) is effective. \(T_2\) is defined as a tree on \(2 \times u_\omega\) where

\[
(s, (\alpha_0, \ldots, \alpha_{n-1})) \in T_2
\]

iff \(s \in T\), \(\text{lh}(s) = n\), and for any \(k, l < n\), for any order preserving \(\sigma : \{1, \ldots, f(k)\} \to \{1, \ldots, f(l)\}\),

1. if \(\tau_k(x, c_{\sigma(1)}, \ldots, c_{\sigma(f(k))}) = \tau_l(x, c_1, \ldots, c_{f(l)})\) is true in \(s\), then \(j^\sigma(\alpha_k) = \alpha_l\);
2. if \(\tau_k(x, c_{\sigma(1)}, \ldots, c_{\sigma(f(k))}) < \tau_l(x, c_1, \ldots, c_{f(l)})\) is true in \(s\), then \(j^\sigma(\alpha_k) < \alpha_l\);
3. if \(\tau_k(x, c_{\sigma(1)}, \ldots, c_{\sigma(f(k))}) > \tau_l(x, c_1, \ldots, c_{f(l)})\) is true in \(s\), then \(j^\sigma(\alpha_k) > \alpha_l\).

In essence, the second coordinate of \(T_2\) attempts to verify the wellfoundedness of the EM blueprint coded in the first coordinate. From \(T_2\) we compute \(\hat{T}_2\), a tree on \(\omega \times (\omega \times u_\omega)\) that projects to a good universal \(\Pi^1_2\) set. By Shoenfield absoluteness, if \(\varphi(v)\) is a \(\Pi^1_2\) formula, effectively from \(\neg \varphi^\Gamma\) we can compute a unary Skolem term \(\tau_{\varphi^\Gamma}\) such that \(\tau_{\varphi^\Gamma}^\Gamma(x) = 0\) iff \(\varphi(x)\) holds.

Define \((\tau_{\varphi^\Gamma}(v), (s, \alpha)) \in \hat{T}_2\) iff \((s, \alpha) \in T_2\) and

1. if \(\tau_{\varphi^\Gamma}(m) = n\) is true in \(s\), then \(v(m) = n\);
2. \(\tau_{\varphi^\Gamma}(x) \neq 0\) is not true in \(s\).

So \(p[\hat{T}_2] = \{\tau_{\varphi^\Gamma}(x) : \varphi(x)\}\).

### 3.3 The tree \(S_2\)

In this section, we redefine the tree \(S_2\) introduced in [26, Section 2] in the language of trees of uniform cofinalities in [14].

A tree of level-1 trees is a tree \(T\) on \(\omega^{<\omega}\) (i.e., \(T \subseteq (\omega^{<\omega})^{<\omega}\) and closed under \(\subseteq\)) and such that for any \(s \in T\), \(\{a \in \omega^{<\omega} : s^\frown (a) \in T\}\) is a level-1 tree.

A level-2 tree of uniform cofinalities, or level-2 tree, is a function \(Q\) such that \(\text{dom}(Q)\) is a tree of level-1 trees, \(\emptyset \in \text{dom}(Q)\) and for any \(q \in \text{dom}(Q)\), \((Q(q \upharpoonright l))_{l \in \text{lh}(q)}\) is a partial level \(\leq 1\) tower of discontinuous type. In particular, \(Q(\emptyset) = (\emptyset, (\emptyset, (\emptyset))\).

We denote \(Q(q) = (Q_{\text{tree}}(q), Q_{\text{node}}(q))\) and \(Q[q] = (Q_{\text{tree}}(q), (Q_{\text{node}}(q \upharpoonright l))_{l \leq \text{lh}(q)})\). So \(Q[q]\) is a potential partial level \(\leq 1\) tower of discontinuous
type. Denote $Q\{q\} = \{a \in \omega^\omega : q^-(a) \in \dom(Q)\}$, which is a level-1 tree. The cardinality of $Q$ is $\card(Q) = \card(\dom(Q))$. $\card(Q)$ could be finite or $\aleph_0$.

For $Q$ a level-2 tree, Let

$$\dom^*(Q) = \dom(Q) \cup \{q^-(1) : q \in \dom(Q)\}.$$ 

Here $1$ is a distinguished element which is $<_{BK}$-smaller than any node in $\omega^\omega$. So $<_{BK} \dom^*(Q)$ extends $<_{BK} \dom(Q)$ where $q^-(1)$ comes before any $q^-(s) \in \dom(Q)$. If $q \neq \emptyset$, denote $Q\{q, -\} = \{q^-(1)\} \cup \{q^-(a) : Q_{\tree}(q^-(a)) = Q_{\tree}(q) \wedge a <_{BK} q(lh(q) - 1)\}$, $Q\{q, +\} = \{q^-(1)\} \cup \{q^-(a) : Q_{\tree}(q^-(a)) = Q_{\tree}(q) \wedge a >_{BK} q(lh(q) - 1)\}$. For $q \in \dom^*(Q)$, $q$ is of discontinuous type if $q \in \dom(Q)$; $q$ is of continuous type if $q \in \dom^*(Q) \setminus \dom(Q)$. In particular, $\{\emptyset, (-1)\} \subseteq \dom^*(Q)$. Put $Q[q^-(1)] = (P, (Q_{\node}(q \mid i))_{i \leq lh(q)})$, where $P$ is the completion of $Q(q)$. So $Q[q^-(1)]$ is a potential partial level $\leq 1$ tower of continuous type.

**Definition 3.7.** Suppose $Q$ is a level-2 tree. A $Q$-description is a triple

$$q = (q, P, \bar{p})$$

such that $q \in \dom^*(Q)$ and $(P, \bar{p}) = Q[q]$. $\desc(Q)$ is the set of $Q$-descriptions. A $Q$-description $(q, P, \bar{p})$ is of (dis-)continuous type iff $q$ is of (dis-)continuous type. The constant $Q$-description is $(\emptyset, \emptyset, \emptyset)$.

If $q = (q, P, \bar{p}) \in \desc(Q)$ is of discontinuous type, put $q^-(1) = (q^-(1), P^+, \bar{p})$ where $P^+$ is the completion of $(P, \bar{p})$. If $\bar{\alpha} = (\alpha_p)_{p \in N}$ is a tuple indexed by $N$, $q \in \dom^*(Q)$, $\dom(Q(q^-)) \subseteq N$ if $q \neq \emptyset$, we put

$$\bar{\alpha} \oplus_Q q = (\alpha_{p_0}, q(0), \ldots, \alpha_{p_{lh(q)} - 1}, q(lh(q) - 1)),$$

where $p_i = Q_{\node}(q \mid i)$.

The ordinal representation of $Q$ is the set

$$\rep(Q) = \{\bar{\alpha} \oplus_Q q : q \in \dom(Q), \bar{\alpha} \text{ respects } Q_{\tree}(q)\} \cup \{\bar{\alpha} \oplus_Q q^-(1) : q \in \dom(Q), \bar{\alpha} \text{ respects } Q(q)\}.$$ 

$\rep(Q)$ is endowed with the $<_{BK}$ ordering:

$$<^Q = <_{BK} \rep(Q).$$

Thus, the $<^Q$-greatest element is $\emptyset = \emptyset \oplus_Q \emptyset$, and the set $\{(\beta, -1) : \beta < \omega_1\}$ is $<^Q$-cofinal below $\emptyset$. In general, if $q \in \dom(Q)$ and $\bar{\alpha}$ respects $Q_{\tree}(q)$, then $\bar{\alpha} \oplus_Q q$ is the $<^Q$-sup of $\bar{\alpha}^-(\beta) \oplus q^-(1) \in \rep(Q)$. The fact that $(0)$
is the \(<_{BK}\)-maximum node of a nonempty regular level-1 tree implies that if \((q, P) \in \text{desc}(Q), q \neq \emptyset, (\alpha_p)_{p \in P} \text{ respects } P\), then \(\alpha_{(q)}\) is bigger than \(\alpha_p\) for any \(p \in P \setminus \{(0)\}\). Hence, when \(Q\) is finite, \(<Q\) has order type \(\omega_1 + 1\). If \(B \in L\) is a subset of \(\omega_1\), we put

\[ f \in B^{Q_1} \]

iff \(f \in L\) is an order preserving, continuous function from \(\text{rep}(Q)\) to \(B \cup \{\omega_1\}\). If \(f \in B^{Q_1}\), for each \(q \in \text{dom}(Q)\), letting \(P_q = Q_{\text{tree}}(q)\), \(f_q\) is the function on \([\omega_1]^{P_q}_\uparrow\) that sends \(\bar{\alpha}\) to \(f(\bar{\alpha} \oplus Q \ q)\), and

\[ [f]^Q = ([f]^Q)_q \in \text{dom}(Q) \]

where \([f]^Q = [f_q]^i\). A tuple \(\bar{\beta}\) respects \(Q\) iff \(\bar{\beta} = [f]^Q\) for some \(f \in \omega_1^{Q_1}\); \(\bar{\beta}\) weakly respects \(Q\) iff \(\beta_0 = \omega_1\) and for any \(q \in \text{dom}(Q) \setminus \{\emptyset\}\), \(\beta_q < f_{Q_{\text{tree}}(q)}(\bar{\beta}_q)\).

If \(y \in [\text{dom}(Q)]\), let \(Q(y) = \text{DEF} \cup_{n<\omega} Q_{\text{tree}}(y \upharpoonright n)\) be an infinite level-1 tree. \(Q\) is \(\Pi_1^1\)-wellfounded iff

1. \(\forall q \in \text{dom}(Q) \ Q \{q\}\) is \(\Pi_1^1\)-wellfounded,
2. \(\forall y \in [\text{dom}(Q)] \ Q(y)\) is not \(\Pi_1^1\)-wellfounded.

In particular, finite level-2 trees are \(\Pi_1^1\)-wellfounded. \(\Pi_1^1\)-wellfoundedness of a level-2 tree is a \(\Pi_1^1\) property in the real coding the tree.

A level-2 tree \(Q\) is a called a subtree of \(Q'\) iff \(Q\) is a subfunction of \(Q'\). A finite level-2 tower is a (possibly empty) sequence \((Q_i)_{1 \leq i \leq n}\) such that \(Q_i\) is a level-2 tree for \(1 \leq i \leq n\), \(\text{card}(Q_i) = i\) and \(i < j \rightarrow Q_i\) is a subtree of \(Q_j\). An infinite level-2 tower is a sequence \(Q = (Q_n)_{1 \leq n < \omega}\) such that for each \(n\), \((Q_1)_{1 \leq i \leq n}\) is a finite level-2 tower. A level-2 system is \((Q_s)_{s \in \omega}^\omega\) such that for each \(s\), \((Q_{sh})_{1 \leq i < \text{lh}(s)}\) is a finite level-2 tower. Associated to a \(\Pi_1^1\) set \(A\) we can assign a level-2 system \((Q_s)_{s \in \omega}^\omega\) so that \(x \in A\) iff the level-2 tower \(Q_x = \text{DEF} (Q_{x^y})_{n < \omega}\) is \(\Pi_1^1\)-wellfounded. If \(A\) is lightface \(\Pi_1^1\), then \((Q_s)_{s \in \omega}^\omega\) can be picked effective.

In our language, the level-2 tree \(S_2\), originally defined in \([26\text{, Section 2]}\), takes the following form.

**Definition 3.8.** Assume \(\Pi_1^1\)-determinacy.

1. \(S_2^-\) is the tree on \(V_\omega \times u_\omega\) such that \((\emptyset, \emptyset) \in S_2^-\) and a nonempty node

\[ (\emptyset, \emptyset) \neq (\bar{Q}, \bar{\alpha}) = ((Q_i)_{1 \leq i \leq n}, (\alpha_i)_{1 \leq i \leq n}) \in S_2^- \]

iff \(\bar{Q}\) is a finite level-2 tower, and putting \(Q_0 = \emptyset, q_i \in \text{dom}(Q_{i+1}) \setminus \text{dom}(Q_i), \beta_{q_i} = \alpha_i\), then \((\beta_{q_i})_{q \in \text{dom}(Q_n)}\) respects \(Q_n\).
2. \( S_2^- \) is the tree on \( V_\omega \times u_\omega \) such that \((\emptyset, \emptyset) \in S_2^-\) and a nonempty node

\[(\emptyset, \emptyset) \neq (\vec{Q}, \vec{a}) = ((Q_i)_{1 \leq i \leq n}, (\alpha_i)_{1 \leq i \leq n}) \in S_2^-\]

iff \( \vec{Q} \) is a finite level-2 tower, and putting \( Q_0 = \emptyset, q_i \in \text{dom}(Q_{i+1}) \setminus \text{dom}(Q_i), \beta_{q_i} = \alpha_i \), then \( (\beta_{q_i})_{q \in \text{dom}(Q_n)} \) weakly respects \( Q_n \).

By Theorem 3.3,

\[ p[S_2^-] = p[S_2] = \{ \vec{Q} : \bigcup \vec{Q} \text{ is } \Pi_1^2\text{-wellfounded} \}. \]

The (non-regular) \( u_\omega \)-scale associated to \( S_2 \) is \( \Delta^1_3 \) (cf. [26]).

A level \( \leq 2 \) tree is a pair \( Q = (\{Q\}, \vec{Q}) \) such that \( \vec{Q} \) is a level-d tree for \( d \in \{1, 2\} \). Its cardinality is \( \text{card}(Q) = \sum_d \text{card}(\{Q\}) \). We follow the convention that \( \{Q\} \) always stands for the level-d component of a level \( \leq 2 \) tree \( Q \). \( Q \) is a level \( \leq 2 \) subtree of \( \vec{Q} \) iff \( \vec{Q} \) is a level-d subtree of \( \{Q\} \) for \( d \in \{1, 2\} \).

\[ \text{rep}(Q) = \bigcup_d (\{d\} \times \text{rep}(\{Q\})). \]

iff \( d = 2 \) and \( \vec{Q} \) is essentially the concatenation of \( <^Q \) and \( ^<\vec{Q} \). \( \text{dom}(Q) = \bigcup_d (\{d\} \times \text{dom}(\{Q\})) \), \( \text{dom}^*(Q) = \bigcup_d (\{d\} \times \text{dom}^*(\{Q\})) \), where \( \text{dom}^*(\{Q\}) = \text{dom}(\{Q\}) = \{Q\} \). \( \text{desc}(Q) = \bigcup_d (\{d\} \times \text{desc}(\{Q\})) \) is the set of \( Q \)-descriptions. \( (d, \vec{q}) \in \text{desc}(Q) \) is of continuous type iff \( d = 2 \) and \( \vec{q} \) is of continuous type; otherwise, \( (d, \vec{q}) \) is of discontinuous type. \( Q \) is \( \Pi_1^1 \)-wellfounded iff \( \{Q\} \) is \( \Pi_1^1 \)-wellfounded and \( \vec{Q} \) is \( \Pi_1^2 \)-wellfounded.

By virtue of the Brouwer-Kleene ordering, the next proposition is a corollary of Theorem 3.3.

**Proposition 3.9.** Let \( Q \) be a level \( \leq 2 \) tree. Then \( Q \) is \( \Pi_1^2 \)-wellfounded iff \( <^Q \) is a wellordering on \( \text{rep}(Q) \).

As a corollary, if \( Q \) is \( \Pi_1^1 \)-wellfounded, then \( \text{o.t.}(<^Q) = \omega_1 + 1 \).

If \( f \) is a function on \( \text{rep}(Q) \), let \( ^d f \) be the function on \( \text{rep}(\{Q\}) \) that sends \( v \) to \( f(d, v) \). If \( B \in \mathbb{L} \) is a subset of \( \omega_1 \), we put

\[ f \in B^{Q\uparrow} \]

iff \( f \in \mathbb{L} \) is an order preserving, continuous function on \( \text{rep}(Q) \), and \( ^d f \in B^{Q\uparrow} \) for \( d \in \{1, 2\} \). \( f \) represents a \( \text{card}(Q) \)-tuple of ordinals

\[ [f]^Q = (^d [f]^Q)_{(d, q) \in \text{dom}(Q)} \]

where \( ^d [f]^Q = [f]^{Q^d}_q \). In particular, we must have \( ^2 [f]^Q = \omega_1 \). Let

\[ [B]^{Q\uparrow} = \{ [f]^Q : f \in B^{Q\uparrow} \}. \]

The properties of a tuple \( [f]^Q \) for \( f \in \omega_1^{Q\uparrow} \) are analyzed in [26, 46]. We restate the key results in the effective context.
**Definition 3.10.** Suppose $Q$ is a level $\leq 2$ tree. An extended $Q$-description is either a $Q$-description or of the form $(2, (q, P, \bar{p}))$ such that $(2, (q^-(1), P, \bar{p}))$ is a $Q$-description of continuous type. $\text{desc}^*(Q)$ is the set of extended $Q$-descriptions. $(d, q) \in \text{desc}^*(Q)$ is regular iff either $(d, q) \in \text{desc}(Q)$ of discontinuous type or $(d, q) \notin \text{desc}(Q)$.

Suppose $(2, q) = (2, (q, P, \bar{p})) \in \text{desc}^*(Q)$. If $f \in \omega_1^{Q^T}$, $2^f_q$ is the function on $[\omega_1]^{P^T}$ defined as follows: $2^f_q = \exists^f_q$ if $(2, q) \in \text{desc}(Q)$; $2^f_q(\bar{a}) = 2^f_q(\bar{a} \downarrow \bar{a})$ if $(2, q) \notin \text{desc}(Q)$. If $\bar{\beta} = (4^{\beta_d})_{(d, q) \in \text{dom}(Q)} \in [\omega_1]^{P^{Q^T}}$, we define $4^{\beta_d}$ for $(d, q) \in \text{desc}^*(Q)$: if $d = 2$, $q = (q, P, \bar{p})$, put $4^{\beta_d} = [4^{\beta_d}]_{\mu^P}$ where $\bar{\beta} = [f]^{Q^T}$. Clearly, $2^{\beta_d} = 2^{\beta_d}$ if $(2, q) \in \text{desc}(Q)$ of discontinuous type, $2^{\beta_d} = 4^{\beta_d}$ if $(2, q) \notin \text{desc}(Q)$. The next lemma computes the remaining case when $q \in \text{desc}(Q)$ is of continuous type, justifying that $4^{\beta_d}$ does not depend on the choice of $f$.

**Lemma 3.11.** Suppose $Q$ is a level $\leq 2$ tree. Suppose $\bar{\beta} = (4^{\beta_d})_{(d, q) \in \text{dom}(Q)} \in [\omega_1]^{P^{Q^T}}$, $(2, q) = (2, (q, P, \bar{p})) \in \text{desc}(Q)$ is of continuous type, $P^- = Q_{\text{tree}(q^-)}$, then $2^{\beta_d} = 4^{\beta_d}$.

*Proof.* Let $\bar{\beta} = [f]^{Q^T}, f \in [\omega_1]^{P^{Q^T}}$. Let $v = p_{h(q^-)} - 1$. So $P$ is the completion of $Q(q^-) = (P^-, v)$.

Suppose $\gamma = [g]_{\mu^P} < 2^{\beta_d}$, $g \in \mathbb{L}$. So for $\mu^P$-a.e. $\bar{\alpha}, g(\bar{\alpha}) < 2^f_q(\bar{\alpha}) = \sup_{\xi < \alpha_v} 2^f_q(\bar{\alpha} \downarrow (\bar{\xi}))$, where $\bar{\alpha} \downarrow (\bar{\xi})$ is the extension of $\bar{\alpha}$ whose entry indexed by $v$ is $\xi$. Let $h(\bar{\alpha})$ be the least $\xi < \alpha_v$ such that $g(\bar{\alpha}) < 2^f_q(\bar{\alpha} \downarrow (\xi))$. Then $h \in \mathbb{L}$. By the remarkability of level-1 sharps, we get $C \in \mu_\mathbb{L}$ such that for any $\bar{\alpha} \in [C]^{P^T}$, $h(\bar{\alpha}) = g(\bar{\alpha}) \downarrow (\text{dom}(P^-)) < \alpha_v$. Hence for any $\bar{\alpha} \in [C]^{P^T}$, $g(\bar{\alpha}) \downarrow (\text{dom}(P^-)) < 2^f_q(\bar{\alpha})$. Hence $j^{P^-}(\gamma) < 2^{\beta_d}$.

Suppose on the other hand $\gamma = [g]_{\mu^P} < 2^{\beta_d}$. Then for $\mu^P$-a.e. $\bar{\alpha}, g(\bar{\alpha}) < 2^f_q(\bar{\alpha}) = \sup_{\xi < \alpha_v} 2^f_q(\bar{\alpha} \downarrow (\text{dom}(P^-) \downarrow (\xi)))$. Let $h(\bar{\alpha})$ be the least $\xi < \alpha_v$ such that $g(\bar{\alpha}) < 2^f_q(\bar{\alpha} \downarrow (\text{dom}(P^-) \downarrow (\xi)))$. By remarkableness, we get $C \in \mu_\mathbb{L}$ and $h' \in \mathbb{L}$ such that for any $\bar{\alpha} \in [C]^{P^T}$, $h(\bar{\alpha}) = h'(\bar{\alpha} \downarrow \{p : p < P^v\})$. Hence, $g(\bar{\alpha}) < 2^f_q(\bar{\alpha} \downarrow (\text{dom}(P^-) \downarrow h'(\bar{\alpha} \downarrow \{p : p < P^v\}))) = j^{P^-} P^- (\eta)$, where $\eta = [\bar{\alpha} \rightarrow 2^f_q(\bar{\alpha} \downarrow h'(\bar{\alpha} \downarrow \{p : p < P^- v\}))]_{\mu^P}$. Clearly, $\eta < 2^{\beta_d}$. So $\gamma < j^{P^-} P^- (\beta_d)$. $\square$

A tuple $\bar{\beta} = (4^{\beta_d})_{(d, q) \in \text{dom}(Q)}$ respects $Q$ iff $\bar{\beta} \in [\omega_1]^{P^{Q^T}}$. In particular, if $\bar{\beta}$ respects $Q$, then $2^{\beta_d} = \omega_1$. $\bar{\beta}$ weakly respects $Q$ iff $(1^{\beta_d})_{q \in Q}$ respects $Q$ and $(2^{\beta_d})_{q \in Q}$ weakly respects $Q$.

The relation of weak respectability is clearly $\Delta_3$. It is essentially shown in [46] that respectability is also $\Delta_3$. We restate the relevant definitions in a more applicable fashion.
Definition 3.12. Suppose $W$ is a finite level-1 tree, $\bar{w} = (w_i)_{i<m}$ is a distinct enumeration of a subset of $W$. Suppose $f : [\omega_1]^{W \uparrow} \to \omega_1$ is a function which lies in $\mathbb{L}$. The signature of $f$ is $\bar{w}$ iff there is $C \in \mu_1$ such that

1. for any $\vec{\alpha}, \vec{\beta} \in [C]^{W \uparrow}$, if $(\alpha_{w_0}, \ldots, \alpha_{w_{m-1}}) <_{BK} (\beta_{w_0}, \ldots, \beta_{w_{m-1}})$ then $f(\vec{\alpha}) < f(\vec{\beta})$;

2. for any $\vec{\alpha}, \vec{\beta} \in [C]^{W \uparrow}$, if $(\alpha_{w_0}, \ldots, \alpha_{w_{m-1}}) = (\beta_{w_0}, \ldots, \beta_{w_{m-1}})$ then $f(\vec{\alpha}) = f(\vec{\beta})$.

In particular, $f$ is constant on a $\mu^W$-measure one set iff the signature of $f$ is $\emptyset$.

Suppose the signature of $f$ is $\bar{w} = (w_i)_{i<m}$. $f$ is essentially continuous iff $m > 0$ and for $\mu^W$ a.e. $\vec{\alpha}$, $f(\vec{\alpha}) = \sup \{f(\vec{\beta}) : (\beta_{w_0}, \ldots, \beta_{w_{m-1}}) < (\alpha_{w_0}, \ldots, \alpha_{w_{m-1}})\}$. Otherwise, $f$ is essentially discontinuous. Put $[B]^{W \uparrow-1} = [B]^{W \uparrow} \times \omega$. For $w \in \text{dom}(W)$, put $[B]^{W \uparrow w} = \{ (\vec{\beta}, \gamma) : \vec{\beta} \in [B]^{W \uparrow}, \gamma < \beta_w \}$. For $v \in \{-1\} \cup W$, say that the uniform cofinality of $f$ is $v$ iff there is $g : [\omega_1]^{W \uparrow v} \to \omega_1$ such that $g \in \mathbb{L}$ and for $\mu^W$-a.e. $\vec{\alpha}$, $F(\vec{\alpha}) = \sup \{ G(\vec{\alpha}, \beta) : (\vec{\alpha}, \beta) \in [\omega_1]^{W \uparrow v} \}$ and the function $\beta \mapsto G(\vec{\alpha}, \beta)$ is order preserving. It is essentially shown in [16] that every $f : [\omega_1]^{W \uparrow} \to \omega_1$ in $\mathbb{L}$ has a unique signature and uniform cofinality. Let $(P_i, p_i)_{i<m}(P_m)$ be the partial level $\leq 1$ tower of continuous type and let $\sigma$ factor $(P_m, W)$ such that $\sigma(p_i) = w_i$ for each $i < m$. Note that $w_i \prec^W w_0$ for $0 < i < m$, so each $P_i$ is indeed a regular level-1 tree. $\vec{P} = (P_i)_{i \leq m}$ is called the level-1 tower induced by $f$, and $\sigma$ is called the factoring map induced by $f$. Note that $\sigma|P_i$ factors $(P_i, W)$ for each $i$. The potential partial level $\leq 1$ tower induced by $f$ is

1. $(P_m, (p_i)_{i<m})$, if $f$ is essentially continuous;

2. $(P_m, (p_i)_{i<m}(-1))$, if $f$ is essentially discontinuous and has uniform cofinality $-1$;

3. $(P_m, (p_i)_{i<m}((p^+) \sim))$, if $f$ is essentially discontinuous and has uniform cofinality $w_* \in W$, $(P_m, p^+)$ is a partial level $\leq 1$ tree, $\sigma((p^+) \sim) = w_*$.

In particular, if $w_* \in W$, $f(\vec{\alpha}) = \alpha_{w_*}$ is the projection map, then the potential partial level $\leq 1$ tower induced by $f$ is $(\emptyset, (0, 0))$. The approximation sequence of $f$ is $(f_i)_{i \leq m}$ where $\text{dom}(f_i) = [\omega_1]^{P_i \uparrow}$, $f_0$ is the constant function with value $\omega_1$, $f_i(\vec{\alpha}) = \sup \{ f(\vec{\beta}) : \vec{\beta} \in [\omega_1]^{P_i \uparrow}, (\beta_{w_0}, \ldots, \beta_{w_{i-1}}) = (\alpha_{p_0}, \ldots, \alpha_{p_{i-1}}) \}$ for $1 \leq i \leq m$. In particular, $f_m(\vec{\beta}_\alpha) = f(\vec{\beta})$ for $\mu^W$-a.e. $\vec{\beta}$.

Note that all the relevant properties of $f$ depend only on the value of $f$ on a $\mu^W$-measure one set. We will thus be free to say the signature, etc. of $f$ when $f$ is defined on a $\mu^W$-measure one set.

34
Definition 3.13. Suppose $\omega_1 \leq \beta < u_\omega$ is a limit ordinal. Suppose $W$ is a finite level-1 tree, $\beta = [f]_{\mu W} < u_{\text{card}(W)+1}$, the signature of $f$ is $(w_i)_{i<m}$, the approximation sequence of $f$ is $(f_i)_{i \leq m}$, the level-1 tower induced by $f$ is $(P_i)_{i \leq m}$, the factoring map induced by $f$ is $\sigma$. Then the signature of $\beta$ is $(\gamma_i)_{i<m}$, the approximation sequence of $\beta$ is $([f_i]_{\nu_i})_{i \leq m}$, $\beta$ is essentially continuous iff $f$ is essentially continuous. The uniform cofinality of $\beta$ is $\omega$ if $f$ has uniform cofinality $\nu$, seed $w_s \in W \cup \{\emptyset\}$. The potential partial level $\leq 1$ tower induced by $\beta$ is the potential partial level $\leq 1$ tower induced by $f$.

The uniform cofinality of $\beta$ is exactly $\text{cf}^\beta(\beta)$. The signature, approximation sequence and essential continuity of $\beta$ are independent of the choice of $(W, f)$ in Definition 3.13 and moreover $\Delta_3^1$ in $\beta$ uniformly.

Suppose the signature of $\beta$ is $(u_i)_{i<m}$, the approximation sequence of $\beta$ is $(\gamma_i)_{i<m}$. For $i \leq m$, let $\tau_{i,m} : \{1, \ldots, i+1\} \to \{l_0, \ldots, l_i\}$ be order preserving. For $i < k < m$, let $\tau_{i,k} = \tau_{i,m}^{-1} \circ \tau_{k,m}$. A straightforward analysis on the representative function of $\beta$ yields the following:

1. For $i < k < m$, $j_{\tau_{i,k}}(\gamma_i) < \gamma_k < j_{\tau_{i,k}}(\gamma_k)$.
2. For $i < m$, $j_{\tau_{i,m}}(\gamma_i) \leq \gamma_m < j_{\tau_{i,m}}(\gamma_i)$.
3. For $i < m$, $j_{\tau_{i,m}}(\gamma_i) = \gamma_m$ iff $i = m - 1$ and $\beta$ is essentially continuous.
4. $\beta = j_{\tau_{m,m}}(\gamma_m)$.

The next lemma is a version of the "converse direction". In its statement, the inequality $j_{\tau_{i,m}}(\gamma_i) < \gamma' < j_{\tau_{i,m}}(\gamma)$ forces $\pi$ to move the signature of $\gamma$ to a proper initial segment of that of $\gamma'$, and forces the approximation sequence of $\gamma$ to be a proper initial segment of that of $\gamma'$. It will be useful in the analysis of descriptions and tree factoring maps in Sections 4.4 and 5.4. The proof is again based on an analysis of the representative function of $\gamma$ and $\gamma'$.

Lemma 3.14. Suppose $A$ is a finite subset of $\omega$. Let $\pi : \{1, \ldots, \text{card}(A)\} \to A$ be order preserving. Suppose that $\gamma < u_{\text{card}(A)+1}$ and $j_{\tau_{i,m}}(\gamma) < \gamma' < j_{\tau_{i,m}}(\gamma)$. Let $(u_k)_{k<v}, (\gamma_k)_{k\leq v}, (P, \vec{p})$ be the signature, approximation sequence and potential partial level $\leq 1$ tower induced by $\gamma$ respectively. Let $(u'_k)_{k<v'}, (\gamma'_k)_{k\leq v'}$, $(P', \vec{p}')$ be the signature, approximation sequence and potential partial level $\leq 1$ tower induced by $\gamma'$ respectively. Let $\text{cf}^\beta(\gamma) = u_l$. Then

1. $v < v'$, $\pi(l_k, \gamma_k) = (l'_k, \gamma'_k)$. $\gamma$ is essentially discontinuous $\rightarrow \gamma_v = \gamma_{v'}$.

$\gamma$ is essentially continuous $\rightarrow \gamma_v < \gamma_{v'}$. 35
2. \( l'_k \not\in A \) for \( v \leq k < v' \).

3. For any \( k < v \), \( l'_k < \pi(l_k) \leftrightarrow l_s \leq l_k \).

4. \( P \) is a proper subtree of \( P' \) and \( \bar{p} \) is an initial segment of \( \bar{p}' \).

Moreover, if \( \gamma' < \gamma'' < j^\pi(\gamma) \) and \((\gamma'_k)_{k < v''}\) is the approximation sequence of \( \gamma'' \), then \( \gamma'_v < \gamma''_v \).

The next few lemmas are essentially part of effectivized Kunen’s analysis of tuples of ordinals in \( u_\omega \). The proofs are rather routine.

Suppose \( E \) is a club in \( \omega_1 \). For a partial level \( \leq 1 \) tree \((P, t)\), put \( \vec{\alpha} = (\alpha_p)_{p \in P, \uparrow t} \in [E]^{(P, t)} \uparrow \) iff \( \vec{\alpha} \) respects \((P, t)\), \((\alpha_p)_{p \in P} \in [E]^{P \uparrow}\), and \( t \neq 1 \rightarrow \alpha_t \in E \). For a level \( \leq 2 \) tree \( Q \), put

\[
\text{rep}(Q) \uparrow E = \{ \vec{\alpha} \oplus Q q : q \in \text{dom}(Q), \vec{\alpha} \in [E]^{Q_{\text{tree}}(q)} \uparrow \}
\]

\[
\cup \{ \vec{\alpha} \oplus Q q^{-1} : q \in \text{dom}(Q), \vec{\alpha} \in [E]^{Q_{\text{tree}}(q)} \uparrow \}.
\]

Put \( \text{rep}(Q) \uparrow E = (\{1\} \times \text{rep}(Q)) \cup (\{2\} \times \text{rep}(Q) \uparrow E) \). Then \( \text{rep}(Q) \uparrow E \) is a closed subset of \( \text{rep}(Q) \) (in the order topology of \( <Q \)).

**Lemma 3.15.** Suppose \( Q \) is a finite level \( \leq 2 \) tree, \( C \in \mu_L \) is a club. Then \( \vec{\beta} \in [C]^Q \uparrow \) iff there exist \( f \in \omega_1^{Q \uparrow} \) and \( E \in \mu_L \) such that \( \vec{\beta} = [f]^Q \) and for any \( q \in \vec{Q} \), \( f(q) \) is a limit point of \( C \); for any \( q \in \text{dom}(\vec{Q}) \), for any \( \vec{\alpha} \in [E]^{Q_{\text{tree}}(q)} \uparrow \), \( 2^f_q(\vec{\alpha}) \) is a limit point of \( C \).

**Proof.** The nontrivial direction is \( \Leftarrow \). Suppose \( f \in \omega_1^{Q \uparrow} \) and \( E \in \mu_L \) are as given. For \( q \in \text{dom}(\vec{Q}) \setminus \{0\} \), let \( \vec{Q}(q) = (P_q, p_q) \), and let \( q^* \) be the \( <_{BK} \)-maximum of \( \vec{Q}(q, -) \).

**Claim 3.16.** There is \( E' \in \mu_L \) such that \( E' \subseteq E \) and for any \( q \in \text{dom}(\vec{Q}) \setminus \{0\}, \) for any  \( \vec{\alpha} \in [E']^{P_q \uparrow} \), if \( p_q \neq -1 \) then \( C \cap (2^{f_q}(\vec{\alpha}), 2^{f_q}(\vec{\alpha})) \) has order type \( \alpha_{p_q^-} \).

**Proof of Claim 3.16.** Otherwise, there is \( q \in \text{dom}(\vec{Q}) \setminus \{0\} \) such that \( p_q \neq -1 \) and for \( \mu^{P_q \text{-a.e.}}. \), \( C \cap (2^{f_q}(\vec{\alpha}), 2^{f_q}(\vec{\alpha})) \) has order type smaller than \( \alpha_{p_q^-} \). However, by assumption, \( C \cap (2^{f_q}(\vec{\alpha}), 2^{f_q}(\vec{\alpha})) \) is cofinal in \( 2^{f_q}(\vec{\alpha}) \), and \( 2^{f_q}(\vec{\alpha}) \) witnesses that \( 2^{f_q}(\vec{\alpha}) \) has uniform cofinality \( p_q^- \). This leads to a function \( h \in L \) where for \( \mu^{P_q \text{-a.e.}}. \), \( h(\vec{\alpha}) \) is a cofinal sequence in \( \alpha_{p_q^-} \) of order type \( < \alpha_{p_q^-} \). Hence, \( \text{cf}(\text{seed}(P_q, p_q)) < \text{seed}(P_q, p_q) \) by Loś, which is absurd.

Fix \( E' \) as in Claim 3.16. We are able to define \( f' : \text{rep}(Q) \uparrow E' \rightarrow C \) such that \( f(1, q) = f'(1, q) \) for \( q \in \vec{Q} \), \( f(2, \vec{\alpha} \oplus Q q) = f'(2, \vec{\alpha} \oplus Q q) \) for
\( q \in \text{dom}(\mathcal{Q}) \setminus \{\emptyset\}, \alpha \in [E'']^P \uparrow. \) Let \( \theta : \text{rep}(Q) \to \text{rep}(Q) \mid E' \) be an order preserving bijection. Let \( E'' \in \mu \) where \( \eta \in E'' \) iff \( E'' \cap \eta \) has order type \( \eta. \) It is easy to see that \( \theta \mid (\text{rep}(Q) \mid E'') \) is the identity map. Define \( g = f' \circ \theta. \) Then \( g \in C^Q \) and \( [g]^Q = [f]^Q. \)

**Lemma 3.17.** Suppose \( Q \) is a finite level \( \leq 2 \) tree, \( \mathcal{Q}(q) = (P_q, p_q) \) for \( q \in \text{dom}(Q), E \in \mu \) is a club. Suppose \( f : \text{rep}(Q) \mid E \to \omega_1 + 1 \) satisfies

1. \( f \mid (\{1\} \times \text{rep}(Q)) \) is continuous, order preserving;
2. if \( q \in \text{dom}(\mathcal{Q}), \) then the potential partial level \( \leq 1 \) tower induced by \( 2f_q \) is \( \mathcal{Q}[q], \) the approximation sequence of \( 2f_q \) is \( (2f_q^i) \leq \text{lh}(q), \) and the uniform cofinality of \( 2f_q \) on \( [E]^P \uparrow \) is witnessed by \( 2f_q^{-1}, \) i.e., if \( \alpha \in [E]^P \uparrow, \) then \( 2f_q(\alpha) = \sup\{2f_q^{-1}(\alpha^-)(\beta) : \alpha^-(\beta) \in \text{rep}(2\mathcal{Q}) \mid E\}, \) and the map \( \beta \mapsto 2f_q^{-1}(\alpha^-(\beta)) \) is continuous, order preserving;
3. if \( a, b \in 2\mathcal{Q}\{q\} \) and \( a \prec BK b, \) then \( [f_q^{-1}(a)]_{\mu^P \cdot a^-} < [f_q^{-1}(b)]_{\mu^P \cdot b^-}. \)

Then there is \( E' \in \mu \) such that \( E' \subseteq E \) and \( f \mid (\text{rep}(Q) \mid E') \) is order preserving.

**Proof.** We know by assumption that for \( \mu^P \)-a.e. \( \alpha, f_q(\alpha) = \sup\{f_q^{-1}(\alpha^-)(\beta) : \beta \prec \alpha^-\}. \) Fix for the moment \( q \) such that \( p_q \neq -1. \) For \( \alpha = (\alpha_p)_{p \in P_q} \), put \( \alpha^- = (\alpha_p)_{p < BK P_q}. \) By remarkability of (level-1) sharps, there is a function \( h \in \text{L} \) and \( E'_q \in \mu \) such that for any \( \alpha \in [E'_q]^P \uparrow, h(\alpha^-) < \alpha^- \) and for any \( \beta \in \alpha^- \cap E'_q, \) for any \( a, b \in 2\mathcal{Q}\{q\}, f_q^{-1}(\alpha^-)(\beta) < f_q^{-1}(b)(\alpha^-)(h(\alpha^-))). \) Let \( \eta \in E'' \) iff for any \( \alpha \in [\eta]^P \uparrow, \) if \( \forall p < BK p_k \alpha_p < \eta \) then \( h(\alpha^-) < \eta. \) Finally, let \( E'' = \bigcap\{E'_q : p_q \neq -1\}. \) \( E'' \) works for the lemma.

**Lemma 3.18.** Suppose that \( Q \) is a finite level \( \leq 2 \) tree and \( \mathcal{B} = (d_{\mathcal{B}})_{d,q} \in \text{dom}(\mathcal{Q}) \) is a tuple of ordinals in \( u_\omega. \) Then \( \mathcal{B} \) respects \( Q \) iff all of the following holds:

1. \( (d_{\mathcal{B}})_{q \in Q} \) respects \( \mathcal{Q}. \)
2. For any \( q \in \text{dom}(\mathcal{Q}), \) the potential partial level \( \leq 1 \) tower induced by \( \beta_q = \mathcal{Q}[q], \) and the approximation sequence of \( \beta_q \) is \( (\beta_q)_{l \leq \text{lh}(q)}. \)
3. If \( a, b \in 2\mathcal{Q}\{q\} \) and \( a \prec BK b \) then \( 2\beta_q^{-1}(a) < 2\beta_q^{-1}(b). \)

Moreover, if \( C \in \mu \) is a club, then \( \mathcal{B} \in [C]^Q \) iff \( \mathcal{B} \) respects \( Q \) and letting \( C' \) be the set of limit points of \( C, \) then \( \mathcal{B} \in C' \) for \( q \in \mathcal{Q}, 2\beta_q \in j^{2\mathcal{Q}(\mathcal{Q})(C')} \) for \( q \in \text{dom}(\mathcal{Q}). \)
Lemma 3.19. The relation “$Q$ is a finite level $\leq 2$ tree $\land \vec{\beta}$ respects $Q$” is \(\Delta^1_3\).

Lemma 3.20. Suppose $Q$ and $Q'$ are level $\leq 2$ trees with the same domain. Suppose $\vec{\beta}$ respects both $Q$ and $Q'$. Then $Q = Q'$.

Suppose $Q$ is a finite level $\leq 2$ tree. Suppose $(d, q) \in \text{desc}^*(Q)$, and if $d = 2$ then $q = (q, P, \vec{p})$. Put

\[
[(d, q)]_Q = \begin{cases}
\|1, (q)\|_{<Q} & \text{if } d = 1, \\
[\vec{\alpha} \mapsto \|(2, \vec{\alpha} \oplus_{Q} q)\|_{<Q}]_{\mu} & \text{if } d = 2.
\end{cases}
\]

To save ink, put $[d, q]_Q = [(d, q)]_Q$. If in addition, $d = 2$ and $q \in \text{desc}(Q)$ of discontinuous type, put $[2, q]_Q = [2, q]_{Q}$. It is easy to compute $[d, q]_Q$ only from the syntactics.

Definition 3.21. To every ordinal $\xi < \omega^\omega$ (ordinal arithmetic), we assign $\hat{\xi}$ as follows:

1. $\hat{0} = 0$.
2. $\hat{1} = \omega$.
3. If $0 < \eta = \omega^{n_1} + \cdots + \omega^{n_k} < \omega^\omega$, $\omega > n_1 \geq \cdots \geq n_k$ in the Cantor normal form, then $\hat{\omega} = u_{n_1+1} \cdots u_{n_k+1}$.

4. If $0 < \xi = \omega^{n_1} + \cdots + \omega^{n_k}$, $\omega^\omega > \eta_1 \geq \cdots \geq \eta_k$ in the Cantor normal form, then $\hat{\xi} = \hat{\omega}^{\eta_1} + \cdots + \hat{\omega}^{\eta_k}$.

Then

\[
\{\hat{\xi} : 0 < \xi < \omega^\omega\} = \{[d, q]_Q : Q \text{ finite level } \leq 2 \text{ tree}, (d, q) \in \text{desc}^*(Q)\}
\]

and the relation $\hat{\xi} = [d, q]_Q$ is effective. The ordering among different $[d, q]_Q$ can be computed in the following concrete way. Put $(1, q) = (1, q)$. For $q = (q, P, \vec{p})$, $k = \text{lh}(q)$, $\vec{p} = (p_i)_{i < \text{lh}(\vec{p})}$, put

\[
\langle 2, q \rangle = \begin{cases}
(2, \|p_0\|_{<\tau}, q(0), \ldots, \|p_{k-2}\|_{<\tau}, q(k-2), -1) & \text{if } q \in \text{desc}(\dot{\tau}Q) \text{ of continuous type,} \\
(2, \|p_0\|_{<\tau}, q(0), \ldots, \|p_{k-1}\|_{<\tau}, q(k-1), -1) & \text{if } q \in \text{desc}(\dot{\tau}Q) \text{ of discontinuous type,} \\
(2, \|p_0\|_{<\tau}, q(0), \ldots, \|p_{k-1}\|_{<\tau}, q(k-1), \|p_k\|_{<\tau}) & \text{if } q \notin \text{desc}(\dot{\tau}Q).
\end{cases}
\]
Define
\[(d, q) \prec (d', q')\]
iff \((d, q) <_{BK} (d', q')\). Define
\[(d, q) \sim (d', q')\]
iff \((d, q) = (d', q')\). Then for any finite level \(\leq 2\) tree \(Q\), \([d, q]_Q < [d', q']_Q\)
iff \((d, q) \prec (d', q')\); \([d, q]_Q = [d', q']_Q\) iff \((d, q) \sim (d', q')\). In fact, \((d, q) \sim (d', q')\) iff either \((d, q) = (d', q')\) or \({(d, q), (d', q')}\} = \{(2, (0, 0, 0)), (2, ((-1), (0), ((0))))\}.

Define \(\prec^Q = \prec\) desc\(^*\)(Q), \(\sim^Q = \sim\) desc\(^*\)(Q). It is also easy to verify the next lemma on the order of the entries of \(\vec{\beta}\) which respects \(Q\).

**Lemma 3.22.** Suppose \(Q\) is a level \(\leq 2\) tree and \(\vec{\beta}\) respects \(Q\). Suppose \((d, q), (d', q') \in\) desc\(^*\)(Q). Then \(d\beta_q < d\beta_{q'}\) iff \((d, q) \prec^Q (d', q')\); \(d\beta_q = d\beta_{q'}\) iff \((d, q) \sim^Q (d', q')\).

## 4 The level-2 sharp

### 4.1 The equivalence of \(x^{2\#}\) and \(M^\#_1(x)\)

From now on, we assume \(\Delta^1_3\)-determinacy. By Kechris-Woodin [29], \(\Diamond(\omega^2\cdot\Pi^1_3)\)-determinacy follows. By Neeman [37, 38] and Woodin [30, 48], this is also equivalent to “for every \(x \in \mathbb{R}\), there is an \((\omega,\omega_1)\)-iterable \(M^\#_1(x)\)”.

**Definition 4.1.** Suppose \(\mathcal{X} = \omega^k \times \mathbb{R}^l\) is a product space. Suppose \(x\) is a real and \(\beta \leq u_\omega\). A subset \(A \subseteq \mathcal{X}\) is \(\beta\)-\(\Pi^1_3\) iff there is a \(\Pi^1_3\)(\(x\)) set \(B \subseteq u_\omega \times \mathcal{X}\) such that \(A = \text{Diff } B\). \(A\) is \(\beta\)-\(\Pi^1_3\) iff \(A\) is \(\beta\)-\(\Pi^1_3(0)\). \(A\) is \(\beta\)-\(\Pi^1_3\) iff \(A\) is \(\beta\)-\(\Pi^1_3\) for some real \(x\).

By Theorem [2.1] when \(\beta\) is a limit ordinal, \(A \subseteq \mathcal{X}\) is \(\beta\)-\(\Pi^1_3\)(\(x\)) iff there is a pair of \(\Sigma_1\)-formulas \((\varphi, \psi)\) such that
\[(\vec{n}, \vec{y}) = (n_1, \ldots, n_k, y_1, \ldots, y_l) \in A\]
iff
\[L_{\kappa, x, y}[T_2, x, \vec{y}] = \exists \alpha < \beta(\forall \eta < \alpha \varphi(\eta, \vec{n}, \vec{y}, T_2, x) \land \neg \psi(\alpha, \vec{n}, \vec{y}, T_2, x)).\]

**Lemma 4.2.** Assume \(\Delta^1_3\)-determinacy. Suppose \(n, m\) are positive integers. If \(A\) is \((u_n)\)\(^m\)-\(\Pi^1_3\)(\(x\)), then \(A\) is \(\Diamond^2(\omega n\cdot\Pi^1_3(x))\).
Lemma 4.3. Assume \( \Delta^1_2 \)-determinacy. Let \( n < \omega \). If \( A \in \mathcal{D}(\omega n \Pi^1_1(x)) \), then \( A \subseteq u_{n+2} \Pi^1_3(x) \).

Proof. Without loss of generality, assume \( x = 0 \), \( m = 1 \), and \( A \subseteq \mathbb{R} \). Let \( B \) be a \( \Pi^1_3 \) subset of \( u_n \times \mathbb{R} \) such that \( A = \text{Diff} B \). Let \( B^* = \{(w, y) : (|w|, y) \in B\} \) be the \( \Pi^1_3 \) code set of \( B \). Let \( C \subseteq \mathbb{R}^3 \) be a \( \Sigma^1_2 \) set such that

\[
(\omega, y) \in B^* \iff \forall r(\omega, y, r) \in C.
\]

Consider the game \( H(y) \), where I produces \( w, r \in \mathbb{R} \), II produces \( w', r' \in \mathbb{R} \). The game is won by I iff both of the following hold:

1. \( w \in \text{WO}_n \), \( |w| \) is odd, and \( (w, y, r) \notin C \).
2. If \( w' \in \text{WO}_n \), \( |w'| \) is even, and \( (w', y, r') \notin C \), then \( |w| < |w'| \).

Therefore, \( y \in A \) iff I has a winning strategy in \( H(y) \).

Since \( L[y, w, r, w', r'] \) is \( \Sigma^1_3 \)-absolute, and since the relation \( |w| \leq |w'| \) for \( w, w' \in \text{WO}_n \) is definable over \( L[y, w, r, w', r'] \) from parameters \( u_1, \ldots, u_{n-1} \), the payoff set of the game \( H(y) \) can be expressed as a first order statement over \( L[y, \cdot] \) from parameters \( u_1, \ldots, u_{n-1} \). That is, there is a formula \( \theta \) such that an infinite run

\[
(w, r, w', r')
\]

is won by I iff

\[
L[y, w, r, w', r'] \models \theta(y, w, r, w', r', u_1, \ldots, u_{n-1}).
\]

It follows by Martin [33] that the payoff set of \( H(y) \) is \( \mathcal{D}(\omega n \Pi^1_1(y)) \), uniformly in \( y \), hence determined. Hence \( A \) is in \( \mathcal{D}(\omega n \Pi^1_1) \). \( \square \)

Lemma 4.3. Assume \( \Delta^1_2 \)-determinacy. Let \( n < \omega \). If \( A \in \mathcal{D}(\omega n \Pi^1_1(x)) \), then \( A \subseteq u_{n+2} \Pi^1_3(x) \).

Proof. Without loss of generality, assume \( x = 0 \) and \( A \subseteq \mathbb{R} \). We produce an effective transformation from a \( \mathcal{D}(\omega n \Pi^1_1) \) definition to the desired \( u_{n+2} \Pi^1_3 \) definition. By Martin [33], if \( (y, r) \in \mathbb{R}^2 \), \( C \subseteq \mathbb{R} \) is \( \omega n \Pi^1_1(y, r) \), then there is a formula \( \varphi \) such that Player I has a winning strategy in \( G(C) \) iff

\[
L[y, r] \models \varphi(y, r, u_1, \ldots, u_n).
\]

The transform from the \( \omega n \Pi^1_1(y, r) \) definition of \( C \) to \( \varphi \) is uniform, independent of \( (y, r) \). Suppose \( A = \text{Diff} B \), where \( B \subseteq \mathbb{R}^2 \) is \( \mathcal{D}(\omega n \Pi^1_1) \). Suppose \( \varphi \) is a formula such that

\[
(y, r) \in B \iff L[y, r] \models \varphi(y, r, u_1, \ldots, u_n).
\]

To establish a \( u_{n+2} \Pi^1_3 \) definition of \( A \), we have to decide which player has a winning strategy in \( G(B_y) \), for \( y \in \mathbb{R} \). For ordinals \( \xi_1 < \cdots < \xi_n < \eta < \omega_1 \), we say that \( M \) is a Kechris-Woodin non-determined set with respect to \( (y, \xi_1, \ldots, \xi_n, \eta) \) iff
1. $M$ is a countable subset of $\mathbb{R}$;

2. $M$ is closed under join and Turing reducibility;

3. $\forall \sigma \in M \ \exists v \in M \ L_\eta[y, \sigma \otimes v] = \neg \varphi(y, \sigma \otimes v, \xi_1, \ldots, \xi_n)$;

4. $\forall \sigma \in M \ \exists v \in M \ L_\eta[y, v \otimes \sigma] = \varphi(y, v \otimes \sigma, \xi_1, \ldots, \xi_n)$.

In clause 3, "$\forall \sigma \in M$" is quantifying over all strategies $\sigma$ for Player I that is coded in some member of $M$; $\sigma \otimes v$ is Player I’s response to $v$ according to $\sigma$, and $\sigma \otimes v = (\sigma \ast v) \oplus w$ is the combined infinite run. Similarly for clause 4, roles between two players being exchanged. Say that $z$ is $(y, \xi_1, \ldots, x_n, \eta)$-stable if $z$ is not contained in any Kechris-Woodin non-determined set with respect to $(y, \xi_1, \ldots, \xi_n, \eta)$. $z$ is $y$-stable if $z$ is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable for all $\xi_1 < \ldots < \xi_n < \eta < \omega_1$. The set of $(y, z)$ such that $z$ is $y$-stable is $\Pi^1_3$. By the proof of Kechris-Woodin [29], for all $y \in \mathbb{R}$, there is $z \in \mathbb{R}$ which is $y$-stable.

Note that if $z$ is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable and $z \leq_T z'$, then $z'$ is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable. Let $<_{y_1, \ldots, y_{n-1}}$ be the following wellfounded relation on the set of $z$ which is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable:

$z' <_{y_1, \ldots, y_{n-1}} z \leftrightarrow z$ is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable $\land z \leq_T z' \land$

$\forall \sigma \leq_T z \exists v \leq_T z' \ L_\eta[y, \sigma \otimes v] = \neg \varphi(y, \sigma \otimes v, \xi_1, \ldots, \xi_n)$

$\forall \sigma \leq_T z \exists v \leq_T z' \ L_\eta[y, v \otimes \sigma] = \varphi(y, v \otimes \sigma, \xi_1, \ldots, \xi_n)$.

Wellfoundedness of $<_{y_1, \ldots, y_{n-1}}$ follows from the definition of $(y, \xi_1, \ldots, \xi_n, \eta)$-stability. If $z$ is $(y, \xi_1, \ldots, \xi_n, \eta)$-stable, then $<_{y_1, \ldots, y_{n-1}} \{z' : z' <_{y_1, \ldots, y_{n-1}} z\}$ is a $\Sigma^1_3$ wellfounded relation in parameters $(y, z)$ and the code of $(\xi_1, \ldots, \xi_n, \eta)$, hence has rank $< \omega_1$ by Kunen-Martin. If $z$ is $y$-stable, let $f_y^z$ be the function that sends $(\xi_1, \ldots, \xi_n, \eta)$ to the rank of $z$ in $<_{y_1, \ldots, y_{n-1}}$. Then $f_y^z$ is a function into $\omega_1$. By $\Sigma^1_3$-absoluteness between $V$ and $L[y, z]^{|\Coll(\omega, \eta)}$, we can see $f_y^z \in L[y, z]$. Furthermore, $f_y^z$ is definable over $L[y, z]$ in a uniform way, so there is a $\{\xi\}$-Skolem term $\tau$ such that for all $(y, z) \in \mathbb{R}^2$, if $z$ is $y$-stable, then

$f_y^z(\xi_1, \ldots, \xi_n, \eta) = \tau(L[y, z])$.

Let

$\beta^z_y = \tau(L[y, z](y, z, u_1, \ldots, u_{n+1}))$.

The function

$$(y, z) \mapsto \beta^z_y$$

is $\Delta^1_3$ in the sharp codes. We say that $z$ is $y$-ultrastable iff $z$ is $y$-stable and $\beta^z_y = \min\{\beta^w_y : w$ is $y$-stable$\}$.
Claim 4.4. If \( z \) is \( y \)-ultrastable, then there is \( \sigma \leq_T z \) such that \( \sigma \) is a winning strategy for either of the players in \( G(B_y) \).

Proof of Claim 4.4. Suppose otherwise. For any \( \sigma \leq_T z \) which is a strategy for either player, pick \( w_\sigma \) which defeats \( \sigma \) in \( G(B_y) \). Let \( w \) be a real coding \( \{(\sigma, w_\sigma) : \sigma \leq_T z \} \). By an indiscernability argument, for any \((y, w)\)-indiscernibles \( \xi_1 < \cdots < \xi_n < \eta \), for any \( \sigma \leq_T z \), if \( \sigma \) is a strategy for Player I, then

\[
L_\eta[y, \sigma \otimes w_\sigma] \models \neg \varphi(y, \sigma \otimes w_\sigma, \xi_1, \ldots, \xi_n);
\]

if \( \sigma \) is a strategy for Player II, then

\[
L_\eta[y, w_\sigma \otimes \sigma] \models \varphi(y, w_\sigma \otimes \sigma, \xi_1, \ldots, \xi_n).
\]

This exactly means

\[
w <^y_{\xi_1, \ldots, \xi_n, \eta} z,
\]

and hence

\[
f^w_T(\xi_1, \ldots, \xi_n, \eta) < f^z_T(\xi_1, \ldots, \xi_n, \eta).
\]

Since \( z \) is \( y \)-stable and \( z \leq_T w \), \( w \) is \( y \)-stable. Therefore, \( \beta^w_y \) is defined and \( \beta^w_y < \beta^z_y \), contradicting to \( y \)-ultrastablensens of \( z \). \( \square \)

From Claim 4.4, if Player I (or II) has a winning strategy in \( G(B_y) \), then for any \( y \)-ultrastable \( z \), there is a winning strategy for Player I (or II) in \( G(B_y) \) which is Turing reducible to \( z \). Therefore, Player I has a winning strategy in \( G(B_y) \) iff there is \( \delta < u_{n+2} \) such that

\[
\exists z \ (z \text{ is } y\text{-stable} \land \beta^z_y = \delta) \tag{1}
\]

and

\[
\forall \gamma \leq \delta \ \forall z \ ((z \text{ is } y\text{-stable} \land \beta^z_y = \gamma) \rightarrow
\exists \sigma \leq_T z \ (\sigma \text{ is a winning strategy for I in } G(B_y))). \tag{2}
\]

Note that in (1),

\[
\{(\delta, y) : \exists z \ (z \text{ is } y\text{-stable} \land \beta^z_y = \delta)\}
\]

is a \( \Sigma^1_3 \) subset of \( u_\omega \times \mathbb{R} \), and in (2),

\[
\{(\gamma, y) : \forall z \ ((z \text{ is } y\text{-stable} \land \beta^z_y = \gamma) \rightarrow
\exists \sigma \leq_T z \ (\sigma \text{ is a winning strategy for I in } G(B_y)))\}
\]

is a \( \Pi^1_3 \) subset of \( u_\omega \times \mathbb{R} \). So \( \exists \delta < u_{n+2}((1) \land (2)) \) is a \( u_{n+2} \)-\( \Pi^1_3 \) definition of \( A \). \( \square \)
Lemma 4.2 and Lemma 4.3 are concluded in a simple equality between pointclasses.

**Theorem 4.5.** Assume $\Delta^1_2$-determinacy. Then for $x \in \mathbb{R}$,

$$\mathcal{O}^2(\omega^2 - \Pi^1_1(x)) = \prec u_\omega - \Pi^1_3(x).$$

**Definition 4.6.**

$$\mathcal{O}^{T_2,x} = \{(\varphi, \alpha) : \varphi \text{ is a } \Sigma_1 \text{-formula, } \alpha \prec u_\omega, L_{\kappa_3} \upharpoonright [T_2,x] \models \varphi(T_2,x,\alpha)\}.$$  

$\mathcal{O}^{T_2,x}$ is the $u_\omega$-version of Kleene’s $\mathcal{O}$ relative to $(T_2,x)$. It is called $\mathcal{P}_3$ in 23.

**Definition 4.7.**

$$x_n^{2#} = \{(\varphi, \psi, \eta) : \exists \alpha \prec u_\omega((\varphi, \alpha) \notin \mathcal{O}^{T_2,x} \land \forall \eta \prec \alpha, (\psi, \eta) \in \mathcal{O}^{T_2,x})\}.$$  

$$x^2\# = \{(n, \varphi, \eta) : n \prec \omega \land (\varphi, \eta) \in x_n^{2#}\}.$$  

$\mathcal{O}^{T_2,x}$ splits into $\omega$ many parts $(\mathcal{O}^{T_2,x}(\omega \times u_\omega))_{n<\omega}$. Each part is squeezed into a real $x_n^{2#}$ by applying the difference operator on its second coordinate. The join of $(x_n^{2#})_{n<\omega}$ is $x^2\#$. In particular, $x_n^{2#}$ is $\Pi^1_3$-equivalent to the good universal $\Pi^1_3$ real, which is called the $\Delta^1_3$-jump of $x$. Each $x_n^{2#}$ belongs to $L_{\kappa_3}[T_2,x]$, but $x^2\# \notin L_{\kappa_3}[T_2,x]$. The distinction between $x_n^{2#}$ and $x^2\#$ does not have a lower level analog.

The expression of $0^{2#}$ generalizes Kleene’s $\mathcal{O}$ to the higher level. Note that the transformations between $\mathcal{O}^2(\omega^2 - \Pi^1_1(x))$ and $\prec u_\omega - \Pi^1_3(x)$ definitions in Theorem 1.5 are uniform. Applying Theorem 1.5 to the space $\mathcal{X} = \omega_1$, in combination with Theorem 2.1 we get the equivalence between $x^2\#$ and $M^\#(x)$.

**Theorem 4.8.** Assume $\Delta^1_2$-determinacy. Then $x^2\#$ is many-one equivalent to $M^\#(x)$, the many-one reductions being independent of $x$.

$0^{2#}$ is essentially a fancy way of expressing $y_3$, the leftmost real of $\widehat{T_2}$ which is used in the standard uniformization argument. $T_2$ and $y_3$ are used in 31 to show that every nonempty $\Sigma^1_1$ set of reals contains a member which is recursive in $y_3$, or in our terminology, recursive in $0^{2#}$. Basis theorems can also be proved with inner model theory. If $M^\#_{2n-1}$ exists, then every nonempty $\Sigma^1_{2n+1}$ set of reals contains a member recursive in $M^\#_{2n-1}$ (cf. 47 51). At higher levels, the leftmost real basis arguments are investigated in 27. It is shown by Harrington (modulo Neeman 37 38) that under $\Delta^1_{2n}$-determinacy, there is a $\Delta^1_{2n+1}$-scale on a $\Delta^1_{2n+1}$ set whose leftmost real $y_{2n+1}$ is $\Delta^1_{2n+1}$-equivalent to $M^\#_{2n-1}$ and such that every nonempty $\Sigma^1_{2n+1}$ set contains a real recursive in $y_{2n+1}$. It is asked in 27 Conjecture 11.2 whether $y_{2n+1}$ is Turing equivalent to $M^\#_{2n-1}$. Theorem 4.8 solves this conjecture in the $n = 1$ case in an effective manner.

43
4.2 Homogeneity properties of $S_2$

By [27, Lemma 14.2], $L_{\delta_1^3}[T_2]$ is admissibly closed. We shall define a system of $L_{\delta_1^3}[T_2]$-measures on finite tuples in $u_\omega$. This system of $L_{\delta_1^3}[T_2]$-measures will witness $S_2$ being $L_{\delta_1^3}[T_2]$-homogeneous. Under AD, these $L_{\delta_1^3}[T_2]$-measures are total measures induced from the strong partition property on $\omega_1$ (cf. [26]). These measures enable the Martin-Solovay tree construction of $S_3$ projecting to the universal $\Pi^1_3$ set, to be redefined in Section 4.3. In our situation, we must recast the effective version of the proof of the strong partition property on $\omega_1$. Let $X^\uparrow$ be the set of strictly increasing functions $f : \omega_1 \to X$ that belong to $L$. Only functions in $L$ will be partitioned, and the partition must be guided by a $\Delta^1_3$ surjection from $\omega_1^\uparrow$ onto $u_\omega$ and a subset $A \subseteq u_\omega$ which lies in $L_{\delta_1^3}[T_2]$.

Every function $f \in \omega_1^\uparrow$ is of the form $\alpha \mapsto \tau L[x](x,\alpha)$ where $x \in R$ and $\tau$ is a Skolem term. Thus, sharp codes for increasing functions is a good coding system for $\omega_1^\uparrow$.

Definition 4.9. $\omega_1$ has the level-2 strong partition property iff for every function $\psi : \omega_1^\uparrow \to u_\omega$ such that the relation “$(\tau, x^\#)$ is a sharp code for an increasing function, $\alpha = \psi(\tau L[x](x,\cdot))$” is $\Delta^1_3$, for every $B \in L_{\delta_1^3}[T_2]$, there is $X \subseteq \omega_1$ such that $\omega_1 = o.t.(X) = X \in L$ and either $\psi''X^\uparrow \subseteq B$ or $\psi''X^\uparrow \subseteq u_\omega \setminus B$.

In most applications, $\psi$ will have the property that $\psi(f) = \psi(g)$ whenever $\forall \alpha < \omega_1 \sup f''\alpha = \sup g''\alpha$. The partition will be essentially on continuous functions only. In this case, when $X$ is the homogeneous set produced by Definition 4.9 so is the set of limit points of $X$. We will henceforth demand that the homogeneous set is a club in $\omega_1$.

Martin’s proof of the strong partition property on $\omega_1$ under AD carries over in a trivial way. For the reader’s convenience, we include a proof.

Theorem 4.10 (Martin). Assume $\Delta^1_2$-determinacy. Then $\omega_1$ has the level-2 strong partition property.

Proof. We imitate the proof in [19, Theorem 28.12], which builds on partially iterable sharps. We are given $\psi : \omega_1^\uparrow \to u_\omega$ whose complexity is $\Delta^1_3$ (in the sharp codes for increasing functions) and the target of the partition $B \in L_{\delta_1^3}[T_2]$. Define the game $G$ in which I produces $\langle \tau \rangle$, II produces $\langle \sigma \rangle$, $\langle \tau \rangle \in \omega_1^\uparrow$ and either $\psi''X^\uparrow \subseteq B$ or $\psi''X^\uparrow \subseteq u_\omega \setminus B$. An infinite run is won by Player II iff

1. If $\langle \tau \rangle$ is a putative sharp code for an increasing function, then so is $\langle \sigma \rangle$. Moreover, for any $\eta < \omega_1$, if $a^\ast$ is $\eta$-wellfounded $\land \tau^{M_{a^\ast}, \eta}(\eta) \in \text{wfp}(M_{a^\ast}, \eta)$
then
\[ b^* \text{ is } \eta\text{-wellfounded } \land \sigma^{M_{b^*,\eta}}(\eta) \in \operatorname{wfp}(M_{b^*,\eta}). \]

2. If \( \langle \tau, a^* \rangle, \langle \sigma, b^* \rangle \) are true sharp codes for increasing functions, \( a^* = a^\#, b^* = b^\# \), letting \( h(\eta) = \sup\{ \tau L[a](\omega\eta + n), \sigma L[b](\omega\eta + n) : n < \omega \} \) for \( \eta < \omega_1 \), then \( \psi(h) \in B \).

The payoff set of \( G \) is in \( L_{\aleph_1}[T_2] \) by Theorem 2.1 hence in \( \mathcal{E}(\omega^2, \Pi_1^1) \) by Lemma 2.11. Hence \( G \) is determined.

Suppose that player I has a winning strategy \( \varphi \) in \( G \). Let \( C \) be the set of \( \varphi \)-admissibles and limits of \( \varphi \)-admissibles. Similarly to the proof of Lemma 2.11 using boundedness, if \( \langle \tau, \varphi^\# \rangle \) is a true sharp code for an increasing function such that \( \forall \beta < \omega_1 \ \sigma L[\varphi](\beta) \in C \), then \( \langle \tau, \varphi^\# \rangle =_{\text{DEF}} f * \langle \tau, \varphi^\# \rangle \) is a true sharp code for an increasing function, and for any \( \eta \in C \), for any \( \beta \leq \eta \) such that \( \forall \beta < \beta \ \sigma L[\varphi](\beta) \leq \eta \), we have \( \tau L[\varphi](\beta) < \min(C \setminus \eta + 1) \).

Let \( e : \omega_1 \to C \) enumerate \( C \) in the increasing order and let \( X = \{ \sup_{n<\omega} \xi : \xi < \omega_1 \} \). We show that \( \varphi^\prime X \cup_{\omega_1} B \). Suppose that \( f \in X^\uparrow \). By definition of \( X \) there is a function \( g \in C^\uparrow \) such that \( f(\alpha) = \sup_{n<\omega} g(\omega\alpha + n) \) for any \( \alpha < \omega_1 \). Let \( b \in \mathbb{R} \) and \( \sigma \) be such that \( g(\eta) = \tau L[b](\eta) \) for any \( \eta < \omega \). Let \( g(\eta) = \tau L[b](\eta) \). Then the response \( \langle \tau, a^\# \rangle =_{\text{DEF}} f * \langle \tau, b^\# \rangle \) is a true sharp code for an increasing function, and for any \( \alpha < \omega_1 \), for any \( n < \omega \),
\[
\tau L[a](\omega\alpha + n) < \min(C \setminus (g(\omega\alpha + n) + 1)) \leq g(\omega\alpha + n + 1),
\]
where the last inequality follows from the fact \( g \in C^\uparrow \). Let \( h \) be given as in the definition of \( G \). Then \( h = f \). Since \( \varphi \) is a winning strategy for Player I, \( \psi(h) \notin B \).

A symmetrical argument shows that if Player II has a winning strategy in \( G \), then there is \( X \in \mathbb{L} \) which is cofinal in \( \omega_1 \) such that \( \varphi^\prime X \cup_{\omega_1} B = \emptyset \).

**Definition 4.11.** Let \( Q \) be a finite level \( \leq 2 \) tree. We define
\[
A \in \mu^Q
\]
iff there is \( C \in \mu_L \) such that
\[
[C]^{Q \uparrow} \subseteq A.
\]

\( \mu^Q \) is easily verified to be a countably complete filter concentrating on \( [\omega_1]^{Q \uparrow} \). In particular, when \( \operatorname{card}(Q) = 1 \), \( \mu^Q \) is the principal measure concentrating on \( \{ (\omega_1)(2,0) \} \). Noticing the facts that \( \operatorname{rep}(Q) \) has order type
ω₁ + 1, and that [f]Q depends only on \( \{ f(v) : \| v \| _{<\omega} \text{ is a limit ordinal} \} \). Theorem 4.10 implies that

\[ \mu^Q \text{ is an } \mathbb{L}^\delta_3[T_2]-\text{measure.} \]

Let \( j^Q = j^\mu_{\mathbb{L}^\delta_3[T_2]} \) be the restricted ultrapower map of \( \mu^Q \) on \( \mathbb{L}^\delta_3[T_2] \). Put

\[ [f]_{\mu^Q} = [f]_{\mu^Q_{\mathbb{L}^\delta_1[T_2]}} \text{ for } f \in \mathbb{L}^\delta_3[T_2]. \]

Loś' theorem reads: for any first order formula \( \varphi \), for any \( x \in \mathbb{R} \), for any \( f_i \in \mathbb{L}^\delta_3[T_2], 1 \leq i \leq n \),

\[ j^Q(L_{\kappa_3}[T_2, x]) \models \varphi([f_1]_{\mu^Q}, \ldots, [f_n]_{\mu^Q}) \]

iff

for \( \mu^Q\text{-a.e. } \vec{\xi}, L_{\kappa_3}[T_2, x] \models \varphi(f_1(\vec{\xi}), \ldots, f_n(\vec{\xi})). \)

**Lemma 4.12.** Assume \( \Delta^1_2 \)-determinacy. If \( P, W \) are finite level-1 trees, \( A \subseteq [\omega_1]^{W+} \times \mathbb{R} \) is \( \Pi^1_3 \) (or \( \Sigma^1_3, \Delta^1_3 \) resp.), then so are

\[ B = \{ x : \text{for } \mu^W\text{-a.e. } \vec{\alpha}, (\vec{\alpha}, x) \in A \}, \]

\[ C = \{ (\vec{\beta}, x) : \vec{\beta} \in j^P(A_x) \}, \]

where \( A_x = \{ \vec{\alpha} : (\vec{\alpha}, x) \in A \}. \)

**Proof.** If \( x \in B \) iff \( \exists y \forall \vec{\alpha} \in [\omega_1]^{W+} (\vec{\alpha} \text{ are } y\text{-admissibles } \rightarrow (\vec{\alpha}, x) \in A) \).

If \( x \notin B \) iff \( \exists y \forall \vec{\alpha} \in [\omega_1]^{W+} (\vec{\alpha} \text{ are } y\text{-admissibles } \rightarrow (\vec{\alpha}, x) \notin A) \). The quantifier \( \forall \vec{\alpha} \in [\omega_1]^{W+} \) does not increase the complexity due to Corollary 2.3. The complexity of \( C \) follows from that of \( B \) and Loś. \( \square \)

A purely descriptive set theoretic proof of Lemma 4.12 is given in [13, Lemma 4.40].

**Lemma 4.13.** Assume \( \Delta^1_2 \)-determinacy. Suppose \( Q \) is a finite level \( \leq 2 \) tree. If \( A \subseteq \omega_1 \times \mathbb{R} \) is \( \Pi^1_3 \) (or \( \Sigma^1_3, \Delta^1_3 \) resp.), then so is

\[ B = \{ (\vec{\beta}, x) : \vec{\beta} \in [A_x]^{Q_1} \}, \]

where \( A_x = \{ \alpha : (\alpha, x) \in A \}. \)

**Proof.** Put \( A'_x = \text{the set of limit points of } A_x \). By Lemma 3.15, \( \vec{\beta} \in [A_x]^{Q_1} \)

iff \( \vec{\beta} \) respects \( Q \) and for any \( q \in \downarrow Q \), \( \downarrow \beta_q \in A'_x \), for any \( q \in \text{dom}(Q), \)

\( \downarrow \beta_q \in j_{\text{tree}}(q)(A'_x). \)

Now apply Lemma 4.12 and Lemma 3.19. \( \square \)
Lemma 4.14. Assume $\Delta^1_2$-determinacy. Suppose $Q$ is a finite level $\leq 2$ tree. If $A \subseteq [\omega_1]^\omega \times \mathbb{R}$ is $\Pi^1_3$ (or $\Sigma^1_3$, $\Delta^1_3$ resp.), then so is

$$B = \{x : \text{for } \mu^Q \text{-a.e. } \beta, (\overline{\beta}, x) \in A\}.$$ 

Proof. Let $C = \{(y, \alpha) : \alpha < \omega_1 \wedge \alpha \text{ is } y \text{-admissible}\}$. $C$ is $\Delta^1_3$. Then $x \in B$ iff $\exists y \forall \overline{\beta} (\overline{\beta} \in [C_y]^\omega \rightarrow (\overline{\beta}, x) \in A)$. $x \notin B$ iff $\exists y \forall \overline{\beta} (\overline{\beta} \in [C_y]^\omega \rightarrow (\overline{\beta}, x) \notin A)$. Use Lemma 4.13 \qed

Corollary 4.15. Assume $\Delta^1_2$-determinacy. Suppose $Q$ is a finite level $\leq 2$ tree. Then $j^Q(\alpha) < \delta^1_3$ for any $\alpha < \delta^1_3$. $j^Q(T_2) \in \mathbb{L}_{\delta^1_3}[T_2]$.

Proof. By Lemma 4.14 for any $\alpha < \delta^1_3$, $j^Q(\alpha)$ is the length of a $\Delta^1_3$ prewellordering on $\mathbb{R}$. $j^Q(T_2) \in L_{\kappa_\alpha^\#}[T_2, M^\#]$ by Corollary 2.15 \qed

Corollary 4.15 is the effective version of [46 Corollary 3.9]. Actually, $j^Q(T_2)$ is $\Delta^1_3$ in the sharp codes, a fact to be shown in Section 4.5.

For $Q$ a finite level $\leq 2$ tree, by Corollary 4.15, $\mathbb{L}_{\delta^1_3}[j^Q(T_2)] = \text{Ult}(\mathbb{L}_{\delta^1_3}[T_2], \mu^Q)$.

If $Q$ is a subtree of $Q'$, both finite, then $\mu^Q$ projects to $\mu^{Q'}$ via the map that sends $(\beta_i)_{i<\omega} \in \text{dom}(Q')$ to $(\beta_i)_{i<\omega} \in \text{dom}(Q)$. Let

$$j^{Q,Q'} : \text{Ult}(\mathbb{L}_{\delta^1_3}[T_2], \mu^Q) \rightarrow \text{Ult}(\mathbb{L}_{\delta^1_3}[T_2], \mu^{Q'})$$

be the induced factor map. If $\overline{Q} = (Q_n)_{n<\omega}$ is a level $\leq 2$ tower, the associated $\mathbb{L}_{\delta^1_3}[T_2]$-measure tower $(\mu^{Q_n})_{n<\omega}$ is easily seen close to $\mathbb{L}_{\delta^1_3}[T_2]$.

If $(P, \overline{p}) = (P, (p_i)_{i<m})$ is a potential partial level $\leq 1$ tower, let $f \in B^{(P,\overline{p})}$ iff $f : [\omega_1]^\omega \rightarrow B$ is a function and

1. if $(P, \overline{p})$ is of continuous type, then the signature of $f$ is $(p_i)_{i<m}$, $f$ is essentially continuous;

2. if $(P, \overline{p})$ is of discontinuous type, then the signature of $f$ is $(p_i)_{i<m-1}$, $f$ has uniform cofinality $\text{ucf}(P, \overline{p})$.

Let $\beta \in [B]^{(P,\overline{p})}$ iff $\beta = [f]_{\mu^P}$ for some $f \in B^{(P,\overline{p})}$.

$$\mu^{(P,\overline{p})}$$

is the $\mathbb{L}_{\delta^1_3}[T_2]$-measure where $A \in \mu^{(P,\overline{p})}$ iff there is $E \subseteq \mu_{\mathbb{L}}$ such that $[E]^{(P,\overline{p})} \subseteq A$. $\beta$ respects $(P, \overline{p})$ iff $\beta \in [\omega_1]^{(P,\overline{p})}$. Let $j^{(P,\overline{p})} = j^{(P,\overline{p})}_{\mathbb{L}_{\delta^1_3}[T_2]}$ be the induced $\mathbb{L}_{\delta^1_3}[T_2]$-ultrapower map. Let $\text{seed}^{(P,\overline{p})}$ be represented modulo $\mu^{(P,\overline{p})}$ by the identity map.
If \((d, q) \in \text{desc}(Q)\), let \(\mu^{(d,q)} = \mu_{\|} \) if \(d = 1\); \(\mu^{(d,q)} = \mu^{(P,\bar{p})}\) if \(d = 2\), \(q = (q, P, \bar{p})\). Thus, \(\mu^{(d,q)}\) projects to \(\mu^{(d,q)}\) via the map \(\bar{\beta} \mapsto d^{\beta}_{q}\), i.e., \(A \in \mu^{(d,q)}\) iff \(\{ \beta : d^{\beta}_{q} \in A \} \in \mu^{Q}\). (Recall the definition of \(d^{\beta}_{q}\) from \(\bar{\beta}\) in Section 3.3.)

Let \((d, q)^{Q}\) be the induced factor map, so that \(j^{Q} = (d, q)^{Q} \circ j_{\mu_{\|}}\) if \(d = 1\), \(j^{Q} = (d, q)^{Q} \circ j^{(P,\bar{p})}\) if \(d = 2\). Let

\[
\text{seed}^{Q}_{(d,q)}
\]

be represented modulo \(\mu^{Q}\) by the map \(\bar{\beta} \mapsto d^{\beta}_{q}\).

The homogeneity property of the Martin-Solovay tree on a \(\Pi^{1}_{2}\) set (cf. [26]) translates to our context:

**Theorem 4.16.** Assume \(\Delta^{1}_{2}\)-determinacy. Let \(\bar{Q} = (Q_{n})_{n<\omega}\) be an infinite level-2 tower. Let \(Q_{\omega} = \cup_{n<\omega}Q_{n}\). The following are equivalent.

1. \(Q_{\omega}\) is \(\Pi^{1}_{1}\)-wellfounded.
2. \(\text{<_Q}\) is a wellordering.
3. There is \(\bar{\beta} = (\beta_{t})_{t \in \text{dom}(Q_{\omega})}\) which respects \(Q_{\omega}\).
4. \((\mu^{Q_{n}})_{n<\omega}\) is \(L_{4}[T_{2}]\)-countably complete.
5. The direct limit of \((j^{Q_{m},Q_{n}})_{m<n<\omega}\) is wellfounded.

**Proof.** 1 \(\iff\) 2: By Proposition 3.9.

2 \(\Rightarrow\) 4: Suppose \(\text{<_Q}\) is a wellordering. Let \((A_{n})_{n<\omega}\) be such that \(A_{n} \in \mu^{Q_{n}} \cap L_{4}[T_{2}]\). Let \(x \in \mathbb{R}\) and \(C \in L[x]\) be a club in \(\omega_{1}\) such that \([C]^{Q_{n}^+} \subseteq A_{n}\) for all \(n\). Let \(f : \text{dom}(\text{<_Q}) \to C\) be given by

\[
f(\bar{\alpha} \oplus_{Q_{\omega}} t) = \|\bar{\alpha} \oplus_{Q_{\omega}} t\|_{<_Q}\text{-th element of }C.
\]

Then \(f \in L[x, Q_{\omega}]\) and is order preserving. Let \(\beta_{n} = [f \upharpoonright \text{rep}(Q_{n})]^{Q_{n}}\). Then for all \(n\), \((\beta_{1}, \ldots, \beta_{n}) \in A_{n}\).

4 \(\Rightarrow\) 3: This follows from the fact that \(\mu^{Q_{n}}\) concentrates on tuples that respect \(Q_{n}\).

3 \(\Rightarrow\) 1: If \(x \in \text{dom}(Q_{\omega})\), then \(j^{Q_{\omega}(x^{R}),Q_{\omega}(x^{R})} (\beta_{x^{R}}) > \beta_{x^{R}}\) for all \(k < l < \omega\). This means the direct limit of \(j^{Q_{\omega}(x^{R}),Q_{\omega}(x^{R})}\) is illfounded. Hence \(Q_{\omega}(x)\) is not \(\Pi^{1}_{2}\)-wellfounded by Theorem 3.3.

4 \(\iff\) 5: By Proposition 2.17.

**Definition 4.17.** \(Q^{0}, Q^{1}, Q^{20}, Q^{21}\) denote the following typical level \(\leq 2\) trees of cardinalities at most 2:
\[ \begin{align*}
\mathcal{Q}^0 &= \emptyset, \\
\mathcal{Q}^1 &= \{0\}, \\
\text{dom}(\mathcal{Q}^0) &= \text{dom}(\mathcal{Q}^1) = \emptyset.
\end{align*} \]

For \( d \in \{0, 1\} \), \( \mathcal{Q}^{2d} = \emptyset \), \( \text{dom}(\mathcal{Q}^{2d}) = \{0, ((0))\} \), \( \mathcal{Q}^{2d}((0)) \) is of degree \( d \).

\( \mu^{\mathcal{Q}^0} \) is a principle measure. \( \mu^{\mathcal{Q}^1} \) is essentially \( \mu_L \). \( \mu^{\mathcal{Q}^{20}} \) and \( \mu^{\mathcal{Q}^{21}} \) are essentially refinements of the \( L_{\delta_1,T_2} \)-club filter on \( u_2 \), the former concentrates on ordinals of \( L_{\delta_1,T_2} \)-cofinality \( \omega \), the latter of \( L_{\delta_1,T_2} \)-cofinality \( \omega_1 \).

### 4.3 The tree \( S_3 \)

A partial level \( \leq 2 \) tree is a pair \((Q, (d, q, P))\) such that \( Q \) is a finite level \( \leq 2 \) tree, and one of the following holds:

1. \((d, q, P) = (0, -1, \emptyset)\), or
2. \( d = 1 \), \( q \notin \mathcal{Q}, \mathcal{Q} \cup \{q\} \) is a level-1 tree, \( P = \emptyset \), or
3. \( d = 2 \), \( q \notin \text{dom}(\mathcal{Q}) \), \( \text{dom}(\mathcal{Q}) \cup \{q\} \) is tree of level-1 trees, \( P \) is the completion of \( \mathcal{Q}(q^-) \). (In particular, \( \mathcal{Q}(q^-) \) must have degree 1.)

The degree of \((Q, (d, q, P))\) is \( d \). We put \( \text{dom}(Q, (d, q, P)) = \text{dom}(Q) \cup \{(d, q)\} \). The cardinality of \((Q, (d, q, P))\) is \( \text{card}(Q, (d, q, P)) = \text{card}(Q) + 1 \). The uniform cofinality of a partial level \( \leq 2 \) tree \((Q, (d, q, P))\) is

\[ \text{ucf}(Q, (d, q, P)), \]

defined as follows.

1. \( \text{ucf}(Q, (d, q, P)) = (0, -1) \) if \( d = 0 \);
2. \( \text{ucf}(Q, (d, q, P)) = (1, q^-) \) if \( d = 1 \), \( \text{lh}(q) > 1 \);
3. \( \text{ucf}(Q, (d, q, P)) = (2, (\emptyset, \emptyset, \emptyset)) \) if \( d = 1 \), \( \text{lh}(q) = 1 \);
4. \( \text{ucf}(Q, (d, q, P)) = (2, (q', P, \vec{p})) \) if \( d = 2 \), \( \mathcal{Q}[q^-] = (P^-, \vec{p}) \), and \( q' \) is the \( <_{BK} \)-least element of \( \mathcal{Q}\{q, +\} \).

So \( \text{ucf}(Q, (d, q, P)) \) is either \((0, -1)\) or a regular extended \( Q \)-description. The cofinality of \((Q, (d, q, P))\) is

\[ \text{cf}(Q, \overrightarrow{(d, q, P)}) = \begin{cases} 0 & \text{if } d = 0, \\
1 & \text{if } d = 1 \text{ and } q = \min(\mathcal{Q}(q)), \\
2 & \text{otherwise.} \end{cases} \]
A tuple $\vec{\beta} = (\beta_t)_{(t,\ell) \in \text{dom}(Q,(d,q,P))}$ respects $(Q,(d,q,P))$ iff $\vec{\beta} \upharpoonright \text{dom}(Q)$ respects $Q$ and $d \beta_q < \omega$ if $d = 0$, $d \beta_q < d \beta_k$ if $d > 0$ and $\text{ucf}(Q,(d,q,P)) = (d,q)$. A partial level $\leq 2$ tree of degree $0$ has no completion. A completion of a partial level $\leq 2$ tree $(Q,(d,q,P))$ of degree $\geq 1$ is a level $\leq 2$ tree $Q^*$ such that $\text{dom}(Q^*) = \text{dom}(Q,(d,q,P))$, $Q^* \upharpoonright \text{dom}(Q) = Q$, and either $d = 1$ or $d = 2 \land Q\text{tree}(t) = P$. For a level $\leq 2$ tree $Q'$, $(Q,(d,q,P))$ is a partial subtree of $Q'$ iff a completion of $(Q,(d,q,P))$ is a subtree of $Q'$.

A partial level $\leq 2$ tower of discontinuous type is a nonempty finite sequence $(Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ such that $\text{card}(Q_1) = 1$, each $(Q_i, (d_i, q_i, P_i))$ is a partial level $\leq 2$ tree, and each $Q_{i+1}$ is a completion of $(Q_i, (d_i, q_i, P_i))$. Its signature is $(d_i, q_i)_{1 \leq i \leq k}$. Its uniform cofinality is $\text{ucf}(Q_k, (d_k, q_k, P_k))$. A partial level $\leq 2$ tower of continuous type is $(Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}^\ast$ such that either $k = 0 \land Q_1$ is the level $\leq 2$ tree of cardinality $1$ or $(Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a partial level $\leq 2$ tower of discontinuous type $\wedge Q_1$ is a completion of $(Q_{k-1}, (d_{k-1}, q_{k-1}, P_{k-1}))$. Its signature is $(d_i, q_i)_{1 \leq i \leq k}$. If $k > 0$, its uniform cofinality is $(1, q_{k-1})$ if $d_{k-1} = 1$, $(2, (q_{k-1}, P, \vec{p}))$ if $d_{k-1} = 2$, $Q[q_{k-1}] = (P, \vec{p})$. For notational convenience, the information of a partial level $\leq 2$ tower is compressed into a potential partial level $\leq 2$ tower. A potential partial level $\leq 2$ tower is $(Q_*, (d,q,P)) = (Q, (d_i, q_i, P_i))_{1 \leq i \leq \text{lh}(\vec{q})}$ such that for some level $\leq 2$ tower $\vec{Q} = (Q_i)_{1 \leq i \leq k}$, either $Q_1 = Q_1 \wedge (\vec{Q}, (d, q, P))$ is a partial level $\leq 2$ tower of discontinuous type or $(\vec{Q}, (d, q, P)) \upharpoonright Q_* (Q_*)$ is a partial level $\leq 2$ tower of continuous type. The signature, (dis-)continuity type, uniform cofinality of $(Q_*, (d,q,P))$ are defined according to the partial level $\leq 2$ tree generating $(Q_*, (d,q,P))$.

$$\text{ucf}(Q_*, (d,q,P))$$

denotes the uniform cofinality of $(Q_*, (d,q,P))$. Clearly, a potential partial level $\leq 2$ tower $(Q_*, (d,q,P))$ is of continuous type iff $\text{card}(Q_*) = \text{lh}(\vec{q})$, of discontinuous type iff $\text{card}(Q_*) = \text{lh}(\vec{q}) - 1$.

**Definition 4.18.** A level-3 tree of uniform cofinality, or level-3 tree, is a function $R$ such that $\emptyset \notin \text{dom}(R)$, $\text{dom}(R) \cup \{\emptyset\}$ is tree of level-1 trees and for any $r \in \text{dom}(R)$, $(R(r \upharpoonright l))_{1 \leq l \leq \text{lh}(r)}$ is a partial level $\leq 2$ tower of discontinuous type. If $R(r) = (Q_r, (d_r, q_r, P_r))$, we denote $R_{\text{tree}}(r) = Q_r$, $R_{\text{node}}(r) = (d_r, q_r)$, $R[r] = (Q_r, (d_r, q_r, P_r))_{1 \leq i \leq \text{lh}(r)}$. $R[r]$ is a potential partial level $\leq 2$ tower of discontinuous type. If $Q$ is a completion of $R(r)$, put $R[r, Q] =$
(Q, (d_q, q_q, P_q))_{1 \leq l \leq \text{lh}(r)}, which is a potential partial level \leq 2 tower of continuous type. For r ∈ dom(R) \cup \{\emptyset\}, put R\{r\} = \{a ∈ \omega^\omega : r^\frown(a) ∈ dom(R)\}, which is a level-1 tree.

The cardinality of R is card(R) = card(dom(R)). R is said to be regular iff ((1)) \notin dom(R). In other words, when R \neq \emptyset, ((0)) is the <_{BK}-maximum of dom(R).

Suppose R is a level-3 tree. Let dom*(R) = dom(R) \cup \{r^\frown(-1) : r ∈ dom(R)\}. For r ∈ dom(R), put R\{r, -\} = \{r^\frown(-1)\} \cup \{r^\frown(a) : R\text{tree}(r^\frown(a)) = R\text{tree}(r), a <_{BK} r(\text{lh}(r) - 1)\}, R\{r, -\} = \{r\} \cup \{r^\frown(a) : R\text{tree}(r^\frown(a)) = R\text{tree}(r), a >_{BK} r(\text{lh}(r) - 1)\}.

If \vec{\beta} = (d_\beta, (d,q))_{\in N} is a tuple indexed by N, r ∈ dom*(R), \text{lh}(r) = k, either k = 1 or dom(R(r^-)) \subseteq N, we put

$$\vec{\beta} \oplus_R r = (r(0), d_\beta q_1, r(1), \ldots, d_\beta q_{k-1}, r(k-1)),\$$

where (d_i, q_i) = R_{\text{node}}(r \upharpoonright i). The ordinal representation of R is the set

$$\text{rep}(R) = \{\vec{\beta} \oplus_R r : r ∈ dom(R), \vec{\beta} \text{ respects } R\text{tree}(r)\}
\cup \{\vec{\beta} \oplus_R r^\frown(-1) : r ∈ dom(R), \vec{\beta} \text{ respects } R(r)\}.$$

\text{rep}(R) is endowed with the <_{BK} ordering

$$<^R = <_{BK}\text{rep}(R).$$

R is \(\Pi_3^1\)-wellfounded iff

1. \(\forall r ∈ \text{dom}(R) \cup \{\emptyset\} \ R\{r\} \text{ is } \Pi_3^1\text{-wellfounded}, \text{ and}\)
2. \(\forall z ∈ [\text{dom}(R)] \ R(z) =_{\text{DEF}} \cup_{n<\omega}( R\text{tree}(z \upharpoonright n))_{1 \leq n < \omega} \text{ is not } \Pi_3^2\text{-wellfounded.}\)

For level-3 trees R and R', R is a subtree of R' iff R is a subfunction of R'. A finite level-3 tower is a sequence \((R_i)_{i \leq n}\) such that n < \omega, each R_i is a regular level-2 tree, \text{card}(R_i) = i + 1 and i < j → R_i is a subtree of R_j. \(\vec{R}\) is regular iff each R_i is regular. An infinite level-3 tower is a sequence \(\vec{R} = (R_n)_{n<\omega}\) such that for each n, \((R_i)_{i \leq n}\) is a finite level-3 tower. \(\Pi_3^1\)-wellfoundedness of a level-3 tower is a \(\Pi_3^1\) property in the real coding the tower. In particular, every finite level-3 tree is \(\Pi_3^1\)-wellfounded. Similarly to Proposition 3.9 we have

**Proposition 4.19.** Assume \(\Delta_3^1\)-determinacy. Suppose R is a level-3 tree. Then R is \(\Pi_3^1\)-wellfounded iff \(<^R\) is a wellordering.
Associated to a $\Pi_3^1$ set $A$ we can assign a level-3 system $(R_s)_{s \in \omega^{<\omega}}$ so that $x \in A$ iff the infinite level-3 tree $R_x = \text{DEF } \bigcup_{n<\omega} R_{2^n}$ is $\Pi_3^1$-wellfounded. If $A$ is lightface $\Pi_3^1$, then $(R_s)_{s \in \omega^{<\omega}}$ can be picked effective.

Suppose $F \in L_{\delta_1^1}[T_2]$ is a function on $\text{rep}(R)$, $r \in \text{dom}(R)$. Then $F_r$ is a function on $\omega_{\text{tree}(r)}^{\uparrow}$ that sends $\vec{\beta}$ to $F(\vec{\beta} \oplus_R r)$. $F$ represents a card($R$)-tuple of ordinals

$$[F]^R = ([F]^R)_r \in \text{dom}(R)$$

where $[F]^R = [F_r]_{\mu_{\text{tree}(r)}}$ for $r \in \text{dom}(R)$. If $B \subseteq \delta_3^1$, put

$$F \in B^{R^\uparrow}$$

iff $F \in L_{\delta_1^1}[T_2]$ and $F$ is an order-preserving continuous function from $\text{rep}(R)$ to $B$ (with respect to $<_R$ and $\prec$). Let

$$[B]^{R^\uparrow} = \{ [F]^R : F \in B^{R^\uparrow} \}.$$ 

A tuple of ordinals $\vec{\gamma} = (\gamma_r)_{r \in \text{dom}(R)}$ is said to respect $R$ iff $\vec{\gamma} \in [\delta_3^1]^{R^\uparrow}$. $\vec{\gamma}$ is said to weakly respect $R$ iff for any $t, t' \in \text{dom}(R)$, if $t$ is a proper initial segment of $t'$, then $j_{R_{\text{tree}(t)}, R_{\text{tree}(t')}}(\gamma_t) > \gamma_{t'}$. By virtue of the order $<_R$, if $\vec{\gamma}$ respects $R$, then $\vec{\gamma}$ weakly respects $R$ and whenever $R_{\text{tree}(t^{-}(p))} = R_{\text{tree}(t^{-}(q))}$ and $p < q$, then $\gamma_{t^{-}(p)} < \gamma_{t^{-}(q)}$.

The trees $S_3^{-}$ and $S_3$ are defined in [26]. They both project to the universal $\Pi_3^1$ set. In our language, they take the following form.

**Definition 4.20.** Assume $\Delta_2^1$-determinacy.

1. $S_3^{-}$ is the tree on $V_\omega \times \delta_3^1$ such that $(\emptyset, \emptyset) \in S_3^{-}$ and

$$(R_t, \vec{\alpha}) = ((R_i)_{i \leq n}, (\alpha_i)_{i \leq n}) \in S_3^{-}$$

iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, $\beta_r = \alpha_{i+1}$, then $(\beta_r)_{r \in \text{dom}(R_n)}$ respects $R_n$.

2. $S_3$ is the tree on $V_\omega \times \delta_3^1$ such that $(\emptyset, \emptyset) \in S_3$ and

$$(R_t, \vec{\alpha}) = ((R_i)_{i \leq n}, (\alpha_i)_{i \leq n}) \in S_3$$

iff $\vec{R}$ is a finite regular level-3 tower and letting $r_i \in \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, $\beta_r = \alpha_{i+1}$, then $(\beta_r)_{r \in \text{dom}(R_n)}$ weakly respects $R_n$.

By Theorem 4.16

$$p[S_3] = p[S_3] = \{ \vec{R} : \vec{R} \text{ is a } \Pi_3^1\text{-wellfounded level-3 tower} \}.$$ 

52
The (non-regular) scale associated to $S_3$ is $\Pi^1_3$. For $\xi < \delta^1_3$, put $(\tilde{R}, \tilde{\alpha}) \in S_3 \upharpoonright \xi$ iff $(\tilde{R}, \tilde{\alpha}) \in S_3$ and $(\tilde{R}, \tilde{\alpha}) \neq (\emptyset, \emptyset) \rightarrow \alpha_0 < \xi$.

The properties of a tuple respecting $R$ is decided by the signature, approximation sequence and relative ordering of its entries, in a parallel way to the level-2 case. It is handled in [12]. We state the results in our language.

For level $\leq 2$ trees $Q, X$, we say that $\pi : \text{dom}(X) \rightarrow \text{dom}(Q)$ factors $(X, Q)$ iff putting $(d, \pi(x)) = \pi(d, x)$ for $(d, x) \in \text{dom}(X)$,

1. $\pi$ factors $(X, Q)$;
2. if $x \in \text{dom}(X)$ then $2^X(x) = 2^Q(\pi(x))$;
3. if $x, x' \in \text{dom}(X)$, then $x <_{BK} x' \rightarrow 2^\pi(x) <_{BK} 2^\pi(x'), x \subseteq x' \rightarrow 2^\pi(x) \subseteq 2^\pi(x')$.

For $d \in \{1, 2\}$, $\delta^1_d$ has this fixed meaning if $\pi$ factors $(Q, X)$. Extend the definition of $\delta^1_\pi$ on dom $^*(Q)$ and desc $^*(Q)$ is the following natural way: if $q^-(-1) \in \text{dom}^*(Q)$, define $2^\pi(q^-(-1)) = \pi(q^-(-1))$; if $q = (q, P, \tilde{p}) \in \text{desc}^*(Q)$, define $2^\pi(q) = (\pi(q), P, \tilde{p})$. If $\tilde{\beta} = (d\beta_q)_{d \in \text{dom}(Q)} \in [\omega_1]^Q$, put $\bar{\beta}_\pi = (d\beta_{\pi,x})_{(d, x) \in \text{dom}(X)} \in [\omega_1]^X$ where $d\beta_{\pi,x} = d\beta_{\pi(x)}$.

**Definition 4.21.** Suppose $Q$ is a finite level $\leq 2$ tree, $(\bar{d}, \bar{q}) = ((d_i, q_i))_{1 \leq i < k}$ is a distinct enumeration of a subset of $Q$ and such that $\{q_i : d_i = 2\} \cup \{\emptyset\}$ forms a tree on $\omega^{<\omega}$. Suppose $F : [\omega_1]^Q \rightarrow \delta^1_3$ is a function which lies in $L_{\delta^1_3}[T_2]$. The signature of $F$ is $(\bar{d}, \bar{q})$ iff there is $E \in \mu_1$ such that

1. for any $\bar{\beta}, \bar{\gamma} \in [E]^Q$, if $(d\beta_{q_0}, \ldots, d_{k-1}\gamma_{q_{k-1}}) <_{BK} (d\beta_{q_0}, \ldots, d_{k-1}\beta_{q_{k-1}})$ then $f(\bar{\beta}) < f(\bar{\gamma})$;
2. for any $\bar{\beta}, \bar{\gamma} \in [E]^Q$, if $(d\beta_{q_0}, \ldots, d_{k-1}\gamma_{q_{k-1}}) = (d\beta_{q_0}, \ldots, d_{k-1}\beta_{q_{k-1}})$ then $f(\bar{\beta}) = f(\bar{\gamma})$.

Clearly the signature of $F$ exists and is unique. In particular, $F$ is constant on a $\mu^Q$-measure one set iff the signature of $F$ is $\emptyset$.

Suppose the signature of $F$ is $(\bar{d}, \bar{q}) = ((d_i, q_i))_{1 \leq i < k}$. $F$ is essentially $\text{continuous}$ if for $\mu^Q$-a.e. $\bar{\beta}$, $F(\bar{\beta}) = \sup\{F(\bar{\gamma}) : \bar{\gamma} \in [\omega_1]^Q, (d\beta_{q_0}, \ldots, d_{k-1}\gamma_{q_{k-1}}) <_{BK} (d\beta_{q_0}, \ldots, d_{k-1}\beta_{q_{k-1}})\}$. Otherwise, $F$ is essentially discontinuous. Put $[\omega_1]^Q \times \omega = [\omega_1]^Q \times \omega$. For $(d, q) \in \text{desc}^*(Q)$ regular, put $[\omega_1]^Q(d, q) = \{(\beta, \gamma) : \bar{\beta} \in [\omega_1]^Q, \gamma < d\beta_q\}$. For $(d, q)$ either $(0, -1)$ or in desc $^*(Q)$ regular, say that the uniform cofinality of $F$ is $\text{ucf}(F) = (d, q)$ iff there is $G : [\omega_1]^Q(d, q) \rightarrow \delta^1_3$ such that $G \in L_{\delta^1_3}[T_2]$ and for any for $\mu^Q$-a.e. $\bar{\beta}$, $F(\bar{\beta}) = \sup\{G(\bar{\beta}, \gamma) : (\beta, \gamma) \in$
and the function $\gamma \mapsto G(\tilde{\beta}, \gamma)$ is order preserving. The cofinality of $F$ is

$$\text{cf}(F) = \begin{cases} 
0 & \text{if } \text{ucf}(F) = (0, -1), \\
1 & \text{if } \text{ucf}(F) = (1, q), q = \text{min}(\omega^Q), \\
2 & \text{otherwise}.
\end{cases}$$

Let $(X_i, (d_i, x_i, W_i)) \sim (X_k)$ be the partial level $\leq 2$ tower of continuous type and let $\pi$ factor $(X_k, Q)$ such that $\pi(d_i, x_i) = (d_i, q_i)$ for $1 \leq i < k$. The potential partial level $\leq 2$ tower induced by $F$ is

1. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k}$, if $F$ is essentially continuous;
2. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k} \sim ((0, -1), \emptyset)$, if $F$ is essentially discontinuous and has uniform cofinality $(0, -1)$;
3. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k} \sim ((1, x^+), \emptyset)$, if $F$ is essentially discontinuous and has uniform cofinality $(1, q_*)$, $(X_k, (1, x^+, \emptyset))$ is a partial level $\leq 2$ tree, $\pi(1, (x^+)^-) = (1, q_*)$;
4. $(X_k, (d_i, x_i, W_i))_{1 \leq i < k} \sim ((2, x^+, P_*), (q_*, P_*, \tilde{\beta}), (X_k, (2, x^+, P_*))))$ is a partial level $\leq 2$ tree, and either

(a) $q_* \in \text{desc}(\omega^Q)$, $x^+ = (x^+)^- \text{ or } \pi(2, (x^+)^- \text{ or } (a^-)) = (2, q_*)$, or

(b) $q_* \notin \text{desc}(\omega^Q)$, $\pi(2, (x^+)^-) = (2, q_*)$.

The approximation sequence of $F$ is $(F_i)_{1 \leq i \leq k}$ where $F_i$ is a function on $[\omega]^X_i$, $F_i(\tilde{\beta}) = \sup\{ F(\gamma) : \gamma \in [\omega]^Q_i, (d_i, \gamma_{q_1}, \ldots, d_i, \gamma_{q_{i-1}}) = (\alpha_{\tilde{\beta}_{x_1}}, \ldots, \alpha_{\tilde{\beta}_{x_{i-1}}}) \}$ for $1 \leq i \leq k$.

The existence and uniqueness of the uniform cofinality of $F$ will be proved in Section 4.5. In particular, if $R$ is a level-3 tree, $H \in (\delta^1_3)^{R_\uparrow}$, then for any $r \in \text{dom}(R)$, $H_r$ has signature $(R_{\text{node}}(r | i))_{1 \leq i \leq \text{lh}(r)}$, is essentially discontinuous, has uniform cofinality $\text{ucf}(R(r))$ and cofinality $\text{cf}(R(r))$, induces the potential partial level $\leq 2$ tower $R[r]$, and $(H_{rf})_{1 \leq i \leq \text{lh}(r)}$ is the approximation sequence of $H_r$. Again, all the relevant properties of $F$ depends only on the value of $F$ on a $\mu^Q$-measure one set. We will thus be free to say the signature, etc. of $F$ when $F$ is defined on a $\mu^Q$-measure one set.

**Definition 4.22.** Suppose $\omega_1 \leq \gamma < \delta^1_3$ is a limit ordinal. Suppose $Q$ is a finite level $\leq 2$ tree, $\gamma = [F]_{\mu^Q}$, the signature of $F$ is $((d_i, q_i))_{1 \leq i < k}$, the approximation sequence of $F$ is $(F_i)_{1 \leq i \leq k}$. Then the $Q$-signature of $\beta$ is $((d_i, q_i))_{1 \leq i < k}$, the $Q$-approximation sequence of $\gamma$ is $([F]_{\mu^Q})_{1 \leq i \leq k}$.
Q-essentially continuous iff $F$ is essentially continuous. The Q-uniform cofinality of $\gamma$ is $\omega$ if $F$ has uniform cofinality $(0, -1)$, seed$^{2}_{d,q}$ if $f$ has uniform cofinality $(d, q) \in \text{desc}^*(Q)$. The $Q$-potential partial level $\leq 2$ tower induced by $\gamma$ is the potential partial level $\leq 2$ tower induced by $F$.

In Section 4.5, we will show that all the relevant properties in Definition 4.22 are independent of the choice of $F$ (but depends on $Q$ of course). We will also show that the $Q$-uniform cofinality of $\gamma$ is exactly $\text{cf}^{|T_2 \cup T^*(T_2)}(\gamma)$, and $\text{cf}^{|\gamma|_{T_2}}(\gamma) = u_{\text{cl}(F)}$, where we set $u_0 = \omega$.

**Definition 4.23.** We fix the notations for all the level-3 trees of cardinality 1. For $d \in \{0, 1, 2\}$, $\text{dom}(R^d) = \{(0)\}$ and $R^d((0))$ is of degree $d$.

### 4.4 Level-2 description analysis

If $Q$ is a level-2 tree, $q = (q, P, \vec{p}) \in \text{desc}(Q)$, $\text{lh}(q) = k$, $\vec{p} = (p_i)_{i < \text{lh}(\vec{p})}$, $\sigma$ is a function whose domain contains $P$, we put

$$\sigma \oplus q = \sigma \oplus Q q = (\sigma(p_0), q(0), \ldots, \sigma(p_{k-1}), q(k-1)).$$

**Definition 4.24.** Suppose $W$ is a finite level-1 tree and suppose $Q$ is a level $\leq 2$ tree. A $(Q, W)$-description is of the form $D = (d, (q, \sigma))$ such that either

1. $d = 1$, $q \in \langle Q \rangle$, $\sigma = \emptyset$, or
2. $d = 2$, $q = (q, P, \vec{p}) \in \text{desc}(\langle Q \rangle)$, $\sigma$ factors $(P, W)$.

$\text{desc}(Q, W)$ is the set of $(Q, W)$-descriptions. A $(Q, *)$-description is a $(Q, W')$-description for some finite level-1 tree $W'$. $\text{desc}(Q, *)$ is the set of $(Q, *)$-descriptions. We sometimes abbreviate $(d, q, \sigma)$ for $(d, (q, \sigma)) \in \text{desc}(Q, W)$ without confusion.

Suppose now $D = (d, q, \sigma)$ and if $d = 2$, then $q = (q, P, \vec{p})$, $\vec{p} = (p_i)_{i < \text{lh}(\vec{p})}$, $\text{lh}(q) = k$. The degree of $D$ is $d$. The level-1 signature of $D$ is

$$\text{sign}_1(D) = \begin{cases} \emptyset & \text{if } d = 1, \\ (\sigma(p_i))_{i < k} & \text{if } d = 2. \end{cases}$$

$D$ is of level-1 continuous type iff $d = 2$ and $q$ is of continuous type; otherwise, $D$ is of level-1 discontinuous type. The level-1 uniform cofinality of $D$ is

$$\text{ucf}_1(D) = \begin{cases} -1 & \text{if } d = 1 \lor (d = 2 \land \text{ucf}(P, \vec{p}) = -1), \\ \sigma(\text{ucf}(P, \vec{p})) & \text{if } d = 2 \land \text{ucf}(P, \vec{p}) \neq -1. \end{cases}$$
The level-2 signature of \( D \) is

\[
\text{sign}_2(D) = \begin{cases} 
((1, q)) & \text{if } d = 1, \\
((2, q \upharpoonright i))_{1 \leq i \leq k-1} & \text{if } d = 2, q \text{ of continuous type}, \\
((2, q \upharpoonright i))_{1 \leq i \leq k} & \text{if } d = 2, q \text{ of discontinuous type}.
\end{cases}
\]

\( D \) is of level-2 \( W \)-continuous type iff \( d = 2 \) and if \( \text{ucf}(P, \vec{p}) \neq -1 \land \sigma(\text{ucf}(P, \vec{p})) \neq \min(\omega^W) \), then \( \text{pred}_{\omega^W}(\sigma(\text{ucf}(P, \vec{p}))) \in \text{ran}(\sigma) \). Otherwise, \( D \) is of level-2 \( W \)-discontinuous type. The level-2 \( W \)-uniform cofinality of \( D \) is

\[
\text{ucf}_{2}^{W}(D)
\]

defined as follows. If \( d = 1 \), then \( \text{ucf}_{2}^{W}(D) = (1, q) \). If \( d = 2 \), \( q \) is of continuous type,

1. if \( D \) is of level-2 \( W \)-continuous type, then \( \text{ucf}_{2}^{W}(D) = (2, (q^-, P \setminus \{p_{k-1}\}, \vec{p})) \);
2. if \( D \) is of level-2 \( W \)-discontinuous type, then \( \text{ucf}_{2}^{W}(D) = (2, (q^-, P, \vec{p})). \)

If \( d = 2 \), \( q \) is of discontinuous type,

1. if \( D \) is of level-2 \( W \)-continuous type, then \( \text{ucf}_{2}^{W}(D) = (2, q) \);
2. if \( D \) is of level-2 \( W \)-discontinuous type, then \( \text{ucf}_{2}^{W}(D) = (2, (q, P \cup \{p_{k}\}, \vec{p})). \)

The constant \((Q, \ast)\)-description is \((2, (\emptyset, \emptyset, \emptyset), \sigma_{0})\) where \( \sigma_{0} \) is the unique that factors \((\emptyset, \ast)\), i.e., \( \sigma_{0}(\emptyset) = \emptyset \).

Note that if \( D \in \text{desc}(Q, W) \) and \( W \) is a subtree of \( W' \), then \( D \in \text{desc}(Q, W') \), but \( \text{ucf}_{2}^{W}(D) \) could be different from \( \text{ucf}_{2}^{W'}(D) \). If \( Q \) is finite, there are in total

\[
\text{card}(Q) + \sum_{q \in \text{dom}(Q)} \left( \frac{\text{card}(W)}{\text{lh}(q)} \right) + \sum_{Q(q) \text{ of degree 1}} \left( \frac{\text{card}(W)}{\text{lh}(q) + 1} \right)
\]

many \((Q, W)\)-descriptions. We shall establish an exact correspondence between \( \text{desc}(Q, W) \) and uniform indiscernibles \( \leq j^{Q} \circ j^{W}(\omega_{1}) \).

Suppose \( \tilde{D} = (d, q, \sigma) \in \text{desc}(Q, W) \), and if \( d = 2 \), then \( q = (q, P, \vec{p}), \vec{p} = (p_{i})_{i < \text{lh}(\vec{p})}, \text{lh}(q) = k \). For \( g \in \omega_{1}^{Q^{\uparrow}} \), let

\[
g^{W}_{\tilde{D}} : [\omega_{1}]^{W^{\uparrow}} \to \omega_{1} + 1
\]

56
be the function as follows: if \( d = 1 \), then \( g^W_D(\vec{\alpha}) = \frac{1}{\min(\vec{\alpha})} \) when \( \min(\vec{\alpha}) > 1 \); otherwise \( g^W_D(\vec{\alpha}) = \|(1, (q))\| < q \) when \( \min(\vec{\alpha}) > 1 \). In particular, if \( D \) is the constant \((Q, *)\)-description, then \( g^W_D \) is the constant function with value \( \omega_1 \). Clearly, the signature of \( g^W_D \) is \( \text{sign}^1(D) \); \( D \) is of level-1 continuous type iff \( g^W_D \) is essentially continuous; the uniform cofinality of \( g^W_D \) is \( ucf^1(D) \). Suppose additionally that \( Q \) is finite. Let

\[
\text{id}^Q_W
\]

be the function \([g]^Q \mapsto [g^W_D]_µ^W\), or equivalently, \( \vec{\beta} \mapsto \sigma^W(d_{\vec{\beta}}) \), where \( \emptyset^W \) is interpreted as \( j^W \). Clearly, the signature of \( \text{id}^Q_W \) is \( \text{sign}_2^W(D) \); \( \text{id}^Q_W \) is essentially continuous iff \( D \) is of level-2 \( W \)-continuous type; the uniform cofinality of \( \text{id}^Q_W \) is \( ucf^2_W(D) \). Let

\[
\text{seed}^Q_W \in \mathbb{L}_{δ^1}(j^Q \circ j^W[T_2])
\]

be the element represented modulo \( µ^Q \) by \( \text{id}^Q_W \). In particular, if \( d = 1 \) then \( \text{seed}^Q_W = \text{seed}^Q_{(1, q)} \); if \( d = 2 \), \( P = W \) and \( \sigma = \text{id}_P \), then \( \text{seed}^Q_W = \text{seed}^Q_{(2, q)} \). By Loš, if \( D \) is not the constant \((Q, *)\)-description, for any \( A \in µ_L \), \( \text{seed}^Q_W \in j^Q \circ j^W(A) \). Thus, we can define

\[
D^Q_W : \mathbb{L}_{δ^1}(j^Q(T_2)) \to \mathbb{L}_{δ^1}(j^Q \circ j^W[T_2])
\]

by sending \( j^Q(h)(\omega_i) \) to \( j^Q \circ j^W(h)(\text{seed}^Q_W) \).

If \( Q \) is a level-2 tree, \( \vec{q} = (q, P, \vec{p}) \in \text{desc}(Q), l \leq \text{lh}(q) \), define

\[
\vec{q} | l = (q | l, \{p_i : i < l\}, (p_i)_{i<l}).
\]

which is a \( Q \)-description. If \( D = (2, q, \sigma) \in \text{desc}(Q, *), \vec{q} = (q, P, \vec{p}), l \leq \text{lh}(q) \), define

\[
D | l = (2, q | l, \sigma | \{p_i : i < l\})
\]

which is a \((Q, *)\)-description. Define

\[
D \triangleleft D'
\]

iff \( D = D' | l \) for some \( l < \text{lh}(D') \). Define \( \triangleleft^Q_W = \triangleleft | \text{desc}(Q, W) \).

The ordering of \( \text{seed}^Q_W \) is definable in the following concrete way. Put

\[
\langle D \rangle = \begin{cases} (1, \vec{q}) & \text{if } d = 1, \\ (2, \sigma \oplus \vec{q}) & \text{if } d = 2. \end{cases}
\]

\( ^1 \)the split in definition is insignificant, only to ensure Lemma 4.25.
Define

\[ \mathbb{D} \prec \mathbb{D}' \]

iff \( \langle \mathbb{D} \rangle <_{BK} \langle \mathbb{D}' \rangle \), the ordering on subcoordinates in \( \omega^\omega \) again according to \( <_{BK} \). For example, the constant \((Q, \ast)\)-description \( \mathbb{D}_0 \) is the \( \prec \)-maximum, and we have \( \langle \mathbb{D}_0 \rangle = (2, \emptyset) \). When \( 1 \leq \text{card}(\mathbb{Q}) < \aleph_0 \), the \( \prec \)-least \((Q, \ast)\)-description is \( (1, q, \emptyset) \), where \( q \) is the \( <_{BK} \)-least node in \( \mathbb{Q} \). When \( \mathbb{W} \neq \emptyset \), the \( \prec \)-least \((\mathbb{Q}, \mathbb{W})\)-description of degree 2 is \( \mathbb{D}_\mathbb{W} = (2, ((-1), \{(0\}, ((0))\}), \sigma_\mathbb{W}) \), where \( \sigma_\mathbb{W}((0)) \) is the \( <_{BK} \)-least node in \( \mathbb{W} \), and we have \( \langle \mathbb{D}_\mathbb{W} \rangle = (2, (\sigma_\mathbb{W}(1), -1)) \). Define \( \prec_{\mathbb{Q}, \mathbb{W}} = \prec \text{ desc}(\mathbb{Q}, \mathbb{W}) \). \( \prec_{\mathbb{Q}, \mathbb{W}} \) exactly determines the order of the \( \text{seed}_{\mathbb{D}}^{\mathbb{Q}, \mathbb{W}} \)'s, as in the following lemma. It is parallel to Lemma 3.22.

Lemma 4.25. Suppose \( \mathbb{D}, \mathbb{D}' \in \text{desc}(\mathbb{Q}, \mathbb{W}) \) and \( \mathbb{D} \prec_{\mathbb{Q}, \mathbb{W}} \mathbb{D}' \). Then

1. For any \( g \in \omega_1^Q \), for any \( \bar{\alpha} \in \omega_1^W \), \( g_{\mathbb{D}}^W(\bar{\alpha}) < g_{\mathbb{D}'}^W(\bar{\alpha}) \).

2. Suppose \( \mathbb{Q} \) is finite. Then \( \text{seed}_{\mathbb{D}}^{\mathbb{Q}, \mathbb{W}} < \text{seed}_{\mathbb{D}'}^{\mathbb{Q}, \mathbb{W}} \). Moreover, for any \( \beta < u_2 \), \( \mathbb{D}_{\mathbb{Q}, \mathbb{W}}(\beta) < \text{seed}_{\mathbb{D}'}^{\mathbb{Q}, \mathbb{W}} \).

Proof. 1. Simple computation. 2. Note that \( \mathbb{D}_{\mathbb{Q}, \mathbb{W}}(\omega_1) = \text{seed}_{\mathbb{D}}^{\mathbb{Q}, \mathbb{W}} \). We directly prove the “moreover” part. We are given \( \beta = j_{\omega_1}(h)(\omega_1), \) where \( h \) is a function into \( \omega_1 \). Let \( E \in \mu_\omega \) such that for any \( \alpha \in E, h(\alpha) < \min(E \setminus \alpha + 1) \). We have \( \mathbb{D}_{\mathbb{Q}, \mathbb{W}}(\beta) = j_{\mathbb{Q}, \mathbb{W}}(h)(\text{seed}_{\mathbb{D}}^{\mathbb{Q}, \mathbb{W}}) \). By L"os, it suffices to show that for any \( g \in E^\omega \), \( j_{\mathbb{Q}, \mathbb{W}}(h)([g_{\mathbb{D}}^W]_{\mu_\omega}) < [g_{\mathbb{D}'}^W]_{\mu_\omega} \). By L"os again, it suffices to show that for any \( \bar{\alpha} \in [\omega_1]^W, h(g_{\mathbb{D}}^W(\bar{\alpha})) < g_{\mathbb{D}'}^W(\bar{\alpha}) \). We already know that \( g_{\mathbb{D}}^W(\bar{\alpha}), g_{\mathbb{D}'}^W(\bar{\alpha}) \in E \). By our choice of \( E \), it suffices to show that \( g_{\mathbb{D}}^W(\bar{\alpha}) < g_{\mathbb{D}'}^W(\bar{\alpha}) \). This is exactly part 1.

Suppose \( \mathbb{W} \) is a level-1 proper subtree of \( \mathbb{W}' \), \( \mathbb{W}' \) is finite, \( w \in \mathbb{W} \cup \{\emptyset\} \), \( w' \in \mathbb{W}' \setminus \mathbb{W} \). Define

\[ w \prec_{\mathbb{W}}^W w' \]

iff \( w' <_{BK} w \) and \( \{w^* \in \mathbb{W} : w' <_{BK} w^* <_{BK} w\} = \emptyset \). \( \prec_{\mathbb{W}}^W \) inherits the following trivial continuity property.

Lemma 4.26. Suppose \( \mathbb{W} \) is a level-1 proper subtree of \( \mathbb{W}' \), \( \mathbb{W}' \) is finite, \( w \in \mathbb{W}, w' \in \mathbb{W}' \setminus \mathbb{W} \), \( w \prec_{\mathbb{W}}^W w' \). Suppose \( C \in \mu_\omega \) is a club, \( C' \) is the set of limit points of \( C \). Then for any \( \bar{\alpha} \in [C]^W \),

\[ \alpha_w = \sup\{\beta_w : \bar{\beta} \in [C]^W, \bar{\beta} \text{ extends } \bar{\alpha}\}. \]
Suppose \( W \) is a proper level-1 subtree of \( W' \). For \( D \in \text{desc}(Q,W) \) and \( D' \in \text{desc}(Q,W') \setminus \text{desc}(Q,W) \), define the level-1 end extension relation
\[
D \prec_{1}^{Q,W} D'
\]
iff \( D' \prec D \) and \( \{ D^* \in \text{desc}(Q,W) : D' \prec D^* \prec D \} = \emptyset \). Thus, \( D \prec_{1}^{Q,W} D' \) iff both \( D, D' \) are of degree 2 and letting \( D = (2,(q,P,\bar{p}),\sigma), D' = (2,(q',P',\bar{p}'),\sigma') \), \( \text{lh}(q) = k, \bar{p} = (p_i)_{i<\text{lh}(\bar{p})} \), then either
1. \( q \) is of continuous type (hence \( \text{lh}(\bar{p}) = k \)), \( D^- \prec D' \), \( \sigma(p_{k-1}) \prec_{1}^{W} \sigma'(p_{k-1}) \), or
2. \( q \) is of discontinuous type (hence \( \text{lh}(\bar{p}) = k + 1 \)), \( D \prec D' \), \( \sigma(p_{k}) \prec_{1}^{W} \sigma'(p_{k}) \).

As a corollary to Lemma 4.26, \( \prec_{1}^{Q,W} \) inherits the following continuity property.

**Lemma 4.27.** Suppose \( W \) is a proper subtree of \( W' \), \( D \in \text{desc}(Q,W) \), \( D' \in \text{desc}(Q,W') \). Suppose \( C \in \mu_\omega \) is a club, \( C' \) is the set of limit points of \( C \). Then for any \( g \in \omega_{3}^{Q} \), for any \( \vec{\alpha} \in [C']^{W_1} \),
\[
g_{D}^{W}(\vec{\alpha}) = \sup \{ g_{D'}^{W}(\vec{\beta}) : \vec{\beta} \in [C']^{W_1}, \vec{\beta} \text{ extends } \vec{\alpha} \}.
\]

Suppose \( Q \) is a proper subtree of \( Q' \), both finite. For \( (d,q) \in \text{desc}^*(Q) \), \( (d',q') \in \text{desc}^*(Q') \), define the level-2 extension relation
\[
(d,q) \prec_{2}^{Q} (d',q')
\]
iff \( (d',q') \prec (d,q) \) and \( \{(d^*,q^*) \in \text{desc}^*(Q) : (d',q') \prec (d^*,q^*) \prec (d,q) \} = \emptyset \). Thus, \( (d,q) \prec_{2}^{Q} (d',q') \) iff either
1. \( d = d' = 1, q \prec_{1}^{Q} q' \), or
2. \( d' = 1, 0 \prec_{1}^{Q} q', d = 2, q \in \{((-1),\{(0)\}),((0)),(\emptyset,\emptyset,\emptyset)\} \), or
3. \( d = d' = 2 \), letting \( q = (q,P,\bar{p}), \bar{p} = (p_i)_{i<\text{lh}(\bar{p})} \), \( q = (q',P',\bar{p}'), \bar{p}' = (p_i')_{i<\text{lh}(\bar{p}')}, \text{lh}(q) = k \), then either
   (a) \( q \in \text{desc}(Q) \) is of continuous type, \( k \geq 2 \), \( (P,\bar{p}) = (P',\bar{p}' \downarrow k) \), \( (q^-)^- \subseteq q', q(k-2) \prec_{1}^{Q,(q^-)^-} q'(k-2) \), or
   (b) \( q \in \text{desc}(Q) \) is of discontinuous type, \( (P,\bar{p} \downarrow k) = (P',\bar{p}' \downarrow k) \), \( q^- = (q^-)^-, q(k-1) \prec_{1}^{Q,(q^-)} q'(k-1) \), or

59
Suppose continuity property. Direction relies on the assumption that \( \eta \) iff \( D \)
\( \{ q' \in \text{desc}(Q) : D' \prec \text{desc}(Q) \} \).

Lemma 4.28. Suppose \( C \in \mu_L \) is a club. Let \( \eta \in C' \) iff \( C \cap \eta \) has order type \( \eta \).

Then for any \( \beta \in [C]'^{Q'} \),
\[
\beta^Q = \sup \{ \gamma' : \gamma' \in [C]'^{Q'}, \gamma' \text{ extends } \beta \}.
\]

In the proof of Lemma 4.28, the construction of \( \gamma \) that witnesses the \( \preceq \)
relation relies on the assumption that \( \eta \in C' \) iff \( C \cap \eta \) has order type \( \eta \).

Suppose \( Q \) is a proper subtree of \( Q' \), both finite. For \( D \in \text{desc}(Q,W) \),
\( D' \in \text{desc}(Q',W) \setminus \text{desc}(Q,W) \). Define the level-2 end extension relation
\[
D \prec_2^{Q,W} D'
\]
iff \( D' \prec D \) and \( \{ D^* \in \text{desc}(Q,W) : D' \prec D^* \prec D \} = \emptyset \). Thus, putting
\( D = (d, q, \sigma), D = (d', q', \sigma'), D \prec_2^{Q,W} D' \) iff either

1. \( d = d' = 1, q \prec_1^Q q' \), or
2. \( d' = 1, \emptyset \prec_1^Q q', d = 2, q = ((-1), \{(0)\}, ((0))), \sigma((0)) = \min(\prec^W) \), or
3. \( d = d' = 2, \) letting \( q = (q, P, \bar{p}), \bar{p} = (p_i)_{i \prec \text{lh}(\bar{p})}, q = (q', P', \bar{p}'), \bar{p}' = (p'_i)_{i \prec \text{lh}(\sigma)}, \text{lh}(q) = k \), then either
   \( (a) \) \( q \) is of continuous type (hence \( \text{lh}(\bar{p}) = k \)), \( D^- \prec D' \), \( \emptyset \prec_1^Q q' \)
   \( q'(k-1) \), either \( p_{k-1} = -1 \) or \( \sigma'(p_{k-1}) = \text{pred}_W(\sigma(p_{k-1})) \), or
   \( (b) \) \( q \) is of discontinuous type (hence \( \text{lh}(\bar{p}) = k+1 \)), \( D \prec D' \), \( \emptyset \prec_1^Q q' \)
   \( q'(k) \), \( \sigma'(p_k) = \text{pred}_W(\sigma(p_k)) \).

In particular, \( D \prec_2^{Q,W} D' \) implies that \( D \) is of level-2 \( W \)-discontinuous type.
\( \prec_2^{Q,W} \) inherits the following continuity property.

Lemma 4.29. Suppose \( C \in \mu_L \) is a club. Let \( \eta \in C' \) iff \( \eta \in C \) and \( C \cap \eta \)
has order type \( \eta \). Suppose \( Q \) is a proper subtree of \( Q' \), both finite, \( D = (d, q, \sigma) \in \text{desc}(Q,W) \),
\( D' = (d', q', \sigma') \in \text{desc}(Q',W) \), \( D \prec_2^{Q,W} D' \). Then
for any \( \beta \in [C]'^{Q'} \),
\[
\sigma^W(\beta^Q) = \sup \{ (\sigma')^W(\beta^Q) : \gamma' \in [C]'^{Q'}, \gamma' \text{ extends } \beta \}.
\]
Proof. The $\geq$ direction follows from Lemma 4.25. We show the $\leq$ direction. When $d = d' = 1$, both sides are equal to $\beta_q$ by Lemma 4.26. When $d = 2 \land d' = 1$, both sides are equal to $\omega_1$ by Lemma 4.26 again. Suppose now $d = d' = 2$. Let $q = (q, P, \bar{p}), \bar{p} = (p_i)_{i < \text{lh} (\bar{p})}$, $q = (q', P', \bar{p})$, $\bar{p}' = (p_i')_{i < \text{lh} (\bar{p})}$, \text{lh} (q) = k.

Case 1: $q$ is of continuous type.

Let $P^- = P \setminus \{p_{k-1}\}$. So $Q(q^-) = (P^-, p_{k-1})$. Let $q'' = q^' \upharpoonright k$, $\sigma'' = \sigma' \upharpoonright P$, $p'' = Q \text{node}(q'')$. Then $(2, (q'', P, \bar{p}^-(p''))) <_{\mathcal{D}} (2, (q^-, P, \bar{p}))$. By Lemma 4.28

$$j_{P^-, P}(\beta_q^-) = \sup \{2^{\gamma_{q''}} : \gamma \in [C]^{|q|}, \gamma \text{ extends } \bar{\beta}\}.$$ 

It suffices to show that

$$\sigma^W \circ j_{\sup} (\beta_q^-) = (\sigma'')^W \circ j_{P^-, P}(\beta_q^-).$$

This is exactly Lemma 4.35 using the fact $c^{\bar{\beta}_q} = \text{seed}_{\bar{P}_{k-1}}^{P^-}$.

Case 2. $q$ is of discontinuous type.

Let $P^+$ be the completion of $(P, p_k)$ if $p_k \neq -1$, $P^+ = P$ if $p_k = -1$. Let $q'' = q^' \upharpoonright k + 1$, $\sigma'' = \sigma' \upharpoonright P^+$. Then $(2, (q'', P^+, \bar{p}')) <_{\mathcal{D}} (2, (q^-, P, \bar{p}))$. By Lemma 4.28

$$j_{P^+, P}(\beta_q^+)^W = \sup \{2^{\gamma_{q''}} : \gamma \in [C]^{|q|}, \gamma \text{ extends } \bar{\beta}\}.$$ 

It remains to show

$$\sigma^W (\beta_q^+) = (\sigma'')^W \circ j_{P^+, P}(\beta_q).$$

This is exactly Lemma 4.36 using the fact $c^{\bar{\beta}_q} = \text{seed}_{\bar{P}_k}^{P^-}$ when $p_k \neq -1$, $c^{\bar{\beta}_q} = \omega$ when $p_k = -1$.  

\[\square\]

Definition 4.30. Suppose $S$ is a finite regular level-1 tree and $Q$ is a level $\leq 2$ tree. Suppose $\tau : S \cup \{\emptyset\} \to \text{desc}(Q, *)$ is a function. Then $\tau$ factors $(S, Q, *)$ iff

1. $\tau(\emptyset)$ is the constant $(Q, *)$-description.
2. If $s \prec^S s'$, then $\tau(s) \prec \tau(s')$.

For a level-1 tree $W$, $\tau$ factors $(S, Q, W)$ iff $\tau$ factors $(S, Q, *)$ and $\text{ran}(\tau) \subseteq \text{desc}(Q, W)$. In particular, if every $\tau(s)$ is of degree 1, then $\tau$ factors $(S, Q, 0)$.

If $S$ is a level-1 tree, then

$$\text{id}_{S}(s), (S, Q_0, S), \text{ where } id_{S}(s) = (2, ((-1), \{(0)\}), \sigma_s), \sigma_s(0) = s.$$
Suppose \( \tau \) factors \((S, Q, W)\). For \( g \in \omega_1^{Q^1} \), let

\[
g^W_\tau : [\omega_1]^{W^1} \rightarrow [\omega_1]^{S^1}
\]

be the function sending \( \vec{\alpha} \) to \((g^W_\tau(s)(\vec{\alpha}))_{s \in \text{dom}(S)}\). Lemma 4.25 ensures that \( g^W_\tau \)

is indeed a function into \([\omega_1]^{S^1}\). In particular, \( g^S_{\text{id} \times S} \) is the identity map on \([\omega_1]^{S^1}\).

\[
\text{id}^Q_{\tau, W}
\]

is the map sending \([g]^Q\) to \([g]^W_{\mu, W}\). So \( \text{id}^Q_{\tau, W}(\vec{\beta}) = (\text{id}^W_{\tau(s)}(\vec{\beta}))_{s \in S} \). Put

\[
\text{seed}^Q_{\tau, W} = [\text{id}^Q_{\tau, W}]_{\mu, Q}
\]

By Lemma 4.25 and Loš, for any \( A \in \mu^S \), \( \text{seed}^Q_{\tau, W} \in j^Q \circ j^W(A) \). Hence, we can unambiguously define

\[
\tau^{Q, W} : \mathbb{L}_{\delta^1_3}[j^S(T_2)] \rightarrow \mathbb{L}_{\delta^1_3}[j^Q \circ j^W(T_2)]
\]

by sending \( j^S(h)(\text{seed}^S) \) to \( j^Q \circ j^W(h)(\text{seed}^Q_{\tau, W}) \). \( \tau^{Q, W} \) is the unique map

such that for any \( z \in \mathbb{R}, \tau^{Q, W} \) is elementary from \( L_{\kappa_3}^{S}[j^S(T_2), z] \) into \( L_{\kappa_3}^{C}[j^Q \circ j^W(T_2), z] \)

and for any \( s \in S, \tau^{Q, W} \circ s^S = \tau(s)q^{Q, W} \).

**Lemma 4.31.** Suppose \( Q, W \) are finite.

1. If \( D = \min(\kappa^{Q, W}) \), then \( \text{seed}^{Q, W}_D = \omega_1 \). Hence \( D^{Q, W} \) is the identity on \( \omega_1 + 1 \).

2. If \( E = \text{pred}_{\omega^Q, W}(D) \), then \( (E^{Q, W})^{\mu, \omega_2} \) is a cofinal subset of \( \text{seed}^{Q, W}_D \).

**Proof.** We only prove the case when \( 1^Q = \emptyset \). The general case takes an analogous additional argument.

**Case 1:** \( W = \emptyset \).

The only \((Q, W)\)-description is the constant \((Q, *)\)-description \( D_0 \). We only have to prove part 1. For any \( x \in \mathbb{R}, D^{Q, W}_0 = j^Q \circ j^W \) is elementary from \( L_{\kappa_3}^{S}[T_2, x] \) into \( L_{\kappa_3}^{C}[j^Q \circ j^W(T_2), x] \). It follows that \( D^{Q, W}_0 \upharpoonright \omega_1 \) is the identity map. It remains to show that \( \text{seed}^{Q, W}_D = \omega_1 \). We already know that \( \text{seed}^{Q, W}_D = j^Q(\omega_1) \). Suppose \([g]_{\mu, Q} < j^Q(\omega_1) \) and we try to show that \([g]_{\mu, Q} \leq \omega_1 \). Let \( Q' \) be the completion of the partial level \( \leq 2 \) tree \((Q, (1, (0), \emptyset))\). Let \( D' = (1, (0), \emptyset) \in \text{desc}(Q', W) \). Then \( D_0 \lhd^{Q, W} D' \). We partition functions \( f \in \omega_1^{Q^1} \) according to whether \( 1^Q f \upharpoonright (0) \leq g((f \upharpoonright \text{rep}(Q))^Q) \). We obtain a club \( C \in \mu_L \) which is homogeneous for this property. Let \( \eta \in C' \) iff \( \eta \in C \) and \( C \cap \eta \) has order type \( \eta \). If the homogeneous side satisfies \( 1^Q f \upharpoonright (0) > g((f \upharpoonright \text{rep}(Q))^Q) \), we let \( \alpha_0 = \text{the } \omega\text{-th element of } C \), and so every \( f \in C''^{Q^1} \) is
extendable to \( f' \in C^{Q_1} \) so that \( 1[\tilde{f}]_{[0]}^Q = \alpha_0 \). Therefore, for every \( \tilde{\xi} \in [C']^{Q_1}, g(\tilde{\xi}) < \alpha_0 \). Hence by \( \text{Lo"{s}} \), \( [g]_{[\mu]Q} < j^Q(\alpha_0) = \alpha_0 \) and we are done. If the homogeneous side satisfies \( 1[\tilde{f}]_{[0]}^Q \leq g([f \restriction \text{rep}(Q)]^Q) \), then by Lemma 4.29 \( \omega_1 = 2[|\text{rep}(Q)|]_0^Q \leq g([f \restriction \text{rep}(Q)]^Q) \), contradicting to the assumption on \( g \).

Case 2: \( W \neq \emptyset \).

We firstly prove part 1. The \( \prec^{Q,W} \)-minimum is \( D_0 = (2, q, \sigma) \), where \( q = (((-1), \{(0)\}), \{(0)\})) \). \( \sigma((0)) \) is the \( \prec_{BK} \)-least node in \( W \). \( \text{seed}_{D_0}^{Q,W} \) is represented modulo \( \mu \) by the function that sends \( \beta \) to \( \sigma^W(\beta_q) = \sigma^W(\omega_1) = \omega_1 \). Hence, \( \text{seed}_{D_0}^{Q,W} = j^Q(\omega_1) \). Work with the same \( Q \) as in Case 1 and argue with the same partition arguments.

Next, we prove part 2. Let \( D = (2, q, \sigma), q = (q, P, \tilde{p}), E = (2, r, \tau), r = (r, Z, \tilde{z}) \). Then \( q \neq (-1) \). Put \( Q' = \{a \in \omega^{<\omega} : \nu^\downarrow(a) \in \text{dom}(\tilde{Q})\} \) for \( v \in \text{dom}(\tilde{Q}) \).

Subcase 2.1: \( r \) is of discontinuous type.

Let \( Q' \) be the level \( \leq 2 \) tree extending \( Q \) such that \( \text{dom}(Q') \setminus \text{dom}(Q) = \{(2, r')\}, r' = r^\downarrow(a), \emptyset <_{Q[r]} a, Q'(r') = Q(r) \). Let \( r' = (r', Z, \tilde{z}) \). Then \( D <_{2}^{Q,W} E' \). Our partition arguments will be based on \( Q' \).

Suppose \( [g]_{[\mu]Q} < \text{seed}_{D}^{Q,W} \) and we seek \( \eta_0 < \nu_2 \) such that \( [g]_{[\mu]Q} < E_{Q,W}^{Q,W}(\eta_0) \). We partition functions \( f \in \omega_{Q_1}^\downarrow \) according to whether \( \tau^W([2][f]_{Q_1}^Q) \leq g([f \restriction \text{rep}(Q)]^Q) \). By Theorem 4.10, we obtain a club \( C \in \mu \), which is homogeneous for this property. Let \( \eta \in C \) iff \( \eta \in C \) and \( C \cap \eta \) has order type \( \eta \). If the homogeneous side satisfies \( \tau^W([2][f]_{Q_1}^Q) > g([f \restriction \text{rep}(Q)]^Q) \), we let \( \eta_0 = j^P(h_0)(\omega_1), h_0(a) = \min(C \setminus (\alpha + 1)) \). This allows us to extend every \( f \in (\omega)^{<\omega} \) to \( f' \in C^{Q_1} \) so that \( 2[\tilde{f}]_{[0]}^Q = j^P(h_0)([f]_Q^Q) \). Therefore, for every \( \tilde{\xi} \in [C']^{Q_1}, g(\tilde{\xi}) < \tau^W(j^Q(h_0)(\xi)) = j^W(h_0)(\tau^W(\xi)) \). Hence \( [g]_{[\mu]Q} < j^Q \circ j^W(h_0)(\text{seed}_{E}^{Q,W}) = E_{Q,W}(j^P(h_0)(\omega_1)) \). Hence \( [g]_{[\mu]Q} < E_{Q,W}(\eta_0) \).

If the homogeneous side satisfies \( \tau^W([2][f]_{Q_1}^Q) \leq g([f \restriction \text{rep}(Q)]^Q) \), then by Lemma 4.29 \( \omega_1 = 2[|\text{rep}(Q)|]_0^Q \leq g([f \restriction \text{rep}(Q)]^Q) \). This contradicts our assumption on \( g \).

Subcase 2.2: \( r \) is of continuous type.

Let \( Q' \) be the level \( \leq 2 \) tree extending \( Q \) such that \( \text{dom}(Q') \setminus \text{dom}(Q) = \{(2, r')\}, r' = r^\downarrow(a), \emptyset <_{1}^{Q[r]} a, Q'(r') = Q(r) \). Let \( r' = (r'^\downarrow(-1), Z, \tilde{z}) \). Then \( D <_{2}^{Q,W} E' \). The rest is similar to Subcase 2.1. \( \square \)

At this point, it is convenient to label the nodes of a tree of uniform cofinalities using arbitrary sets instead of elements in \( \omega^{<\omega} \) and \( \omega^{<\omega} \). Suppose \( Q \) is a level \( \leq 2 \) tree and \( W \) is a level-1 tree. A representation of \( Q \otimes W \) is a pair \((S, \tau)\) such that \( S \) is a level-1 tree, \( \tau \) factors \((S, Q, W)\),
ran(τ) = desc(Q, W), and s ≺ S s′ iff τ(s) ≺ Q,W τ(s′). Representations of
Q ⊗ W are clearly mutually isomorphic. We shall informally regard

Q ⊗ W = desc(Q, W) \ {the constant (Q, W)-description}

as a “level-1 tree” by identifying it with S via τ. We put seedQ⊗W = seedS−1D for D ∈ desc(Q, W). If τ′ factors (S′, Q, W), then τ′ also factors “level-1 trees” (S′, Q ⊗ W), and (τ′)Q⊗W makes sense. That is, (τ′)Q⊗W = (τ−1 o τ′)S, where τ−1 o τ′ factors (S′, S). The identity map idQ⊗W : D ⊢ D factors (Q ⊗ W, Q, W). If Q is a subtree of Q′ and W is a subtree of W′, then Q ⊗ W is regarded as a subtree of Q′ ⊗ W′, and the map jQ⊗W. Q⊗W′ makes sense. In other words, let (S, τ) be a representation of Q ⊗ W and (S′, τ′) be a representation of Q′ ⊗ W′ such that S is a subtree of S′ and τ ⊆ τ′, then jQ⊗W. Q⊗W′ = jS′.S′. If π factors level ≤ 2 trees (Q, T), then

π ⊗ W

factors level-1 trees (Q ⊗ W, T ⊗ W), where π(d, q, σ) = (d, 4π(q), σ).

Lemma 4.32. Suppose Q is a finite level ≤ 2 tree, W is a finite level-1 tree.

1. If D ∈ desc(Q, W), then seedDQ⊗W = seedDQ,W.

2. (idQ⊗W)Q,W is identity on jQ o jW(ω1 + 1).

3. If S is a level-1 tree, τ factors (S, Q, W), then τQ,W = τQ⊗W.

Proof. Let D 0, . . . , D m enumerate desc(Q, W) in the ≺Q,W -ascending order. We prove by induction on l ≤ m that seedDQ⊗W = seedDiQ,W for any i ≤ l.

Suppose l = 0. The fact seedDQ,W = ω1 follows from Lemma 4.31. Suppose the induction hypothesis holds at l < m. That is, (D l)Q⊗W(ω1) = seedDlQ,W = ui+1 for l < m. By Loś, (idQ⊗W)Q,W is identity on ui+2. But ((D l)Q,W)′ u2 is a cofinal subset of seedDl+1Q,W by Lemma 4.31. Hence, seedDl+1Q,W = ui+2. This proves part 1. Parts 2-3 are immediate corollaries.

D ∈ desc(Q, W) is direct iff either D is of degree 1 or D is of the form (2, (q, P, ⃗p), idP). Lemma 4.32 has the following corollary on representations of uniform indiscernibles in the μQ-ultrapower.

Lemma 4.33. Suppose Q is a finite level ≤ 2 tree. Then

{un : 1 ≤ n < ω} = {seedDQ,W : W finite, D ∈ desc(Q, W) is direct}.
For \((d, q) \in \text{dom}(Q)\), define

\[
\text{cf}^Q(d, q) = \begin{cases} 
0 & \text{if } d = 1 \lor (d = 2 \land \exists Q(q) \text{ of degree } 0), \\
1 & \text{if } d = 2 \land \text{ucf}(Q(q)) = \min(\prec 2Q_{\text{tree}}(q)), \\
2 & \text{otherwise}.
\end{cases}
\]

By Lemma 4.31 if \(\beta\) respects \(Q\), then

\[
\text{cf}^{T_{\delta_1}[T_2]}(d_{\beta q}) = u_{\text{cf}^Q(d, q)}
\]

where \(u_0 = \omega\).

### 4.5 Approximations of \(S_3\) in \(L_{\delta_1}[T_2]\)

**Lemma 4.34.** Suppose \(Q\) is a level \(\leq 2\) tree, \(W\) is a level-1 subtree of \(W'\), all trees are finite. Then \(\text{j}^Q(j^W \upharpoonright j^W(\omega_1 + 1)) = j^{Q \otimes W \otimes W'} \downharpoonright (j^{Q \otimes W}(\omega_1 + 1))\), and hence \(\text{j}^Q(j^{W' \upharpoonright} j^W(\omega_1 + 1)) = j^{Q \otimes W \otimes W'}(j^{Q \otimes W}(\omega_1 + 1))\) by sufficient elementarity of \(j^Q\).

**Proof.** By Lemma 4.32, \(\text{seed}^{Q \otimes W} = \text{seed}^{Q \otimes W}_D\) for \(D \in \text{desc}(Q, W)\), and similarly for \(W'\). So \(j^{Q \otimes W \otimes W'}(\text{seed}^{Q \otimes W}_D) = \text{seed}^{Q \otimes W'}_D\) for \(D \in \text{desc}(Q, W)\). Since \(j^Q(j^{W'})\) is elementary from \(L[z]\) to \(L[z]\) for any \(z \in \mathbb{R}\), it suffices to show that \(j^Q(j^{W' \upharpoonright} j^W(\omega_1 + 1))(\text{seed}^{Q \otimes W}_D) = j^{Q \otimes W \otimes W'}(\text{seed}^{Q \otimes W}_D)\) for any \(D \in \text{desc}(Q, W)\). Fix \(D \in \text{desc}(Q, W)\). Suppose the typical case when \(D = (2, q, \sigma)\) is of degree 2. Then by Loś,

\[
j^Q(j^{W' \upharpoonright} j^W(\omega_1 + 1))(\text{seed}^{Q \otimes W}_D) = j^Q(j^{W' \upharpoonright} j^W(\omega_1 + 1))(\bar{\xi} \mapsto \sigma^W(\bar{\xi}_q))\mu_q
\]

\[
= \bar{\xi} \mapsto j^{W' \upharpoonright} j^W(\omega_1 + 1)(\bar{\xi})\mu_q
\]

\[
= \bar{\xi} \mapsto \sigma^W(\bar{\xi}_q)\mu_q
\]

\[
= \text{seed}^{Q \otimes W'}_D
\]

\[
= j^{Q \otimes W \otimes W'}(\text{seed}^{Q \otimes W}_D).
\]

**Lemma 4.35.** Suppose \(\pi\) factors finite level \(\leq 2\) trees \((Q, T)\) and \(W\) is a finite level-1 tree, all trees are finite. Then \(\pi^T \upharpoonright j^Q \circ j^W(\omega_1 + 1) = (\pi \otimes W)^{Q \otimes W, T \otimes W} \upharpoonright j^{Q \otimes W}(\omega_1 + 1)\).

**Proof.** Apply Lemma 4.32 and use the fact that \(\pi^T(\text{seed}^{Q \otimes W}_D) = \text{seed}^{T \otimes W, \pi W}_D\) for \(D \in \text{desc}(Q, W)\).
Lemma 4.36. Suppose Q is a finite level ≤ 2 tree. Then

1. \( j^Q \restriction \{ u_n : n < \omega \} \) is \( \Delta^1_3 \), uniformly in Q.
2. \( j^Q(u_\omega) = u_\omega \).
3. \( j^Q \restriction u_\omega \) is \( \Delta^1_4 \), uniformly in Q.
4. Suppose \( P, P' \) are finite level-1 trees and \( \pi \) factors \( (P, P') \). Then \( j^Q(\pi^{P'} \restriction u_\omega) \) is \( \Delta^1_3 \), uniformly in \( (Q, P, P', \pi) \).
5. \( j^Q(T_2) \) is \( \Delta^1_4 \), uniformly in Q.

Proof. 1 and 2. By Lemma 4.32.
3. \( j^Q(\tau^{L[\xi]}(u_1, \ldots, u_n)) = \tau^{L[\xi]}(j^Q(u_1), \ldots, j^Q(u_n)) \).
4. By Lemma 4.34.
5. by 4. \( \square \)

The following lemma refines Corollary 4.15.

Lemma 4.37. Assume \( \Delta^1_3 \)-determinacy. Suppose \( x \in \mathbb{R} \). Then for any finite level ≤ 2 tree Q, \( j^Q(\kappa_3^x, \lambda_3^x) = (\kappa_3^x, \lambda_3^x) \). Moreover, \( S_3 \restriction \kappa_3^x \) and \( S_3 \restriction \lambda_3^x \) are both uniformly \( \Delta^1_3 \)-definable over \( L_{\kappa_3^x}[T_2, x] \) from \( \{ T_2, x \} \).

Proof. By elementarity, \( j^Q(\kappa_3^x) \) is the least \( \gamma \) for which \( L_\gamma[j^Q(T_2), x] \) is admissible. But \( j^Q(T_2) \in L_{\kappa_3^x}[T_2, x] \) by Lemma 4.36. Consequently, \( L_{\kappa_3^x}[j^Q(T_2), x] \) is admissible. Since \( j^Q \) is non-decreasing on ordinals, we must have \( j^Q(\kappa_3^x) = \kappa_3^x \). Similarly, \( \lambda_3^x \), being the sup of the ordinals \( \Delta^1_3 \)-definable over \( L_{\kappa_3^x}[T_2, x] \) from \( \{ T_2, x \} \), is also fixed by \( j^Q \).

To define \( S_3 \restriction \kappa_3^x \), it is of course enough to establish a uniformly \( \Delta^1_3 \) definition of \( j^Q \wedge \kappa_3^x \) over \( L_{\kappa_3^x}[T_2, x] \), for Q a level ≤ 2 subtree of Q. Note that every element of \( L_{\kappa_3^x}[T_2, x] \) is \( \Delta^1_3 \)-definable over \( L_{\kappa_3^x}[T_2, x] \) from \( u_\omega \cup \{ T_2, x \} \), and hence by Loś, every ordinal in \( j^Q(L_{\kappa_3^x}[T_2, x]) \) is \( \Delta^1_3 \)-definable over \( j^Q(L_{\kappa_3^x}[T_2, x]) \) from parameters in \( u_\omega \cup \{ j^Q(T_2), x \} \). The lemma follows immediately from Lemma 4.37 and Loś:

\[
j^Q(\gamma) = \gamma'
\]

iff for some \( \xi < \kappa_3^x \), some \( \Sigma_1 \)-formula \( \varphi \), some ordinal \( \alpha < u_\omega \),

\[
L_{j^Q(\xi)}[j^Q(T_2), x] = \forall \delta \ (\delta = \gamma \leftrightarrow \varphi(\delta, j^Q(T_2), x, \alpha)),
\]

and

\[
L_{j^Q(\xi)}[j^Q(T_2), x] = \forall \delta \ (\delta = \gamma' \leftrightarrow \varphi(\delta, j^Q(T_2), x, j^Q(\alpha))).
\]

This is a \( \Sigma_1 \) definition of \( j^Q(\gamma) = \gamma' \) over \( L_{\kappa_3^x}[T_2, x] \) from \( \{ T_2, x \} \). In a similar way, we can write down a \( \Sigma_1 \) definition of \( j^Q(\gamma) \neq \gamma' \). The definition of \( S_3 \restriction \lambda_3^x \) is similar. \( \square \)
In light of Lemma 4.37, $L_{\kappa_3}[S_3 \upharpoonright \kappa_3]$ is regarded as the “lightface core” of $L_{\kappa_3}[T_2, x]$, analogous to $\mathcal{C}$Liff $x \subseteq CQ$ tree $1 \leq L$ countable countable $L$. The scenario in the AD world suggests that the set of uncountable $F$ : $[\omega_1]^\omega \rightarrow \delta^1_3$ that lie in $L_{\kappa_3}[T_2]$, originally in $[12]$. In particular, we confirm that the set of uncountable $L_{\kappa_3}[T_2]$-regular cardinals is indeed $\{u_1, u_2\}$.

**Theorem 4.38.** Suppose $Q$ is a finite level $\leq 2$ tree, $W$ is a finite level-1 tree. Suppose $D = (d, q, \sigma) \in \text{desc}(Q, W)$. Then

$$\text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{D}^{Q, W}) = \text{seed}_{\text{ucf}_2^{W}(D)}^{Q}.$$

In particular, the set of $L_{\kappa_3}[j^Q(T_2)]$-regular cardinals is exactly

$$\{\text{seed}^{Q}_{(d, q)} : (d, q) \in \text{desc}^*(Q) \text{ is regular}\}.$$

**Proof.** Put $(d, r) = \text{ucf}_2^{W}(D)$. Firstly, we prove $\text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{D}^{Q, W}) = \text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{(d, r)}^{Q})$. There is nothing to prove for $d = 1$. Suppose now $d = 2$.

Case 1: $q$ is of continuous type.

Subcase 1.1: $D$ is of level-2 $W$-continuous type.

In this case, $\sigma$ is continuous at $2\beta_q$. For any $\tilde{\beta} \in [\omega_1]^\omega$, $\sigma^W(2\beta_q) = \sup(\sigma^W \circ j^{P^-}(2\beta_q) - \sigma^W(2\beta_q))$. So $\text{cf}^{L_{\kappa_3}[T_2]}(\sigma^W(2\beta_q)) = \text{cf}^{L_{\kappa_3}[T_2]}(2\beta_q)$. Note that $\text{seed}_{(2, r)}^{Q}$ is represented by the function $\tilde{\beta} \mapsto 2\beta_q$. By Loš, $\text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{D}^{Q, W}) = \text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{(2, r)}^{Q})$.

Subcase 1.2: $D$ is of level-2 $W$-discontinuous type.

Then $\text{pred}_{\omega}(\sigma(p_{k-1}))$ exists and is not in $\text{ran}(\sigma)$. Put $P^- = P \setminus \{p_{k-1}\}$. Let $\sigma'$ factor $(P, W)$ where $\sigma$ and $\sigma'$ agree on $P$ and $\sigma'(p_{k-1}) = \text{pred}_{\omega}(\sigma(p_{k-1}))$. By Lemma 3.5, $\sigma^W(2\beta_q) = (\sigma')^{\sup}(j^{P^-}(2\beta_q) - \sigma^W(2\beta_q))$. So $\text{cf}^{L_{\kappa_3}[T_2]}(\sigma^W(2\beta_q)) = \text{cf}^{L_{\kappa_3}[T_2]}(j^{P^-}(2\beta_q))$. Note that $\text{seed}_{(2, r)}^{Q}$ is represented by the function $\tilde{\beta} \mapsto j^{P^-}(2\beta_q)$. By Loš, $\text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{D}^{Q, W}) = \text{cf}^{L_{\kappa_3}[j^Q(T_2)]}(\text{seed}_{(2, r)}^{Q})$. 67
Case 2: \( q \) is of discontinuous type.

Subcase 2.1: \( D \) is of level-2 \( W \)-continuous type.

Then \( \sigma \) is continuous at \( \frac{3}{2}q \). Proceed as in Subcase 1.1.

Subcase 2.2: \( D \) is of level-2 \( W \)-discontinuous type.

Then \( \text{pred}_{\leq W} (\sigma (p_k^-)) \) exists and is not in \( \text{ran}(\sigma) \). Put \( P^+ = P \cup \{p_k\} \). Let \( \sigma' \) factor \( (P^+, W) \) where \( \sigma' \supset \sigma \) and \( \sigma' (p_k^-) = \text{pred}_{\leq W} (\sigma (p_k^-)) \). By Lemma 3.6, \( \sigma^W (\frac{3}{2}q) = (\sigma')^W_{\sup} (\{j^P, P^+ \} (\frac{3}{2}q)) \). Proceed as in Subcase 1.2.

Note that by Lemma 4.33, each \( u_n \) (1 \( \leq n < \omega \)) is of the form \( \text{seed}^W_D \) for some finite \( W \) and \( D \in \text{desc}(Q, W) \). In summary, we have proved that every \( \mathbb{L}^{\text{desc}_L[T]} \) is regular. If the homogeneous side satisfies \( \mathbb{L}^{\text{desc}_L[T]} \) has order type \( \eta \), then \( \eta \) is homogeneous for this property. Let \( \text{pred}_{\leq r} (\sigma (\tilde{b}_r)) \) exists and is not in \( \text{ran}(\sigma) \). Put \( \text{pred}_{\leq r} (\sigma (\tilde{b}_r)) \).

Proceed as in Subcase 1.2.

Case 2: \( q \) is of discontinuous type.

Subcase 2.1: \( D \) is of level-2 \( W \)-continuous type.

Then \( \sigma \) is continuous at \( \frac{3}{2}q \). Proceed as in Subcase 1.1.

Subcase 2.2: \( D \) is of level-2 \( W \)-discontinuous type.

Then \( \text{pred}_{\leq W} (\sigma (p_k^-)) \) exists and is not in \( \text{ran}(\sigma) \). Put \( P^+ = P \cup \{p_k\} \). Let \( \sigma' \) factor \( (P^+, W) \) where \( \sigma' \supset \sigma \) and \( \sigma' (p_k^-) = \text{pred}_{\leq W} (\sigma (p_k^-)) \). By Lemma 3.6, \( \sigma^W (\frac{3}{2}q) = (\sigma')^W_{\sup} (\{j^P, P^+ \} (\frac{3}{2}q)) \). Proceed as in Subcase 1.2.

Note that by Lemma 4.33, each \( u_n \) (1 \( \leq n < \omega \)) is of the form \( \text{seed}^W_D \) for some finite \( W \) and \( D \in \text{desc}(Q, W) \). In summary, we have proved that every \( \mathbb{L}^{\text{desc}_L[T]} \) is regular. If the homogeneous side satisfies \( \mathbb{L}^{\text{desc}_L[T]} \) has order type \( \eta \), then \( \eta \) is homogeneous for this property. Let \( \text{pred}_{\leq r} (\sigma (\tilde{b}_r)) \) exists and is not in \( \text{ran}(\sigma) \). Put \( \text{pred}_{\leq r} (\sigma (\tilde{b}_r)) \).

Proceed as in Subcase 1.2.

Let \( Q_2 \) be the level \( \leq 2 \) tree defined in a similar way with \( (g, q', \sigma') \) replaced by \( (r, r') \). Let \( Q' \) be the tree extending both \( Q_1 \) and \( Q_2 \) and \( \text{dom}(Q_1) \cup \text{dom}(Q_2) = \text{dom}(Q') \). Put \( q' = (q', P, \tilde{p}) \), \( r' = (r', Z, \tilde{z}) \). So \( q' \prec q \), \( r' \prec q \). We partition functions \( f \in \omega^{Q_1 \uparrow} \) according to whether \( g([f \mid \text{rep}(Q)]^Q) (\omega^{f |Q '}) \prec 2^{[f]_Q} \). By Theorem 4.10, we obtain a club \( C \in \mu_L \) which is homogeneous for this property.

Let \( \eta \in C' \) iff \( \eta \in C \) and \( C \cap \eta \) has order type \( \eta \). Let \( \eta \in C'' \) iff \( \eta \in C' \) and \( C' \cap \eta \) has order type \( \eta \). If the homogeneous side satisfies \( g([f \mid \text{rep}(Q)]^Q) (\omega^{f |Q '}) \prec 2^{[f]_Q} \), then every function \( f \in (C'')^{Q_1 \uparrow} \) extends to some \( f' \in (C'')^{Q_1 \uparrow} \) by Lemma 3.18 and \( \{f'' : 2^{[f'']_Q} \in C^{Q_1 \uparrow} (f' \subseteq f'') \} \) is cofinal in \( 2^{[f]_Q} \). Hence, \( \sup (g([f \mid \text{rep}(Q)]^Q) (\omega^{f |Q '})) \leq 2^{[f]_Q} \), contradicting to our assumption. If the homogeneous side satisfies \( g([f \mid \text{rep}(Q)]^Q) (\omega^{f |Q '}) \geq [f]_Q^Q \), a similar arguments yields \( \sup (g([f \mid \text{rep}(Q)]^Q) (\omega^{f |Q '}) \geq 2^{[f]_Q} \), contradiction again. \( \square \)
It is easy to deduce the following corollary using Loś:

**Corollary 4.39.** Suppose $\beta < \delta^1_3$ is a limit ordinal. Then $\beta$ has $Q$-uniform cofinality $(d, q)$ iff $\text{cf}^{|Q(T_2)}(\beta) = \text{seed}^Q_{(d, q)}$. In particular, the $Q$-uniform cofinality of $\beta$ exists and is unique.

If $\pi$ factors finite level $\leq 2$ trees $(Q, T)$, then $\pi^T(u_n) = u_m \to \pi^T_{\sup}(u_{n+1}) = u_{m+1}$. Therefore, the continuity of $\pi^T$ is decided by $\pi^T \upharpoonright \{u_n : n < \omega\}$. If $(d, q) \in \text{desc}^*(Q)$ is regular, $\pi$ is continuous at $(d, q)$ iff one of the following holds:

1. $d = 1$, either $\downarrow^T \pi(q) = \min(\downarrow^T \pi)$ or $\text{pred}_{\downarrow^T \pi}(\downarrow^T \pi(q)) \in \text{ran}(\downarrow^T \pi)$.
2. $d = 2$, $q = (\emptyset, \emptyset, ((0)))$, either $\uparrow^T = \emptyset$ or $\max(\downarrow^T \pi) \in \text{ran}(\downarrow^T \pi)$.
3. $d = 2$, $q = (q, P, \tilde{p}) \in \text{desc}^*(\mathcal{Q})$, and letting $t' = \max_{< BK} \exists T(\{\pi(q), -\})$, then either $t' = \mathcal{T}(\pi(q)^-) - 1$ or $t' \in \text{ran}(\pi)$.
4. $d = 2$, $q = (q, P, \tilde{p}) \notin \text{desc}^*(\mathcal{Q})$, and letting $a = \max_{< BK}(\exists T(\{\pi(q)\} \cup \{-1\})$, then either $a = -1$ or $\mathcal{T}(\pi(q)^-)(a) \in \text{ran}(\pi)$.

Thus, $\pi$ is continuous at $(d, q)$ if $\pi^T$ is continuous at $\text{seed}^Q_{(d, q)}$. We obtain the following lemma discussing the continuity behavior of $\pi^T$. It is the level-2 version of Lemma 3.4.

**Lemma 4.40.** Suppose $\pi$ factors finite level $\leq 2$ trees $(Q, T)$, $\gamma < \delta^1_3$ is a limit ordinal, $\text{cf}^{|Q(T_2)}(\gamma) = \text{seed}^Q_{(d, q)}$, $(d, q) \in \text{desc}^*(Q)$ is regular. Then

1. $\pi^T(\gamma) = \pi^T_{\sup}(\gamma)$ iff $\pi$ is continuous at $(d, q)$.
2. Suppose $\pi$ is not continuous at $(d, q)$. Let $Q^+$ be a level $\leq 2$ tree and let $\pi^+$ factor $(Q^+, \pi)$ so that $Q^+$ extends $Q$, $\pi^+$ extends $\pi$, and

   (a) if $d = 1$, then $\text{dom}(Q^+) \setminus \text{dom}(Q) = \{(1, q^+), \pi < 1^Q q^+, \pi^+(q^+) = \text{succ}_{\downarrow^T \pi}(\pi(q^-))\}$;

   (b) if $d = 2$ and $q = (\emptyset, \emptyset, ((0)))$, then $\text{dom}(Q^+) \setminus \text{dom}(Q) = \{(1, q^+)\}$, $\emptyset < 1^Q q^+$, $\pi^+(q^+) = \text{min}_{\downarrow^T \pi}(\{a : \forall \pi \in \text{dom}(Q) \downarrow^T \pi(a)\}$;

   (c) if $d = 2$ and $q = (q, P, \tilde{p}) \in \text{desc}^*(\mathcal{Q})$, then $\text{dom}(Q^+) \setminus \text{dom}(Q) = \{(2, q^+)\}$, $q^+ = \max_{< BK} 2^Q + \{q, -\}$, and $\pi^+(q^+) = \pi(q^-)^-(a)$, $a = \text{min}_{< BK}(\{b : \forall \epsilon \in \mathcal{Q}(q, -) \setminus \{q^-(-1)\} 2^Q(\epsilon) < BK 2^Q(q^-)(-b)\}$;

   (d) if $d = 2$ and $q = (q, P, \tilde{p}) \notin \text{desc}^*(\mathcal{Q})$, then $\text{dom}(Q^+) \setminus \text{dom}(Q) = \{(2, q^+)\}$, $q^+ = q^- \max_{< BK} 2^Q + \{q\}$, $\pi^+(q^+) = \pi(q^-)^-(a)$, $a = \text{min}_{< BK}(\{b : \forall \epsilon \in \mathcal{Q}(q) 2^Q(q^-)(\epsilon) < BK 2^Q(q^-)(-b)\}$.

69
Then $\pi^T_{\sup}(\gamma) = (\pi^+)^T \circ j^Q_{\sup}(\gamma)$.

If $\pi$ factors finite level $\leq 2$ trees $(Q, T)$ and $\pi$ is discontinuous at $(d, q)$, then $\text{pred}(\pi, T, (d, q))$ is a node in $\text{dom}(T)$ defined as follows:

1. If $d = 1$, then $\text{pred}(\pi, T, (d, q)) = (1, \text{pred}_{\sup}(\pi(q)))$.
2. If $d = 2$ and $q = (\emptyset, \emptyset, ((0))))$, then $\text{pred}(\pi, T, (d, q)) = (1, \max_{\sup} l^T)$.
3. If $d = 2$ and $q = (q, P, \tilde{p}) \in \text{desc}(Q)$, $q \neq \emptyset$, then $\text{pred}(\pi, T, (d, q)) = (2, \max_{\sup} 2T\{2\pi(q), \emptyset\})$.
4. If $d = 2$ and $q = (q, P, \tilde{p}) \notin \text{desc}(Q)$, $q \neq \emptyset$, then $\text{pred}(\pi, T, (d, q)) = (2, q^-(a))$.

If $(2, q) = (2, (q, P, \tilde{p})) \in \text{desc}(Q)$ then put $\text{pred}(\pi, T, (2, q)) = \text{pred}(\pi, T, (2, q))$.

The next lemma is the level-2 version of Lemma 3.5 whose proof is similar.

**Lemma 4.41.** Suppose $(Q^-, (d, q, P))$ is a partial level $\leq 2$ tree, $T$ is a finite level $\leq 2$ tree, $\pi$ factors $(Q, T)$, and $\pi$ is discontinuous at $(d, q)$. Let $\tau$ factor $(Q, T)$ where $\tau$ and $\pi$ agree on $\text{dom}(Q) \setminus \{(d, q)\}$, $\tau(d, q) = \text{pred}(\pi, T, (d, q))$.

Suppose $\text{cf}^{\pi_{\sup}}(\text{sup}(Q^-(T_2)))(\gamma) = \text{seed}^{Q^-}_{\text{ucf}(Q^-, (d, q, P))}$. Then

$$\pi^T \circ j^{Q^-\sup}(\gamma) = \tau^{T}_{\sup} \circ j^{Q^-\sup}(\gamma).$$

The level-2 version of Lemma 3.6 is similarly proved.

**Lemma 4.42.** Suppose $(Q, (d, q, P))$ is a partial level $\leq 2$ tree, $\text{ucf}(Q, (d, q, P)) = (d^*, q^*)$, $T$ is a finite level $\leq 2$ tree, $\pi$ factors $(Q, T)$, and $\pi$ is discontinuous at $(d^*, q^*)$. Let $Q^+$ be a completion of $(Q, (d, q, P))$ and let $\tau$ factor $(Q^+, T)$ so that $\tau$ extends $\pi$, $\tau(d, q) = \text{pred}(\pi, T, (d^*, q^*))$. Suppose $\text{cf}^{\pi_{\sup}}(\text{sup}(Q^+(T_2)))(\gamma) = \text{seed}^{Q^+}_{(d^*, q^*)}$. Then

$$\pi^T(\gamma) = \tau^{T}_{\sup} \circ j^{Q^+\sup}(\gamma).$$

Note that in Lemma 4.2 the completion $Q^+$ is decided by $\text{pred}(\pi, T, (d^*, q^*))$. There is no freedom in choosing $Q^+$.

In the same spirit as Lemmas 3.15-3.20 we will obtain a concrete way of deciding whether a tuple $\bar{\gamma}$ respects a level-3 tree $R$.

Suppose $E$ is a club in $\omega_1$. For a partial level $\leq 2$ tree $(Q, (d, q, P))$, put $\bar{\alpha} = (\alpha_{(e, t)})_{(e, t) \in \text{dom}(Q, (d, q, P))} \in [E]^{(Q, (d, q, P))}$ iff $\bar{\alpha}$ respects $(Q, (d, q, P))$. 

70
Let \( \alpha_t \in \text{dom}(Q) \in [E]^\omega \), and \( d = 1 \rightarrow 1^{ \alpha_q } \in E, d = 2 \rightarrow 2^{ \alpha_q } \in j^E(E) \). For a level-3 tree \( R \), put

\[
\text{rep}(R) \upharpoonright E = \{ \bar{\beta} \uplus_R r : r \in \text{dom}(R), \bar{\beta} \in [E]^\text{tree}(r) \uparrow \} \\
\cup \{ \bar{\beta} \uplus_R r^\frown (-1) : r \in \text{dom}(R), \bar{\beta} \in [E]^R(r) \uparrow \}.
\]

Then \( \text{rep}(R) \upharpoonright E \) is a closed subset of \( \text{rep}(R) \) (in the order topology of \( < R \)).

**Lemma 4.43.** Suppose \( R \) is a finite level-3 tree, \( B \in \mathbb{L}_{ \mathfrak{d}_T }[T_2] \) is a closed set of ordinals. Then \( \bar{\gamma} \in [B]^R \) if there is \( F \in (\mathfrak{d}_3)^R \) and \( E \in \mu_L \) such that \( \bar{\gamma} = [F]^R \) and for any \( r \in \text{dom}(R) \), for any \( \bar{\beta} \in [E]^R(r) \uparrow \), \( F_r(\bar{\beta}) \) is a limit point of \( B \).

**Proof.** The nontrivial direction is \( \Leftarrow \). Suppose \( F \in (\mathfrak{d}_3)^R \) and \( E \in \mu_L \) are as given. For \( r \in \text{dom}(R) \), let \( R(r) = (Q_r, (d_r, q_r, P_r)) \), and let \( r^* \) be the \( <_{BK} \)-greatest member of \( R\{r, -\} \). In parallel to Claim 3.16, by Theorem 4.38 and cofinality considerations in \( \mathbb{L}_{ \mathfrak{d}_T }[j^Q(T_2)] \), we have

**Claim 4.44.** There is \( E' \in \mu_L \) such that \( E' \subseteq E \) and for any \( r \in \text{dom}(R) \), for any \( \bar{\beta} \in [E']^P(r) \uparrow \),

1. if \( d_r = 1 \) then \( B \cap (F_r^* (\bar{\beta}), F_r(\bar{\beta})) \) has order type \( 1^{\beta_{q_r}} \);
2. if \( d_r = 2 \), \( \text{ucf}(R[r]) = (2, q_r^* \bar{s}), q_r^* = (q_r^* \bar{s}, P_r^* \bar{s}, P_r, P_r^\frown) \), then \( B \cap (F_r^* (\bar{\beta}), F_r(\bar{\beta})) \) has order type \( 2^{\beta_{q_r \bar{s}}} \).

The rest proceeds as in the proof of Lemma 3.15. \( \square \)

**Lemma 4.45.** Suppose \( R \) is a finite level-3 tree, \( R[r] = (Q_r, (d_r, q_r, P_r)) \) for \( r \in \text{dom}(R) \), \( E \in \mu_L \) is a club. Suppose \( f : \text{rep}(R) \upharpoonright E \rightarrow \delta_3^1 \) satisfies

1. if \( r \in \text{dom}(R) \), then the \( Q_r \)-potential partial level \( \leq 2 \) tower induced by \( F_r \) is \( R[r] \), the approximation sequence of \( F_r \) is \( (F_{\uparrow i})_{1 \leq i \leq \text{lh}(q)} \), and the uniform cofinality of \( F_r \) on \( [E]^Q \uparrow \) is witnessed by \( F_r(-1) \), i.e., if \( \bar{\beta} \in [E]^Q \uparrow \), then \( F_r(\bar{\beta}) = \sup\{ F_r(-1) (\bar{\beta} \frown (\gamma)) : \bar{\beta} \frown (\gamma) \in \text{rep}(R) \upharpoonright E \} \), and the map \( \bar{\beta} \mapsto F_r(-1) (\bar{\beta} \frown (\gamma)) \) is continuous, order preserving.
2. if \( R_{\text{tree}}(r^\frown (a)) = R_{\text{tree}}(r^\frown (b)) \), and \( a <_{BK} b \), then \( [F_r(-a)]_{\mu}^{q_r(-a)} < [F_r(-b)]_{\mu}^{q_r(-b)} \).

Then there is \( E' \in \mu_L \) such that \( E' \subseteq E \) and \( f \upharpoonright (\text{rep}(R) \upharpoonright E') \) is order preserving.
Proof. Put $ucf(R[r]) = (d_r, q_{r,s})$, and if $d_r = 2$ then $q_{r,s} = (q_{r,s}, P_{r,s}, \bar{p}_{r,s})$.

We know by assumption that for $\mu_{Q_r}$-a.e. $\vec{\beta}$, $F_r(\vec{\beta}) = \sup \{F_{r^-}(a)(\vec{\beta}^-(\gamma)) : r^- (a) \in \text{dom}(R), \gamma < a_{\beta_{q_r,s}} \}$. Fix for the moment $r$ such that $d_r \neq 0$.

Similarly to the proof of Lemma 3.17, we need a club $E' \in \mu_L$ such that for any $\vec{\beta} \in [E']^{\mu_{Q^+}}$, for any $\gamma < \gamma'$ both in $j^R(E')$, if $R_{\text{tree}}(\gamma^- (a)) = R_{\text{tree}}(\gamma^- (b))$ then $F_{\gamma^-}(a)(\vec{\beta}^-(\gamma)) < F_{\gamma^-}(b)(\vec{\beta}^-(\gamma'))$.

If $\vec{\beta}$ respects $Q_r$ and $\vec{\beta}^-(\gamma)$ respects $R(r)$, let $g(\vec{\beta}^-(\gamma))$ be the least $\gamma'$ satisfying that whenever $r^- (a), r^- (b) \in \text{dom}(R), \delta \leq \gamma, \delta' \geq \gamma', \vec{\beta}^-(\delta)$ respects $R(r^- (a)), \vec{\beta}^-(\delta')$ respects $R(r^- (b))$, we have $F_{r^-}(a)(\vec{\beta}^-(\delta)) < F_{r^-}(b)(\vec{\beta}^-(\delta'))$.

If $Q^+$ is a completion of $R(r)$, then for $\mu_{Q^+}$-a.e. $\vec{\xi}$, $g(\vec{\xi}) < d_{\xi_{q_r,s}}$. By Lemma 4.31, there is $h_{Q^+} : \omega_1 \to \omega_1$ and $E_{Q^+} \in \mu_L$ such that $h \in L$ and for any $\vec{\xi} \in [E_{Q^+}]^{\mu_{Q^+}}, g(\vec{\xi}) < j^R(h_{Q^+})(d_{\xi_{q_r,s}})$. There are only finitely many completions of $R(r)$. Let $h : \omega_1 \to \omega_1$ where $h(\alpha) = \sup \{h_{Q^+}(\alpha) : Q^+ \text{ is a completion of } R(r) \}$. Let $E' = \\cap \{E_{Q^+} : Q^+ \text{ is a completion of } R(r) \}$. Let $\eta \in E''$ if $h''(\eta \cap E') \subseteq \eta$. $E''$ is as desired. \qed

As corollaries of Lemmas 4.43 and 4.45 we obtain:

**Lemma 4.46.** Suppose that $R$ is a level-3 tree and $\vec{\gamma} = (\gamma_r)_{r \in \text{dom}(R)}$ is a tuple of ordinals in $\delta_3^1$. Then $\vec{\gamma}$ respects $R$ iff the following holds:

1. For any $r \in \text{dom}(R)$, the $R_{\text{tree}}(r)$-potential partial level $\leq 2$ tower induced by $\gamma_r$ is $R[r]$, and the $R_{\text{tree}}(r)$-approximation sequence of $\gamma_r$ is $(\gamma_r \eta)_{1 \leq r \leq \text{lh}(\eta)}$.

2. If $R_{\text{tree}}(\gamma^- (a)) = R_{\text{tree}}(\gamma^- (b))$ and $a < B_k b$ then $\gamma_{\gamma^- (a)} < \gamma_{\gamma^- (b)}$.

Moreover, if $B \in \mathbb{L}_{\delta_3^1}[T_2]$ is a closed set, $B'$ is the set of limit points of $B$, then $\vec{\gamma} \in [B']^{R_{\text{tree}}(r)}$ iff $\vec{\gamma}$ respects $R$ and for each $r \in \text{dom}(R)$, $\gamma_r \in j^{R_{\text{tree}}(r)}(B')$.

In particular, if $\vec{\gamma}$ respects $R$, then $\limsup_{r \in \text{dom}(R)}(\gamma_r) = u_{\limsup_{r \in \text{dom}(R)}}$ for $r \in \text{dom}(R)$, where $u_0 = \omega$.

**Lemma 4.47.** Suppose $R$ and $R'$ are level-3 trees with the same domain. Suppose $\vec{\gamma}$ respects both $R$ and $R'$. Then $R = R'$.

### 4.6 Factoring maps between level-2 trees

**Definition 4.48.** Suppose $I < \omega$. Suppose for each $i < I$, $J_i \leq I < \omega$ and $A_i = (a_{i,j})_{1 \leq j < I}$ is a finite sequence of sets. Then the contraction of $(A_i)_{i < I}$ is $(b_k)_{k < K}$ such that

1. $\{a_{i,j} : i < I, J_i < j < J_i \} = \{b_k : k < K\}$.
2. For each \( k < K \), letting \((i^k, j^k)\) be the \(<_{BK}\)-least \((i, j)\) such that \(a_{i,j} = b_k\), then the map \(k \mapsto (i^k, j^k)\) is order preserving with respect to \(<\) and \(<_{BK}\).

**Definition 4.49.** Suppose \( T, Q \) are level \( \leq 2 \) trees. A \((T, Q, -1)\)-description is of the form

\[
C = (1, (t, \emptyset))
\]

such that \( t \in \mathcal{U}(T) \). Suppose \((W, \vec{w})\) is a potential partial level \( \leq 1 \) tower of discontinuous type, \( \vec{w} = (w_i)_{i \leq m} \). If \( w = 0 \), the only \((T, Q, (W, \vec{w}))\)-description is \((2, ((\emptyset, \emptyset), \tau))\), where \( \tau \) factors \((\emptyset, Q, \emptyset)\), which is called the constant \((T, Q, *)\)-description. If \( w > 0 \), a \((T, Q, (W, \vec{w}))\)-description is of the form

\[
C = (2, (t, \tau))
\]

such that

1. \( t \in \text{desc}(T) \) and \( t \neq (\emptyset, \emptyset, \emptyset) \). Let \( t = (t, S, \vec{s}), \text{lh}(t) = k, \vec{s} = (s_i)_{i < \text{lh}(\vec{s})} \).
2. \( \tau \) factors \((S, Q, W)\).
3. The contraction of \((\text{sign}_1(\tau(s_i)))_{i < k}\) is \((w_i)_{i < m}\).
4. If \( t \) is of continuous type and \( w_{m-1} \) does not appear in the contraction of \((\text{sign}_1(\tau(s_i)))_{i < k-1}\) then \( \tau(s_{k-1}) \) is of level-1 discontinuous type.
5. Either \( \text{ucf}(S, \vec{s}) = w_m = -1 \) or \( \text{ucf}_1(\tau(\text{ucf}(S, \vec{s}))) = \text{ucf}(W, \vec{w}) \).

A \((T, Q, *)\)-description is either a \((T, Q, -1)\)-description or a \((T, Q, (W', \vec{w}'))\)-description for some potential partial level \( \leq 1 \) tower \((W', \vec{w}')\) of discontinuous type. For a level-1 tree \( W \), a \((T, Q, W)\)-description is a \((T, Q, (W, \vec{w}))\)-description for some \( \vec{w}' \). \( \text{desc}(T, Q, -1) \), \( \text{desc}(T, Q, (W, \vec{w})) \), \( \text{desc}(T, Q, *) \), \( \text{desc}(T, Q, W) \) denote the sets of relevant descriptions. We sometimes abbreviate \((d, t, \tau)\) for \((d, (t, \tau)) \in \text{desc}(T, Q, *)\) without confusion.

Recalling our notation of \( Q \otimes W \), we may regard \( \text{desc}(T, Q, -1) \subseteq \text{desc}(T, \emptyset) \), \( \text{desc}(T, Q, W) \subseteq \text{desc}(T, Q \otimes W) \). \( T \otimes (Q \otimes W) \) is also a “level-1 tree”, whose nodes consist of non-constant \((T, Q \otimes W)\)-descriptions, so that \( \text{desc}(T, Q \otimes W) = \text{desc}(T \otimes (Q \otimes W)) \). Every non-constant \((T, Q, W)\)-description is a member of \( T \otimes (Q \otimes W) \). The constant \((T, Q, *)\)-description \( C_0 \) is regarded as the constant \( T \otimes (Q \otimes W)\)-description, to make sense of \( \text{seed}_{c_0}^{T \otimes (Q \otimes W)} \). The degree of \((d, t, \tau) \in \text{desc}(T, Q, *)\) is \( d \). In fact, if \( C \in \text{desc}(T, Q, (W, \vec{w})) \) for which \( C \in \text{desc}(T, Q, (W, \vec{w})) \).
Suppose now \( C = (d, t, \tau) \in \text{desc}(T, Q, \ast) \), and if \( d = 2 \), then \( C \in \text{desc}(T, Q, W, \vec{w}) \), \( t = (t, S, \vec{s}) \), \( \text{lh}(t) = k \), \( \vec{s} = (s_i)_{i < k} \), \( \vec{w} = (w_i)_{i \leq m} \). The level-2 signature of \( C \) is

\[
\text{sign}_2(C) = \begin{cases} 
\emptyset & \text{if } d = 1, \\
\text{the contraction of } (\text{sign}_2^W(\tau(s_i)))_{i < k} & \text{if } d = 2.
\end{cases}
\]

\( C \) is of level-2 continuous type iff \( d = 2 \), \( t \) is of continuous type, and \( \tau(s_{k-1}) \) is of level-2 \( W \)-continuous type; \( C \) is of level-2 discontinuous type otherwise. The level-2 uniform cofinality of \( C = (d, t, \tau) \) is

\[
\text{ucf}_2(C)
\]
defined as follows. If \( d = 1 \), then \( \text{ucf}_2(C) = (0, -1) \). If \( d = 2 \) then

1. if \( \text{ucf}(S, \vec{s}) = -1 \), then \( \text{ucf}_2(C) = (0, -1) \);
2. if \( \text{ucf}(S, \vec{s}) = s_\ast \neq -1 \), then \( \text{ucf}_2(C) = \text{ucf}_2^W(\tau(s_\ast)) \).

If \( w_m \neq -1 \), \( C \) is said to be of level-2+ discontinuous type, and put

\[
\text{ucf}_2^+(C) = \text{ucf}_2^W(\tau(\text{ucf}(S, \vec{s}))),
\]

where \( W^+ \) is the completion of \((W, w_m)\). The level-2+ signature of \( C \) is

\[
\text{sign}_{2+}(C) = \begin{cases} 
((1, t)) & \text{if } d = 1, \\
((2, t \upharpoonright i))_{1 \leq i \leq k-1} & \text{if } d = 2, t \text{ is of continuous type,} \\
((2, t \upharpoonright i))_{1 \leq i \leq k} & \text{if } d = 2, t \text{ is of discontinuous type.}
\end{cases}
\]

\( C \) is of level-2+ \( Q \)-continuous type iff \( d = 2 \) and if \( \text{ucf}(S, \vec{s}) \neq -1 \land \tau(\text{ucf}(S, \vec{s})) \neq \min(\langle Q, W \rangle) \), then \( \text{pred}_{\prec_{Q,W}}(\tau(\text{ucf}(S, \vec{s}))) \in \text{ran}(\tau) \). Otherwise, \( C \) is of level-2+ \( Q \)-discontinuous type. The level-2+ \( Q \)-uniform cofinality of \( C \) is

\[
\text{ucf}_{2+}^Q(C)
\]
defined as follows. If \( d = 1 \), then \( \text{ucf}_{2+}^Q(C) = (1, t) \). If \( d = 2 \), \( t \) is of continuous type,

1. if \( C \) is of \( Q \)-continuous type, then \( \text{ucf}_{2+}^Q(C) = (2, (t^-, S \setminus \{s_{k-1}\}, \vec{s})) \);
2. if \( C \) is of \( Q \)-discontinuous type, then \( \text{ucf}_{2+}^Q(C) = (2, (t^-, S, \vec{s})) \).

If \( d = 2 \), \( t \) is of discontinuous type,

1. if \( C \) is of \( Q \)-continuous type, then \( \text{ucf}_{2+}^Q(C) = (2, t) \);
2. if $C$ is of $Q$-discontinuous type, then $\text{ucf}_{2}^{Q}(C) = (2, (t, S \cup \{s_{k}\}, \bar{s}))$.

For $h \in \omega_{1}^{\omega_{1}}$, if $C = (1, t, \emptyset)$ is a $(T, Q, -1)$-description, then

$$h_{C}^{Q} : [\omega_{1}]^{Q_{1}} \to \omega_{1}$$

is the function sending $\vec{\alpha}$ to $1^{[h]_{t}}$ if $\text{min}(\vec{\alpha}) > 1^{[h]_{t}}$, sending $\vec{\alpha}$ to $\text{min}(1, t)$ otherwise; if $W$ is a (possibly empty) level-1 tree, $C = (2, t, \tau)$ is a $(T, Q, W)$-description, $t = (t, S, \bar{s})$, then

$$h_{C}^{Q} : [\omega_{1}]^{Q_{1}} \to j^{W}(\omega_{1})$$

is the function that sends $[g]^{Q}$ to $[\mathcal{h}_{t} \circ g_{W}^{Q}]_{\mu W}$. Note here that $\mathcal{h}_{t} \circ g_{W}^{Q}$ has signature $\text{sign}(W, \vec{w})$, is essentially discontinuous, and has uniform cofinality $\text{ucf}(W, \vec{w})$. In either case, when $Q$ is finite, we have the following: the signature of $h_{C}^{Q}$ is sign$_{2}(C)$; $h_{C}^{Q}$ is essentially continuous iff $C$ is of level-2 continuous type; the uniform cofinality of $h_{C}^{Q}$ is $\text{ucf}_{2}(C)$. If $W^{+}$ is the completion of $(W, w_{n})$, then $j^{W, W^{+}} \circ h_{C}^{Q}$ is of discontinuous type and has cofinality $\text{ucf}_{2}^{Q}(C)$. Moreover, $\text{ran}(h_{C}^{Q}) \subseteq \text{ran}(h)$ if $d = 1$, $\text{ran}(h_{C}^{Q}) \subseteq j^{W}(\text{ran}(h))$ if $d = 2$. When $T, Q$ are both finite, $C = (d, t, \tau) \in \text{desc}(T, Q, *)$,

$$\text{id}_{C}^{T,Q}$$

is the function $[h]^{T} \mapsto [h_{C}^{Q}]_{\mu^{Q}} \circ \xi$, or equivalently, $\xi \mapsto \xi_{d}^{T,Q}$ when $d = 1$, $\vec{\xi} \mapsto \tau^{Q, W}(\vec{\xi})$ when $d = 2$ and $C \in \text{desc}(T, Q, W)$. The signature of $\text{id}_{C}^{T,Q}$ is sign$_{2}^{Q}(C)$; $\text{id}_{C}^{T,Q}$ is essentially continuous iff $C$ is of level-2* $Q$-continuous type; the uniform cofinality of $\text{id}_{C}^{T,Q}$ is $\text{ucf}_{2}^{Q}(C)$.

$$\text{seed}_{C}^{T,Q} \in \mathbb{L}_{\delta_{\bar{s}}}^{T}[j^{T} \circ j^{Q}(T_{2})]$$

is the element represented modulo $\mu^{T}$ by $\text{id}_{C}^{T,Q}$. Using Loś, it is clear that if $d = 1$, then for any $A \in \mu_{d}$, $\text{seed}_{C}^{T,Q} \in j^{T} \circ j^{Q}(A)$; if $d = 2$, then for any $A \in \mu^{(W, \bar{w})}$, $\text{seed}_{C}^{T,Q} \in j^{T} \circ j^{Q}(A)$. Using Lemma 4.32 we can see that $\text{seed}_{C}^{T,Q} \in \{u_{n} : n < \omega\}$, and $\text{seed}_{C}^{T,Q}$ can be computed in the following concrete way:

- If $d = 1$, then $\text{seed}_{C}^{T,Q} = \text{seed}_{C}^{T,\emptyset} = \text{seed}_{C}^{T \circ \emptyset}$.
- If $d = 2$ and $C \in \text{desc}(T, Q, W)$, then $\text{seed}_{C}^{T,Q} = \text{seed}_{C}^{T, Q \circ W} = \text{seed}_{C}^{T \circ (Q \circ W)}$.

If $C = (1, t, \emptyset)$, let

$$C^{T,Q} : \mathbb{L}_{\delta_{\bar{s}}}^{T}[j_{\mathcal{h}_{t}}(T_{2})] \to \mathbb{L}_{\delta_{\bar{s}}}^{T}[j^{T} \circ j^{Q}(T_{2})]$$

75
where $C^{T,Q}(j_{\mu}(h)(\omega_1)) = j^T \circ j^Q(\text{seed}_{C^{T,Q}})$. If $C = (2, t, \tau)$, let

$$C^{T,Q} : L_{\delta_1}[j^{(W_w)}(T_2)] \rightarrow L_{\delta_1}[j^T \circ j^Q(T_2)]$$

where $C^{T,Q}(j^{(W_w)}(h)(\text{seed}(W_\bar{w}))) = j^T \circ j^Q(\text{seed}_{C^{T,Q}})$.

Suppose $(\bar{W}, \bar{w}) = (W_i, w_i)_{i \leq m}$ is a potential partial level-1 tower. If $C = (2, t, \tau) \in \text{desc}(T, Q, (W_m, \bar{w}))$, $t = (t, S, \bar{s})$, define $\text{lh}(C) = m$. If $\bar{m} \leq m$, then

$$C \upharpoonright \bar{m} \in \text{desc}(T, Q, (W_i, (w_i)_{i \leq i}))$$

is defined by the following: letting $l$ be the least such that $\tau(s_i) \notin \text{desc}(Q, W_m)$, and $D \in \text{desc}(Q, W_m)$ be such that $D <_{L_{\delta}}(W_\bar{w}) \tau(s_i)$, then

1. if $D \neq \tau(s_i^{-1})$, then $C \upharpoonright \bar{m} = (2, t \upharpoonright l^{-1}, \bar{\tau})$, where $\bar{\tau}$ and $\tau$ agree on $\uparrow_{l}T_{\text{tree}}(t \upharpoonright l)$,

2. if $D = \tau(s_i^{-1})$, then $C \upharpoonright \bar{m} = (2, t \upharpoonright l, \tau \upharpoonright \uparrow_{l}T_{\text{tree}}(t \upharpoonright l))$.

Define

$$C \lhd C'$$

iff $C = C' \upharpoonright \bar{m}$ for some $\bar{m} < \text{lh}(C')$. Define $\prec^{T,Q} = \lhd \text{desc}(T, Q, \ast)$. As a corollary to Lemma 4.27, $\prec^{T,Q}$ inherits the following continuity property.

**Lemma 4.50.** Suppose $T, Q$ are finite level $\leq 2$ trees, $W$ is a level-1 proper subtree of $W'$. Suppose $E \in \mu_L$ is a club, $E'$ is the set of limit points of $E$. Suppose $C = (2, t, \tau) \in \text{desc}(T, Q, W)$, $C' = (2, t', \tau') \in \text{desc}(T, Q, W')$, $C \prec^{T,Q} C'$. Then for any $h \in \omega_1^{T_{\text{tree}}}$, for any $g \in \omega_1^{T_{\text{tree}}}$, for any $\bar{\alpha} \in [E']^{W_{\bar{\alpha}}}$,

$$h_{t'} \circ g_{t'}^W(\bar{\alpha}) = \sup\{h_{t'} \circ g_{t'}^W(\bar{\beta}) : \bar{\beta} \in [E']^{W_{\bar{\alpha}}}, \bar{\beta} \text{ extends } \bar{\alpha}\}.$$

Hence, the signature and approximation sequence of $h_{t'} \circ g_{t'}^W$ are proper initial segments of those of $h_{t'} \circ g_{t'}^W$ respectively.

Let

$$\langle C \rangle = \begin{cases} 
(1, t) & \text{if } d = 1, \\
(2, \tau \oplus t) & \text{if } d = 2.
\end{cases}$$

Define

$$C \prec C'$$

iff $\langle C \rangle \prec_{BK} \langle C' \rangle$, the ordering on subcoordinates in $\text{desc}(Q, \ast) \cup \text{desc}(Q', \ast)$ according to $\prec$ acting on $\text{desc}(Q, \ast) \cup \text{desc}(Q', \ast)$. The constant $(T, Q, \ast)$-description, $C_0$, is the $\prec$-greatest, and we have $\langle C_0 \rangle = (2, \emptyset)$. Define $\prec^{T,Q} = \prec \upharpoonright \text{desc}(T, Q, \ast)$. $\prec^{T,Q}$ inherits the following ordering property as a corollary to Lemma 4.25.
Lemma 4.51. Suppose \((\bar{W}, \bar{w}) = (W_i, w_i)_{i \leq k}\) is a partial level \(\leq 1\) tower. Suppose \(C = (2, t, \tau) \in \text{desc}(T, Q, W_k)\), \(C’ = (2, t’, \tau’) \in \text{desc}(T, Q, W_{k’})\), \(k \leq m, k’ \leq m\), \(C \prec^{T, Q} C’\). Then for any \(h \in \omega_{\uparrow}^{T, Q}\), for any \(g \in \omega_{\uparrow}^{Q, T}\), for any \(\bar{\alpha} \in \omega_{\uparrow}^{W_{m}}, h \circ g_{\bar{w}}^{W_{m}}(\bar{\alpha}) < h \circ g_{\bar{w}}^{W_{m}}(\bar{\alpha}).\)

Definition 4.52. Suppose \(X, T, Q\) are level \(\leq 2\) trees. Suppose \(\pi : \text{dom}(X) \to \text{desc}(T, Q, *)\) is a function. \(\pi\) is said to factor \((X, T, Q)\) iff

1. If \((1, x) \in \text{dom}(X)\), then \(\pi(1, x) \in \text{desc}(T, Q, -1) \cup \text{desc}(T, Q, 2X[0]).\)
2. If \((2, x) \in \text{dom}(X)\), then \(\pi(2, x) \in \text{desc}(T, Q, 2X[x]).\)
3. \(\pi(2, \emptyset)\) is the constant \((T, Q, \pi)\)-description.
4. For any \((d, x), (d’, x’) \in \text{dom}(X)\), if \((d, x) <_{BK} (d’, x’)\) then \(\pi(d, x) \prec^{T, Q} \pi(d’, x’).\)
5. For any \(x \in \text{dom}(2X) \setminus \{\emptyset\}\), \(\pi(2, x^-) \prec^{T, Q} \pi(2, x).\)

\(\pi\) is said to factor \((X, T, *)\) iff \(\pi\) factors \((X, T, Q’)\) for some level \(\leq 2\) tree \(Q’\).

Suppose \(T\) is a level \(\leq 2\) tree.

\[
\text{id}_{T,*} \quad \text{factors} \quad (T, T, Q^0) \quad \text{where} \quad \text{id}_{T,*}(1, t) = (1, t, \emptyset), \quad \text{id}_{T,*}(2, t) = (2, (t, S, \bar{s}), \text{id}_{*S}).
\]

\[
\text{id}_{*T} \quad \text{factors} \quad (T, Q^0, T) \quad \text{where} \quad \text{id}_{*T}(1, t) = (2, \bar{q}_0, \tau^1_t), \quad \bar{q}_0 = (\langle 1 \rangle, \langle 0 \rangle, \langle 0 \rangle, \tau^1_t). \]

\[
\tau^1_t(\emptyset) = (1, t, \emptyset), \quad \tau^1_t(2, t) = (2, \bar{q}_0, \tau^2_t) \quad \text{when} \quad \text{id}_{T}(t) = (S, S), \quad \tau^2_t(\emptyset) = (2, (t, S, \bar{s}), \text{id}_{S}).
\]

If \(\pi \neq \emptyset\) factors \((X, T, Q)\), then \(T\) is \(\Pi^1_2\)-wellfounded and \(h \in \omega_{\uparrow}^{T, Q}\), let

\[
h^Q_{\pi} : \omega_{\uparrow}^{T, Q} \to [\omega_{\uparrow}^{X, T}]^{X, T}
\]

be the function that sends \(\bar{\beta}\) to \((h^Q_{(d, x)}(\bar{\beta}))(d, x) \in \text{dom}(X)\). The fact that \(h^Q_{\pi}(\bar{\beta}) \in [\omega_{\uparrow}^{X, T}]^{X, T}\) follows from Lemmas \(\ref{lem:4.18} - \ref{lem:4.50}\). Moreover, for any \(\bar{\beta} \in [\omega_{\uparrow}^{X, T}]^{X, T}\), \(h^Q_{\pi}(\bar{\beta}) \in [\text{ran}(h)]^{X, T}\). In particular, if \(Q = X\) then \(h^Q_{\text{id}_{T,*}}\) is the identity function on \([\omega_{\uparrow}^{X, T}]^{X, T}\). If \(T\) is finite, let

\[
\text{id}_{T, Q}^Q \quad \text{be the function} \quad [h]^{T} \mapsto [h^Q_{\pi}]_{\mu, Q}, \quad \text{or equivalently}, \quad \hat{\xi} \mapsto (\text{id}_{T, Q}(\hat{\xi}))(d, x) \in \text{dom}(X).
\]

Let

\[
\text{seed}_{T, Q} = [\text{id}_{T, Q}]_{\mu, T}.
\]
By Loś and Lemmas 3.18, 4.51-4.50, it is clear that for any $A \in \mu^X$, seed$^T_Q T, Q \in j^T \circ j^Q(A)$. Define

$$\pi^{T,Q} : L_{\delta_1}[j^X(T_2)] \to L_{\delta_1}[j^T \circ j^Q(T_2)]$$

by sending $j^X(h)(\text{seed}^X)$ to $j^T \circ j^Q(h)(\text{seed}^{T,Q})$.

Suppose $T, Q$ are both level $\leq 2$ trees. A representation of $T \otimes Q$ is a pair $(X, \pi)$ such that

1. $X$ is a level $\leq 2$ tree;
2. $\pi$ factors $(X, T, Q)$;
3. $\text{ran}(\pi) = \text{desc}(T, Q, *)$;
4. $(d, x) <_{BK} (d', x')$ iff $\pi(d, x) \prec^{T,Q} \pi(d', x')$.

Representations of $T \otimes Q$ are clearly mutually isomorphic. We shall regard $T \otimes Q$ itself as a “level $\leq 2$ tree” whose level-$d$ component is $(t, \tau)$ for which $(d, (t, \tau)) \in \text{desc}(T, Q, *)$, and whose level-$2$ component sends $(t, \tau)$ to $(W, w_m)$ if $(2, (t, \tau)) \in \text{desc}(T, Q, (W, (w_i)_{i \leq m}))$. In this way, $\pi$ is a “level $\leq 2$ tree isomorphism” between $X$ and $T \otimes Q$. All the relevant terminologies of level $\leq 2$ trees carry over to $T \otimes Q$ in the obvious ways. In particular, if $W$ is a finite level-1 tree, a $(T \otimes Q, W)$-description takes one of the following forms (recall that $(d, (t, \tau))$ is simply an abbreviation of $(d, (t, \tau))$):

1. $(1, (t, \emptyset), \emptyset)$ for $(1, (t, \emptyset)) \in \text{desc}(T, Q, -1)$;
2. $(2, ((t, \tau), Z, \bar{z}, \psi)$ for $(2, (t, \tau)) \in \text{desc}(T, Q, (Z, \bar{z}))$ and $\psi$ factoring $(Z, W)$;
3. $(2, ((t, \tau) \neg (-1), Z^+, \bar{z}, \psi)$ for $(2, (t, \tau)) \in \text{desc}(T, Q, (Z, \bar{z}))$, $\bar{z} = (z_i)_{i \leq t}$, $Z^+ \cup \{z_l\}$ and $\psi$ factoring $(Z^+, W)$.

$(T \otimes Q) \otimes W$ is thus regarded as a “level-1 tree” whose nodes consists of non-constant $(T \otimes Q, W)$-descriptions. There is a natural isomorphism

$$\iota^{T,Q,W}$$

between “level-1 trees” $(T \otimes Q) \otimes W$ and $T \otimes (Q \otimes W)$, defined as follows.

1. $\iota^{T,Q,W}(1, (t, \emptyset), \emptyset) = (1, t, \emptyset)$.
2. If $(2, t, \tau) \in \text{desc}(T, Q, (Z, \vec{z}))$, $t = (t, S, \vec{s})$, $\psi$ factors $(Z, W)$, define $\nu_{T,Q,W}(2, ((t, \tau), Z, \vec{z}), \psi) = (2, t, (Q \otimes \psi) \circ \tau)$.

3. If $(2, t, \tau) \in \text{desc}(T, Q, (Z, \vec{z}))$, $t = (t, S, \vec{s})$, $\vec{z} = (z_i)_{i \leq l}$, $\vec{s} = (s_i)_{i \leq k}$, $Z^+ = Z \cup \{z_i\}$, $\psi$ factors $(Z^+, W)$,

(a) if $t$ is of discontinuous type, define $\nu_{T,Q,W}(2, ((t, \tau), Z^+, \vec{z}), \psi) = (2, t^\omega(-1), \psi \ast_0 \tau)$, where $\psi \ast_0 \tau$ factors $(S \cup \{s_i\}, Q, W)$, $\psi \ast_0 \tau$ extends $(Q \otimes \psi) \circ \tau$, $\psi \ast_0 \tau(s_k) = (2, (q_0, \sigma))$, $q_0 = ((-1), \{(0)\}, \{(0)\})$, $\sigma((0)) = \psi(z_i)$;

(b) if $t$ is of continuous type, define $\nu_{T,Q,W}(2, ((t, \tau), Z^+, \vec{z}), \psi) = (2, t, \psi \ast_1 \tau)$, where $\psi \ast_1 \tau$ factors $(S, Q, W)$, $\psi \ast_1 \tau$ extends $(Q \otimes \psi) \circ \tau \mid (S \setminus \{s_k\})$, $\psi \ast_1 \tau(s_k) = (2, q^\omega(-1), \sigma^+)$ where $\tau(s_k) = (2, q, \sigma)$, $q = (q, p, (p_i)_{i \leq m})$, $\sigma^+$ extends $\sigma$, $\sigma^+(p_m) = \psi(z_i)$.

The reason why $\nu_{T,Q,W}$ is a surjection is the following. Suppose $C \in T \otimes (Q \otimes W)$ is of degree 2. Put $C = (2, t, \tau)$, $t = (t, S, \vec{s})$, $\vec{s} = (s_i)_{i \leq \text{lh}(\vec{s})}$, $k = \text{lh}(t)$. Let $(w_i)_{i < k}$ be the contraction of $(\sigma_1(\tau(s_i)))_{i < k}$. Then $w_0$ is the $\leq_Bk$-maximum of $\{w_i : i < k\}$. Let $(Z, \vec{z}) = (Z, (z_i)_{i \leq m})$ be the potential partial level $\leq 1$ tower of continuous type and $\psi : Z \to W$ be the level-1 tree isomorphism such that $\psi(z_i) = w_i$ for any $i < m$. If $t$ is of continuous type, $\tau(s_{k-1})$ is of level-1 continuous type, but $w_{m-1}$ does not appear in the contraction of $(\sigma_1(\tau(s_i)))_{i < k-1}$, then

$$C = \nu_{T,Q,W}(2, ((t, \tau)^\omega(-1), Z, \vec{z}), \psi).$$

Otherwise,

$$C = \nu_{T,Q,W}(2, ((t, \tau), Z, \vec{z}^\omega(z_i)), \psi),$$

where $(Z, z_*)$ is a partial level $\leq 1$ tree, $z_* = -1$ if $\text{ucf}(S, \vec{s}) = -1$, $z_* = \text{ucf}_1((\tau(\text{ucf}(S, \vec{s}))))$ if $\text{ucf}(S, \vec{s}) \neq -1$. $\nu_{T,Q,W}$ justifies the associativity of the $\otimes$ operator acting on level $(\leq 2, \leq 2, 1)$-trees.

The identity function $\text{id}_{T \otimes Q}$ factors $(T \otimes Q, T, Q)$. By definitions and Lemmas 4.32 3.11

$$(\text{id}_{T \otimes Q})^{T,Q}(\text{seed}_{C^*}(T \otimes Q \otimes W)) = \text{seed}_{\nu_{T,Q,W}(C^*)}(T \otimes Q \otimes W)$$

for any $C^* \in (T \otimes Q) \otimes W$. Hence, $(\text{id}_{T \otimes Q})^{T,Q}(u_n) = u_n$ for any $n < \omega$. As $(\text{id}_{T \otimes Q})^{T,Q}$ is elementary from $L_{\delta_1}[j^{T \otimes Q}(T_2), x]$ to $L_{\delta_1}[j^{T \otimes Q}(T_2), x]$ for any $x \in \mathbb{R}$, $(\text{id}_{T \otimes Q})^{T,Q}$ is the identity map on $L_{\delta_1}[T_2]$.

Suppose $\pi$ factors level $\leq 2$ trees $(X, T)$ and $Q$ is another level $\leq 2$ tree.

- $\pi \otimes Q$ factors $(X \otimes Q, T \otimes Q)$, defined as follows: $\pi \otimes Q(d, x, \tau) = (d, d'_{\pi}(x), \tau)$. 79
• $Q \otimes \pi$ factors $(Q \otimes X, Q \otimes T)$, defined as follows: $Q \otimes \pi(d, q, \tau) = (d, q, (\pi \otimes W) \circ \tau)$, where $\tau$ factors $(P, X \otimes W)$. 

We effectively obtain the following lemma which reduces finite iterations of level $\leq 2$ ultrapowers to a single level $\leq 2$ ultrapower. The proof is in parallel to Lemma 4.34.

**Lemma 4.53.** Suppose $X, T, Q$ are finite level $\leq 2$ trees. Then

1. $j^T \circ j^Q = j^{T \otimes Q}$.

2. $\pi$ factors $(X, T, Q)$ iff $\pi$ factors $(X, T \otimes Q)$. If $\pi$ factors $(X, T, Q)$ then $\pi^{T,Q} = \pi^{T \otimes Q}$.

3. If $\pi$ factors $X, T$, then
   
   (a) $j^Q(\pi^T | a) = (Q \otimes \pi)^{Q \otimes T} | j^Q(a)$ for any $a \in L_{\delta^1_3}[T_2]$;
   
   (b) $\pi^T | L_{\delta^1_3}[j^{X \otimes Q}(T_2)] = (\pi \otimes Q)^{T \otimes Q}$.

Suppose $T, Q, U$ are level $\leq 2$ trees. There is a natural “level $\leq 2$ tree isomorphism” $\iota_{T,Q,U}$ between $(T \otimes Q) \otimes U$ and $T \otimes (Q \otimes U)$ defined as follows. Suppose $B \in \text{desc}(T \otimes Q, U, *)$.

1. If $B = (1, (t, \emptyset), \emptyset)$, $C = (1, t, \emptyset) \in \text{desc}(T, Q, -1)$, then $C \in \text{desc}(T, Q \otimes U, -1)$ and $\iota_{T,Q,U}(B) = C$.

2. If $B = (2, ((t, \tau), Z, \bar{z}), \psi) \in \text{desc}(T \otimes Q, U, W)$, $C = (2, t, \tau) \in \text{desc}(T, Q, (Z, \bar{z}))$, $t = (t, s, \bar{s})$, $\psi$ factors $(Z, U, W)$, then $\iota_{T,Q,U}(B) = (2, t, \iota_{Q,U,W}^{-1} \circ (Q \otimes \psi) \circ \tau)$.

3. If $B = (2, ((t, \tau), (-1), Z^+, \bar{z}), \psi) \in \text{desc}(T \otimes Q, U, W)$, $C = (2, t, \tau) \in \text{desc}(T, Q, (Z, \bar{z}))$, $t = (t, s, (s_i)_{i \leq k})$, $\bar{z} = (z_i)_{i \leq i}$,
   
   (a) if $t$ is of discontinuous type, then $\iota_{T,Q,U}(B) = (2, t \sim (-1), \psi \ast_0 \tau)$, where $\psi \ast_0 \tau$ factors $(S \cup \{s_k\}, Q \otimes U, W)$, $\psi \ast_0 \tau$ extends $\iota_{Q,U,W}^{-1} \circ (Q \otimes \psi) \circ \tau$, $\psi \ast_0 \tau(s_k) = \iota_{Q,U,W}^{-1}(2, q_0, \sigma)$, $q_0 = ((-1), \{(0)\}, \{(0)\})$, $\sigma(0) = (z_i)$.

   (b) if $t$ is of continuous type, then $\iota_{T,Q,U}(B) = (2, t, \psi \ast_1 \tau)$, where $\psi \ast_1 \tau$ factors $(S, Q \otimes U, W)$, $\psi \ast_1 \tau$ extends $\iota_{Q,U,W}^{-1} \circ (Q \otimes \psi) \circ (\tau \upharpoonright (S \setminus \{s_k\}))$, $\psi \ast_1 \tau(s_k) = \iota_{Q,U,W}^{-1}(2, q^-(-1), \sigma^+)$ where $\tau(s_k) = (2, q, \sigma)$, $q = (q, P, (p_i)_{i \leq m})$, $\sigma^+$ extends $\sigma$, $\sigma^+(p_m) = (z_i)$. 

80
\( \nu_{T,Q,U} \) justifies the associativity of the \( \otimes \) operator acting on level \( (\leq 2, \leq 2, \leq 2) \) trees.

**Lemma 4.54.** Suppose \( X, T \) are level \( \leq 2 \) trees, \( \theta : \text{rep}(X) \to \text{rep}(T) \) is a function in \( L \), order-preserving and continuous. Then there exists a triple 
\[
(Q, \pi, \vec{\gamma})
\]
such that \( Q \) is a level \( \leq 2 \) tree, \( \pi \) factors \( (X, T, Q) \), \( \vec{\gamma} \) respects \( Q \), and 
\[
\forall h \in \omega_1^{T} \quad h_{T,Q}^{\vec{\gamma}}(\vec{\gamma}) = [h \circ \theta]^X.
\]

**Proof.** For \( d \in \{1, 2\} \), let \( A^d = \{ x \in 1^X : \theta(1, (x)) \in \{d\} \times \text{rep}(^dT) \} \). By order preservation and continuity of \( \theta \), \( A^1 \) is a \( \prec^X \)-initial segment of \( 1^X \). For \( x \in A^1 \), let \( t^1_x \in T \) be such that 
\[
\theta(1, (x)) = (1, (t^1_x)).
\]
The existence of \( t^1_x \) follows from the fact that \( (1, (x)) \) has cofinality \( \omega \) in \( \text{rep}(X) \). For \( x \in A^2 \), let \( t^2_x = (t^1_x, S^2_x, \vec{s}^2_x) \in \text{desc}(^2T) \), \( s^2_x = (s^2_x)_{i<\text{lh}(s^2_x)} \) and \( \vec{\beta}^2_x = (\beta^2_{x,s})_{s \in \text{lh}(s^2)} \) be such that 
\[
\theta(1, (x)) = (2, \vec{\beta}^2_x \oplus_{^2T} t^2_x).
\]

For \( x \in \text{dom}(^2X) \), let \( ^2X(x) = (W_x, w_x) \). By order preservation and continuity of \( \theta \), we can find \( t_x = (t_x, S_x, \vec{s}_x) \in \text{desc}(T) \) and \( \theta_x \in L \) such that for \( \mu_{W_x} \)-a.e. \( \vec{\alpha} \), 
\[
\theta(2, \vec{\alpha} \oplus_{^2T} x) = (2, \theta_x(\vec{\alpha}) \oplus_{^2T} t_x).
\]

Let \( \vec{s}_x = (s_x(i))_{i<\text{lh}(s_x)} \), \( s_x = s_x(\text{lh}(s_x)) \). Let \( \theta_x[\mu_{W_x}] = \vec{\beta}_x = (\beta_{x,s})_{s \in S_x \cup \{0\}} \), 
\[
\theta_x(\vec{\alpha}) = (\theta_x(\vec{\alpha})_s)_{s \in S_x \cup \{0\}},
\]
so that \( \beta_{x,s} = [\theta_x]_{\mu_{W_x}} w_x \). In particular, \( t_\emptyset = \emptyset \), \( \beta_{\emptyset,0} = \omega_1 \), and \( t_x \neq \emptyset \) when \( x \neq \emptyset \). Fixing \( x \), the map \( s \mapsto \beta_{x,s} \) is order preserving with respect to \( \prec^{S_x} \) and \( \prec \). Let 
\[
B_x = \{ s \in S_x : \beta_{x,s} < \omega_1 \}.
\]

So \( B_x \) is closed under \( \prec^{S_x} \). For \( s \in S_x \setminus B_x \), let \( (P_{x,s}, \vec{p}_{x,s}) \) be the potential partial level \( \leq 1 \) tower induced by \( \beta_{x,s} \), \( \vec{p}_{x,s} = (p_{x,s,i})_{i<\text{lh}(\vec{p}_{x,s})} \), \( P_{x,s} = p_{x,s,\text{lh}(\vec{p}_{x,s})} \), \( i \). Let \( (\text{seed}^{W_x}_{x,s,i})_{i<v_{x,s}} \) be the signature of \( \beta_{x,s} \), let \( (\gamma_{x,s,i})_{i<v_{x,s}} \) be the approximation sequence of \( \beta_{x,s} \), and let \( \text{cf}^{W_x}(\beta_{x,s}) = \text{seed}^{W_x}_{x,s,i} \) if \( \text{cf}^{W_x}(\beta_{x,s}) > \omega \). Let \( \sigma_{x,s} \) factor \( (P_{x,s}, W_x) \), where \( \sigma_{x,s}(p_{x,s,i}) = w_{x,s,i} \) for \( i < v_{x,s} \). Let 
\[
D_x = \{ s \in S_x \setminus B_x : \beta_{x,s} \text{ is essentially continuous} \},
\]
\[
E_x = S_x \setminus (B_x \cup D_x).
\]
Claim 4.55. Suppose that \( x \in \text{dom}(2X) \).

1. If \( t_x \) is of continuous type, then \( \theta_{x,s_x} \) has uniform cofinality \( \text{ufc}(2X[x]) \).

2. If \( t_x \) is of continuous type and \( x = (x')^- \), then \( \theta_{x,s_x} \) is essentially discontinuous and thus \( s_x \in E_x \).

3. If \( t_x \) is of discontinuous type then
   
   (a) if \( w_x = -1 \), then \( s_x = -1 \);
   
   (b) if \( w_x \neq -1 \), then \( s_x \neq -1 \), \( \theta_{x,s_x} \) has uniform cofinality \( w_x^- \), and thus \( w_{x,s_x} = w_x^- \).

Claim 4.55. Suppose \( x, x' \in \text{dom}(2X) \), \( x = (x')^- \), \( t_x \) is of continuous type, and the contraction of \( ((w_{x,s_x,j,i})_{i<v_{x,s_x,j}})_{j<\text{lh}(t_x)} \) is \( (w_{x\beta})_{i<\text{lh}(x)} \). Then

1. \( t_x = t_{x'} \).

2. For any \( s \in S_x \setminus \{s_x\} \), \( \beta_{x,s} = \beta_{x',s} \).

3. \( (w_{x,s_x,i}, \gamma_{x,s_x,i})_{i<v_{x,s_x}} \) is a proper initial segment of \( (w_{x',s_x,i}, \gamma_{x',s_x,i})_{i<v_{x',s_x}} \).
   
   Hence, \( P_{x,s_x} \) is a proper subtree of \( P_{x',s_x} \) and \( p_{x,s_x} \) is an initial segment of \( p_{x',s_x} \).

4. \( \sigma_{x',s_x}(p_{x,s_x}) = w_x \).
   In particular, the contraction of \( ((w_{x',s_x,j,i})_{i<v_{x',s_x,j}})_{j<\text{lh}(t_x)} \) is \( (w_{x\beta})_{i<\text{lh}(x)} \).

Proof. By order preservation and continuity of \( \theta \), \( t_x = t_{x'} \) and for \( \mu^{W_x} \)-a.e. \( \bar{\alpha} \),

1. for any \( s \in S_x \setminus \{s_x\} \), if \( \bar{\alpha}' \in [\omega_1]^{W_{x'}} \) extends \( \bar{\alpha} \) then \( \theta_{x,s}(\bar{\alpha}) = \theta_{x',s}(\bar{\alpha}') \);

2. \( \theta_{x,s_x}(\bar{\alpha}) = \sup\{\theta_{x',s_x}(\bar{\alpha}') : \bar{\alpha}' \in [\omega_1]^{W_{x'}} \text{ extends } \bar{\alpha}\} \).

Thus, \( \beta_{x,s} = \beta_{x',s} \) for any \( s \in S_x \setminus \{s_x\} \), and \( j_{\text{sup},W_x}(\beta_{x,s_x}) \leq j_{\text{sup},W_{x'}}(\beta_{x',s_x}) \). As \( t_x = t_{x'} \) is of continuous type and \( w_x \) does not appear in \( \text{sign}(\theta_{x',s}) \) for any \( s \in S_{x'} \setminus \{s_{x'}\} \), \( \theta_{x',s_x} \) is essentially discontinuous, giving \( j_{\text{sup},W_x}(\beta_{x,s_x}) \neq \beta_{x',s_x} \). We can then apply Lemma 3.14 to show that the partial finite level \( \leq 1 \) tower induced by \( \beta_{x,s_x} \) is a proper initial segment of that induced by \( \beta_{x',s_x} \), and \( w_{x',s_x,v_{x,s}} = w_x \). \( \square \)
Claim 4.56. Suppose \( x, x' \in \text{dom}(2X) \), \( x = (x')^- \), \( t_x \) is of discontinuous type, and the contraction of \(((w_{x,s,i,j})_{i<v_{x,s,i,j}})_{j<\text{lh}(t_x)}\) is \((w_{\bar{d}i})_{i<\text{lh}(x)}\). Then

1. \( t_x \subseteq t_x' \).
2. for any \( s \in S_x \), \( \beta_{x,s} = \beta_{x',s} \).
3. \((w_{x,s,i,j})_{i<v_{x,s,i,j}}\) is a proper initial segment of \((w_{x',s,i,j})_{i<v_{x',s,i,j}}\).
Hence, \( P_{x,s} \) is a proper subtree of \( P_{x',s} \) and \( \bar{p}_{x,s} \) is an initial segment of \( \bar{p}_{x',s} \).
4. \( \sigma_{x,s}(p_{x,s}) = w_x \). In particular, the contraction of \(((w_{x',s,i,j})_{i<v_{x',s,i,j}})_{j<\text{lh}(t_{x'})}\) is \((w_{\bar{d}i})_{i<\text{lh}(x)}\).

Proof. By order preservation and continuity of \( \theta \), \( t_x \subseteq t_x' \) and for \( \mu^W \)-a.e. \( \bar{\alpha} \),

1. for any \( s \in S_x \), if \( \bar{\alpha}' \) extends \( \bar{\alpha} \) then \( \theta_{x,s}(\bar{\alpha}) = \theta_{x',s}(\bar{\alpha}') \);
2. \( \theta_{x,s}(\bar{\alpha}) = \sup \{ \theta_{x',s}(\bar{\alpha}') : \bar{\alpha}' \) extends \( \bar{\alpha} \} \).

The rest is similar to the proof of Claim 4.55.

Let

\[ \phi^1 : \{ \beta^2_{x,s} : x \in A^2, s \in S^2_x \} \cup \{ \beta_{x,s} : x \in \text{dom}(2X), s \in B_x \} \to Z^1 \]

be a bijection such that \( Z^1 \) is a level-1 tree and \( v < v' \iff \phi^1(v) < Z^1 \phi^1(v') \).

Let

\[ \phi^2 : \{(w_{x,s,i}, \gamma_{x,s,i})_{i<l} : x \in \text{dom}(2X), s \in D_x \cup E_x, l < \text{lh}(\bar{p}_{x,s}) \} \to Z^2 \cup \{ \emptyset \} \]

be a bijection such that \( Z^2 \) is a level-1 tree and \( v \subseteq v' \iff \phi^2(v) \subseteq \phi^2(v') \), \( v <_{BK} v' \iff \phi^2(v) <_{BK} \phi^2(v') \).

Let

\[ Q = (\dot{Q}, \dot{Q}) \]

where \( \dot{Q} = Z^1 \), \( \dot{Q} \) is a level-2 tree, \( \text{dom}(\dot{Q}) = Z^2 \),

\[ 2Q[\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\text{lh}(\bar{p}_{x,s})-1})^{-1}(-1)] = (P_{x,s}, \bar{p}_{x,s}) \quad \text{for} \quad s \in D_x, \]

\[ 2Q[\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i<\text{lh}(\bar{p}_{x,s})-1})] = (P_{x,s}, \bar{p}_{x,s}) \quad \text{for} \quad s \in E_x. \]

Let

\[ \bar{\gamma} = (\ddot{\gamma}_q)_{(d,q) \in \text{dom}(\dot{Q})} \]
where $\gamma_1 = (\phi^1)^{-1}(q), \gamma_2 = \omega_1, \gamma_q = \gamma_{x,s,t}^1$ when $q = \phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i \leq 1})$. For $x \in A^1$, let
\[
\pi(1, x) = (1, t_x^1, \emptyset).
\]
For $x \in A^2$, let
\[
\pi(1, x) = (2, t_x^2, \tau_x^2),
\]
where $\tau_x^2$ factors $(\tau_x^2, Q, \emptyset), \tau_x^2(1, s) = (1, \phi^1(\beta_{x,s}^2), \emptyset)$. For $x \in \text{dom}(2X)$, let
\[
\pi(2, x) = (2, t_x, \tau_x),
\]
where $\tau_x$ factors $(\tau_x, Q, W_x)$, defined as follows:
\[
\tau_x(s) = \begin{cases} 
(1, \phi^1(\beta_{x,s}^1), \emptyset) & \text{if } s \in B_x, \\
(2, (\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i < \text{lh}(\bar{p}_{x,s})})_{-1})(\neg 1), P_{x,s}, \bar{p}_{x,s}, \sigma_{x,s}) & \text{if } s \in D_x, \\
(2, (\phi^2((w_{x,s,i}, \gamma_{x,s,i})_{i < \text{lh}(\bar{p}_{x,s})})_{-1}), P_{x,s}, \bar{p}_{x,s}, \sigma_{x,s}) & \text{if } s \in E_x.
\end{cases}
\]
It is easy to check that $(Q, \pi, \bar{\gamma})$ works for the lemma. \(\square\)

Note that if $\pi$ factors $\Pi^1_2$-wellfounded trees $(X, T)$, then $[d, x]_X \leq [\pi(d, x)]_T$ for any $(d, x) \in \text{dom}(X)$. We say that $\pi$ minimally factors $(X, T)$ iff $\pi$ factors $(X, T)$, $X, T$ are both $\Pi^1_2$-wellfounded and $[d, x]_X = [\pi(d, x)]_T$ for any $(d, x) \in \text{dom}(X)$. In particular, if $T, Q$ are both $\Pi^1_2$-wellfounded, then id$_{T,*,}$ minimally factors $(T, T \otimes Q)$. In the assumption of Lemma 4.54, if $X, T$ are $\Pi^1_2$-wellfounded and the map $\theta$ is a bijection between $\text{rep}(X)$ and $\text{rep}(T)$, its proof constructs $\pi$ which minimally factors $(X, T \otimes Q)$. This entails the comparison theorem between $\Pi^1_2$-wellfounded trees.

**Theorem 4.57.** Suppose $X, T$ are $\Pi^1_2$-wellfounded level $\leq 2$ trees. Then there exists $(Q, \pi)$ such that $Q$ is $\Pi^1_2$-wellfounded and $\pi$ minimally factors $(X, T \otimes Q)$.

We shall see in Section 3 that the minimally of factoring maps between $\Pi^1_2$-wellfounded trees corresponds exactly to the Dodd-Jensen property of iterations of mice.

Suppose $Q, Q'$ are finite level $\leq 2$ trees, $Q$ is a proper subtree of $Q'$, $(W_i, w_i)_{i \leq m}$ is a partial level $\leq 1$ tower, $m \leq m'$, $C \in \text{desc}(T, Q, (W_m, (w_i)_{i \leq m}))$, $C' \in \text{desc}(T, Q', (W_{m'}, (w_i)_{i \leq m'})) \setminus \text{desc}(T, Q, (W_{m}, (w_i)_{i \leq m}))$. Define
\[
C \triangleleft_{T,Q} C'
\]
iff $C' \prec C$ and $\bigcup_{m \leq k \leq m'} \{ C^* \in \text{desc}(T, Q, (W_k, (w_i)_{i \leq k})) : C' \prec C^* \prec C \} = \emptyset$. A purely combinatorial argument shows that $C \triangleleft_{T,Q} C'$ iff $C$ and $C'$ are both of degree 2 and putting $C = (2, t, \tau), C' = (2, t', \tau'), t = (t, S, \bar{s}), t' = (t', S', \bar{s'}), k = \text{lh}(t)$, then either
Lemma 4.58. Suppose $Q, Q', W, W'$ are finite, $Q$ is a level $\leq 2$ proper subtree of $Q'$, $W$ is a (not necessarily proper) level-1 subtree of $W'$. Suppose $C = (2, t, \tau) \in \text{desc}(T, Q, W)$, $C' = (2, t', \tau') \in \text{desc}(T, Q', W')$, $C <^{T_Q} C'$. Suppose $E \in \mu_L$ is a club, $\eta \in E'$ iff $\eta \in E$ and $E \cap \eta$ has order type $\eta$. Then for any $h \in \omega_1^T$, for any $\beta \in [E']^\omega$,

$$j^{W,W'} \circ h^Q_C(\beta) = \sup \{ h_C^Q(\gamma) : \gamma \in [E']^\omega, \gamma \text{ extends } \beta \}.$$ 

4.7 Level-3 description analysis

Definition 4.59. Suppose $R$ is a level-3 tree. The constant $R$-description is $\emptyset$. An $R$-description is either the constant $R$-description or a triple $(r, Q, (d, q, P))$ such that either $r \in \text{dom}(R) \land (Q, (d, q, P)) = R[r]$ or $r = r^\sim (-1) \land r^\sim \in \text{dom}(R) \land Q$ is a completion of $R(r^\sim) \land (Q, (d, q, P)) = R[r]$. desc($R$) is the set of $R$-descriptions. $(r, Q, (d, q, P))$ is of discontinuous type if $r \in \text{dom}(R)$, of continuous type otherwise. An extended $R$-description is either an $R$-description or a triple $(r, Q, (d, q, P))$ such that $(r^\sim (-1), Q, (d, q, P))$ is an $R$-description of continuous type. desc$^\ast(R)$ is the set of extended $R$-descriptions. An extended $R$-description $r$ is regular iff either $r \in \text{desc}(R)$ of discontinuous type or $r \notin \text{desc}(R)$. A generalized $R$-description is either $(\emptyset, \emptyset, \emptyset)$ or of the form

$$A = (r, \pi, T)$$

so that $r = (r, Q, (d, q, P)) \in \text{desc}(R) \setminus \{\emptyset\}$, $T$ is a finite level $\leq 2$ tree, $\pi$ factors $(Q, T)$. desc$^{**}(R)$ is the set of generalized $R$-descriptions.

Suppose $(Q, (d, q, P))$ is a partial level $\leq 2$ tree. We define

$$\text{ucf}^\ast(Q, (d, q, P)) = \begin{cases} (0, -1, \emptyset) & \text{if } \text{ucf}(Q, (d, q, P)) = (0, -1), \\ (1, q^*, \emptyset) & \text{if } \text{ucf}(Q, (d, q, P)) = (1, q^*), \\ (2, q^*, \text{id}_{\text{tree}(q^*)}) & \text{if } \text{ucf}(Q, (d, q, P)) = (2, q^*), \\ q^* = (q^*, P^*, \bar{p}^*). & \end{cases}$$
Thus, $\text{ucf}^*(Q, (d, q, P)) \in \{(0, -1, \emptyset)\} \cup \text{desc}(Q, P)$, and $\text{cf}(Q, (d, q, P)) = 1$ iff $\text{ucf}^*(Q, (d, q, P)) = \min(\limits_{<Q} F)$. If $\text{cf}(Q, (d, q, P)) = 2$, let

$$\text{ucf}^-(Q, (d, q, P)) = \text{pred}_{<Q, r}(\text{ucf}^*(Q, (d, q, P))).$$

$\text{ucf}^-(Q, (d, q, P))$ can be computed in the following concrete way. If $d = 1$, then $\text{ucf}^-((Q, (1, q, \emptyset)) = (1, \text{pred}_{\text{ucf}}(q, \emptyset))$. If $d = 2$, then $\text{ucf}^-((Q, (2, q, P)) = (2, \text{ucf}(q', P, \bar{p}), \text{id}_P)$, where $q' = (q', P, \bar{p}) \in \text{desc}(Q)$, $q'$ is the $<_B$-maximum of $\check{2}Q\{q, -\}$. If $Q^*$ is a completion of $(Q, (d, q, P))$ and $D = (1, q, \emptyset)$ if $d = 1$, $D = (2, (q, P, \bar{p}), \text{id}_P)$ if $d = 2$ and $\check{2}Q^*[q] = (P, \bar{p})$, then

$$D = \text{pred}_{<Q^*, r}(\text{ucf}^*(Q, (d, q, P))).$$

and

$$\text{ucf}^-(Q, (d, q, P)) = \text{pred}_{<Q^*, r}(D).$$

Suppose $r = (r, Q, (d, q, P)) \in \text{desc}^*(R)$, $lh(r) = k_1, (d, q, P) = (d_i, q_i, P_i)_{1 \leq i \leq k}$. For $F \in (\check{d})_{1}^{R^*}$, define $F_r$ to be a function on $\omega_{Q_r}^{R_r}$: if $r \in \text{desc}(R)$, then $F_r = F_r$; if $r \notin \text{desc}(R)$, then $F_r(\check{\beta}) = F_r(\check{\beta} | R_{\text{tree}}(r))$. If $\check{\gamma} = (\check{\gamma}_r)_{r \in \text{dom}(R)} \in [\check{d}]_{1}^{R^*}$, put $\check{\gamma}_r = [F_r]_{\mu_{Q}}$. If $r \in \text{desc}(R)$ and $A = (r, \pi, T) \in \text{desc}^*(R)$, put $\check{\gamma}_A = \pi^T(\check{\gamma}_r)$. Thus, if $r \in \text{desc}(R)$ is of discontinuous type, then $\check{\gamma}_r = \check{\gamma}$; if $r \notin \text{desc}(R)$, then $\check{\gamma}_r = j_{\text{tree}(r), Q}(\check{\gamma}_r) = \check{\gamma}(\mu_{\text{id}_R \text{tree}(r)}, Q)$. The next lemma computes the remaining case when $r \in \text{desc}(R)$ is of continuous type, justifying that $\check{\gamma}_r$ does not depend on the choice of $F$.

**Lemma 4.60.** Suppose $R$ is a level-3 tree, $\check{\gamma} \in [\check{d}]_{1}^{R^*}$, $r = (r, Q, (d, q, P)) \in \text{desc}(R)$ is of continuous type. Then $\check{\gamma}_r = j_{\text{tree}(r), Q}(\check{\gamma}_r)$.

**Proof.** Suppose $\check{\gamma} = [F]^{R}$, $F \in (\check{d})_{1}^{R^*}$. Put $lh(r) = k_1 + 1, (d, q, P) = (d_i, q_i, P_i)_{1 \leq i \leq k}$. We prove the case when $\text{cf}(R(r^*)) = 2$, the other case being similar. Put $R(r^-) = (Q^-, (d, q, P))$, $\pi^- = \pi | \text{dom}(Q^-)$, so that $Q$ is a completion of $R(r^-)$, $(d, q, P) = (d_k, q_k, P_k)$. Put $\text{ucf}(R(r^-)) = (d', q^{'}, q^{'})$, $\text{ucf}^-(R(r^-)) = (e, z, id_P)$.

We firstly show the $\geq$ direction. Suppose $\delta = \text{sup}_{\check{x} < \check{\beta}} F_r(\check{\beta}^-(\check{x}))$, where $\check{\beta}^-(\check{x})$ is a tuple extending $\check{\beta}$ whose entry indexed by $(d, q)$ is $x$. Let $H(\check{\beta})$ be the least $\xi < \check{d}_{Q^*}$ satisfying $G(\check{\beta}) < F_r(\check{\beta}^-(\check{\xi}))$. By Lemmas 4.31 and 4.25, there is $h: \omega_1 \to \omega_1$ such that $h \in L$ and for $\mu^{Q^*}$-a.e. $\check{\beta}$, $H(\check{\beta}) < j^P(h)_{\check{d}_{Q^*}} < \check{d}_{Q^*}$. Thus, for $\mu^{Q^*}$-a.e. $\check{\beta}$, $G(\check{\beta}) < F_r(\check{\beta})$. Thus, $j^{Q^*}Q^*(\delta) < [F_r]_{\mu_{Q}}$.

We secondly show the $\leq$ direction. Suppose $\delta = \text{sup}_{\check{x} < \check{\beta}} F_r(\check{\beta}^-(\check{x}))$. Then for $\mu^{Q^*}$-a.e. $\check{\beta}$, $G(\check{\beta}) < F_r(\check{\beta})$.
Let $H(\tilde{\beta})$ be the least $\xi < 2^{\beta_v}$ satisfying $G(\tilde{\beta}) < F_\gamma(\tilde{\beta} \restriction \text{dom}(Q)^{-}\tilde{\beta}(\xi))$. By Lemmas 4.32 and 4.25 again, there is $h : \omega_1 \to \omega_1$ such that $h \in \mathbb{L}$ and for $\mu^Q$-a.e. $\tilde{\beta}$, $H(\tilde{\beta}) < j_P(h)(\tilde{\beta}_z) < d_\eta$. Thus, for $\mu^Q$-a.e. $\tilde{\beta}$, $G(\tilde{\beta}) < j^Q_\eta(\eta)$, where $\eta$ is represented modulo $\mu^Q$ by the function $\tilde{\beta} \mapsto F_\gamma(\tilde{\beta} \restriction j_P(h)(\tilde{\beta}_z))$. Since $\eta < \gamma_r$, we have $\delta < j^Q_\eta(\gamma_r)$.

Define

$$C^* = \{ \xi < \delta^3_3 : \text{for any finite level } \leq 2 \text{ tree } Q, j^Q_\sup(\xi) = \xi \}.$$  

Assuming $\Delta^4_2$-determinacy, Lemma 4.37 implies that $C^* \cap \kappa^3_3$ has order type $\kappa^3_3$, and hence $C^*$ has order type $\delta^3_3$. A tuple $\tilde{\gamma}$ is said to strongly respect $R$ iff $\tilde{\gamma} \in [C^*]^R$. In most applications, we are only concerned with $\tilde{\gamma}$ strongly respecting $R$. In that case, the techniques in Section 4.6 helps to decide the ordering of $\gamma_r$ for different $r \in \text{desc}^*(R)$. The results are in parallel to Lemma 3.22.

Define $\langle (\emptyset, \emptyset, \emptyset) \rangle = \langle \emptyset \rangle = \emptyset$. For $A = (r, \pi, T) \in \text{desc}^{**}(R)$, $r = (r, Q, (d, q, P))$, $\text{lh}(r) = k$, define

$$\langle A \rangle = \begin{cases} 
(r(0), [\pi(d_1, q_1)]_T, r(1), \ldots, [\pi(d_{k-2}, q_{k-2})]_T, r(k-2), -1) 
& \text{if } r \text{ is of continuous type, } \pi \text{ is continuous at } (d_{k-1}, q_{k-1}), \\
(r(0), [\pi(d_1, q_1)]_T, r(1), \ldots, [\pi(d_{k-2}, q_{k-2})]_T, r(k-2), [\text{pred}(\pi, T, (d_{k-1}, q_{k-1}))]_T) 
& \text{if } r \text{ is of continuous type, } \pi \text{ is discontinuous at } (d_{k-1}, q_{k-1}), \\
(r(0), [\pi(d_1, q_1)]_T, r(1), \ldots, [\pi(d_{k-1}, q_{k-1})]_T, r(k-1), -1) 
& \text{if } r \text{ is of discontinuous type, } \pi \text{ is continuous at } \text{ucf}(R(r)), \\
(r(0), [\pi(d_1, q_1)]_T, r(1), \ldots, [\pi(d_{k-1}, q_{k-1})]_T, r(k-1), [\text{pred}(\pi, T, \text{ucf}(R(r)))]_T) 
& \text{if } r \text{ is of discontinuous type, } \pi \text{ is discontinuous at } \text{ucf}(R(r)).
\end{cases}$$

and define $\langle r \rangle = \langle (r, Q, \text{id}_Q) \rangle$. If $r$ is of discontinuous type and $Q^+$ is a completion of $Q$, define $\langle (r, Q^+, (d, q, P)) \rangle = \langle (r, Q^+, \text{id}_Q) \rangle$. For $A, A' \in \text{desc}^{**}(R)$, define

$$A < A'$$

iff $\langle A \rangle <_{BK} \langle A' \rangle$; define

$$A \sim A'$$

iff $\langle A \rangle = \langle A' \rangle$. For $r, r' \in \text{desc}^*(R)$, define

$$r < r'$$

iff $\langle r \rangle <_{BK} \langle r' \rangle$; define

$$r \sim r'$$

87
Suppose Lemma 4.61. Then $A <^R A'$ iff $\gamma_A < \gamma_{A'}$; $A \sim^R A'$ iff $\gamma_A = \gamma_{A'}$.

**Proof.** Put $A = (r, T, \pi)$, $A' = (r', T', \pi')$. Recall our convention that $(\gamma_{(0,0,0)}) = \delta_3$. The lemma is trivial if $r = \emptyset$ or $r' = \emptyset$. Assume now $r, r' \neq \emptyset$. Put $r = (r, Q, (d, q, P))$, $(\overline{d}, \overline{q}) = (d_i, q_i, P_i)_{1 \leq i \leq \text{lh}(q)}$, $k = \text{lh}(r)$, $r' = (r', Q', (d', q', P'))$, $(\overline{d}', \overline{q}') = (d'_i, q'_i, P'_i)_{1 \leq i \leq \text{lh}(q')}$, $k' = \text{lh}(r')$. Assume $\gamma = [F]^R$, $F \in (C^*)^R$.

Firstly, we prove that $A \sim A'$ implies $\gamma_A = \gamma_{A'}$.

**Case 1:** $r = r'$ is of continuous type.

Put $Q^- = R_{\text{tree}}(r^-)$.

**Subcase 1.1:** $\pi$ is continuous at $(d_{k-1}, q_{k-1})$.

Then $[\mu(d, q)]_T = [\nu(d, q)]_{T'}$ for any $(d, q) \in \text{dom}(Q^-)$. Put $\tau = \pi | \text{dom}(Q^-)$, $\tau' = \pi' | \text{dom}(Q^-)$. By Lemma 4.60, $\gamma_A = \tau_{\text{sup}}(\gamma_r) = \tau'_{\text{sup}}(\gamma_r)$ and $\gamma_{A'} = (\tau')_{\text{sup}}(\gamma_{r'})$. Given $\delta = [G]_{\mu_{Q^-}} < \gamma_{r'}$, we need to show that $\tau^T(\delta) < \gamma_{A'}$. By Theorem 4.57, there exist $X$ and $\psi$ minimally factoring $(T, T' \otimes X)$. So $\psi \circ \pi(d, q) = \text{id}_{T_{\text{sup}}} \circ \pi'(d, q)$ for any $(d, q) \in \text{dom}(Q^-)$. We shall actually show that $\psi^T_{\text{sup}} \circ \tau^T(\delta) < \gamma_{A'}$, i.e., $(\psi \circ \tau)^{T'}_{\text{sup}}(\delta) < \gamma_{A'}$. By Loš, it suffices to show that for $\mu^T$-a.e. $\tilde{\beta}$, $j^X(G(\tilde{\beta})) < F_r(\tilde{\beta}_r)$. The minimality of $\psi$ implies that $\text{id}_{T_{\text{sup}}}(\tilde{\beta}) = j^X(\tilde{\beta}_r)$. It suffices to show that for $\mu^T$-a.e. $\tilde{\beta}$, $j^X(G(\tilde{\beta}_r)) < F_r(\tilde{\beta}_r)$. Hence, it suffices to show that $\mu^Q$-a.e. $\tilde{\beta}$, $j^X(G(\tilde{\beta})) < F_r(\tilde{\beta})$. As $F_r(\tilde{\beta}) \in C^*$, this inequality is a consequence of $G(\tilde{\beta}) < F_r(\tilde{\beta})$, which holds true for $\mu^Q$-a.e. $\tilde{\beta}$ by assumption.

**Subcase 1.2:** $\pi$ is discontinuous at $(d_{k-1}, q_{k-1})$.

Then $[\mu(d, q)]_T = [\nu(d, q)]_{T'}$ for any $(d, q) \in \text{dom}(Q^-)$. Let $\tau$ factor $(Q, T)$ where $\tau$ and $\pi$ agree on $\text{dom}(Q^-)$ and $\tau(d_{k-1}, q_{k-1}) = \text{pred}(\pi, T, (d_{k-1}, q_{k-1}))$, and likewise define $\tau'$ which factors $(Q', T')$. By Lemma 4.41, $\gamma_A = \tau_{\text{sup}} \circ j^{Q^-}_\text{sup}(\gamma_{r-})$ and $\gamma_{A'} = (\tau')_{\text{sup}} \circ j^{Q^-}_\text{sup}(\gamma_{r'}-)$ and $\gamma_{A'} = (\tau')_{\text{sup}} \circ j^{Q^-}_\text{sup}(\gamma_{r-})$. Work with $X$ and $\psi$ minimally factoring $(T, T' \otimes X)$ and argue similarly to Subcase 1.1.

**Case 2:** $r = r'$ is of discontinuous type.

**Subcase 2.1:** $\pi$ is continuous at $\text{ucf}(R(r))$.

Then $[\mu(d, q)]_T = [\nu(d, q)]_{T'}$ for any $(d, q) \in \text{dom}(Q)$ and $\gamma_A = \tau_{\text{sup}}(\gamma_r)$, $\gamma_{A'} = (\tau')_{\text{sup}}(\gamma_r)$. Argue similarly to Case 1.

**Subcase 2.2:** $\pi$ is discontinuous at $\text{ucf}(R(r))$.

Let $Q^+$ be a completion of $R(r)$ and let $\tau$ factor $(Q^+, T)$ so that $\tau$ extends $\pi$, $\tau(d_k, q_k) = \text{pred}(\pi, T, (d_k, q_k))$, and likewise define $\tau'$ which fac-
tors \((Q^+, T)\). Then \([τ(d, q)]_T = [τ'(d, q)]_{T'}\) for any \((d, q) \in \text{dom}(Q^+)\). By Lemma 4.42, \(γ_A = τ^T_T \circ j^{Q^+}(γ_r)\) and \(γ_{A'} = (τ')^T_{T'} \circ j^{Q^+}(γ_r)\). Argue similarly to Case 1.

Case 3: \(r \neq r'\). Assume \(r = r' \succ (-1)\).

Subcase 3.1: \(τ\) is continuous at \((d_{k-1}, q_{k-1})\).

It follows from Subcase 1.1 and Subcase 2.1 that \(γ_A = τ^T_T(γ_{r'})\) and \(γ_{A'} = (τ')^T_{T'}(γ_{r'})\). Argue similarly as before.

Subcase 3.2: \(τ\) is discontinuous at \((d_{k-1}, q_{k-1})\).

Use a combination of Subcase 1.2 and Subcase 2.2.

Secondly, we prove that \(A \prec A'\) implies \(γ_A < γ_{A'}\).

Case 1: \(⟨A’⟩\) is a proper initial segment of \(⟨A⟩\).

Then \(⟨A’⟩\) does not end with \(-1\). We prove the typical case when \(r'\) is of discontinuous type. So \(r' \subseteq r\). Let \((Q')^+\) be a completion of \(R(r')\) and let \(τ'\) factor \(((Q')^+, T')\) so that \(τ'\) extends \(π', τ'(d_k, q_k) = \text{pred}(π, T, \text{ucf}(R(r')))\).

Then \((Q')^+ = R_{\text{tree}}(r \upharpoonright k')\). We get \(ψ\) minimally factoring \((T, T' \otimes X)\) so that \(ψ \circ π(d, q) = \text{id}_{T', X} \circ τ'(d, q)\) for any \((d, q) \in \text{dom}((Q')^+)\). We shall actually show that \(ψ^{T', X}(γ_A) < γ_{A'}\). By Loś, it suffices to show that for \(μ^{T', \text{a.e.} \tilde{β}}, j_μ^{T'; X}(\tilde{β}) < F_{τ'}(β_{τ'})\). The minimality of \(ψ\) implies that \(\text{id}_{(T'; X)}^{T', X}(\tilde{β})\) agrees with \(j_μ^X(β_{τ'})\) on \(\text{dom}((Q')^+)\). It suffices to show that for \(μ^{Q', \text{a.e.} \tilde{β}}, j_μ^X(F_{τ'}(β_{τ'}) < F_{τ'}(β_{τ'})\). As \(\text{ran}(F) \subseteq C^*\), this would be a consequence of \(F_{τ'}(β_{τ'}) < F_{τ'}(β_{τ'})\). Which follows from order preservation of \(F\).

Case 2: \(⟨A’⟩\) is not a proper initial segment of \(⟨A⟩\).

Similar to Case 1, using the following fact: Suppose \(X, X'\) are level \(≤ 2\) trees and \([d_i, x_i]^X_X = [d_i', x_i']^{X'}_{X'}\) for \(1 \leq i < n\), \([d_n, x_n]^X_X < [d_n', x_n']^{X'}_{X'}\). Then there exist \(U\) and \(ψ\) minimally factoring \((X, X' \otimes U)\), which implies that for any \(β \in [ω]^X\), if \(\text{id}_ω^{X, U}(β) = \tilde{δ}\), then \(δ\) and \(j_μ^U(β)\) agree on \(\{(d_i, x_i) : 1 \leq i < n\}\) and \(d_nδ_{x_n} < j_μ^U(d_n'β_{x_n'})\).

\(\square\)

4.8 Factoring maps between level-3 trees

Put \(π \oplus \emptyset = \emptyset\). Suppose \(Y\) is a level-3 tree, \(y = (y, X, (e, x, W)) \in \text{desc}(Y)\), \(\text{lh}(y) = k\), \((e, x, W) = (e_i, x_i, W_i)_{1 \leq i \leq \text{lh}(y)}, π\) is a function whose domain contains \(\text{dom}(X)\), we put \(π \oplus y = π \oplus Y\).

If \(l < \text{lh}(y)\), then \(y \mid l = (y \mid l, Y_{\text{tree}}(y \mid l), (e_i, x_i, W_i)_{1 \leq i \leq l})\).

**Definition 4.62.** Suppose \(Y\) is a level-3 tree, \(T\) is a level \(≤ 2\) tree. The only \((Y, T, \emptyset)\)-description is \((\emptyset, \emptyset)\), which is called the constant \((Y, T, *)\)-
description. Suppose \((Q, (d, q, P)) = (Q, (d_i, q_i, P_i))_{1 \leq i \leq k}\) is a potential partial level \(\leq 2\) tower of discontinuous type. A \((Y, T, (Q, (d, q, P)))\)-description is of the form

\[
\mathcal{B} = (y, \pi)
\]

with the following properties:

1. \(y \in \text{desc}(Y) \setminus \{\emptyset\}\). Put \(y = (y, X, (e, x, W))\), \(\text{lh}(y) = l\), \((e, x, W) = (e_i, x_i, W_i)_{1 \leq i \leq \text{lh}(y)}\).

2. \(\pi\) factors \((X, T, Q)\).

3. The contraction of \((\text{sign}_2(\pi(e_i, x_i)))_{1 \leq i < l}\) is \(((d_i, q_i))_{1 \leq i < k}\).

4. If \(y\) is of continuous type and \((e_{l-1}, x_{l-1})\) does not appear in the contraction of \((\text{sign}_2(\pi(e_i, x_i)))_{1 \leq i < l}\), then \(\pi(x_{l-1})\) is of level-2 discontinuous type.

5. Put \(\text{ucf}(X, (e, x, W)) = (e_*, x_*)\).

   (a) If \(e_* = 0\) then \(d_k = 0\).

   (b) If \(e_* = 1\) then \(\text{ucf}_2(\pi(1, x_*)) = \text{ucf}(Q, (d, q, P))\).

   (c) If \(e_* = 2\), \(x_* = (x_*, W_*, \vec{w}_*) \in \text{desc}(X)\), then \(\text{ucf}_2(\pi(2, x_*)) = \text{ucf}(Q, (d, q, P))\).

   (d) If \(e_* = 2\), \(x_* = (x_*, W_*, \vec{w}_*) \not\in \text{desc}(X)\), then \(\text{ucf}_2^+(\pi(2, x_*)) = \text{ucf}(Q, (d, q, P))\).

A \((Y, T, Q)\)-description is a \((Y, T, (Q, (d', q', P')))\)-description for some potential partial level \(\leq 2\) tower \((Q, (d', q', P'))\) of discontinuous type. A \((Y, T, *)\)-description is a \((Y, T, Q')\)-description for some level \(\leq 2\) tree \(Q'\) or \(Q' = \emptyset\).

desc\((Y, T, (Q, (d, q, P)))\), desc\((Y, T, Q)\), desc\((Y, T, *)\) denote the sets of relevant descriptions.

Similarly to Definition \[4.49\] if \(\mathcal{B} \in \text{desc}(Y, T, Q)\), then there is at most one \((Q, (d, q, P))\) for which \(\mathcal{B} \in \text{desc}(Y, T, (Q, (d, q, P)))\). Suppose that \(\mathcal{B} = (y, \pi)\) is a \((Y, T, (Q, (d, q, P)))\)-description, \(F \in (\delta^1)_{Y\uparrow}\). Then

\[
F^T_B : [\omega_1]^{T\uparrow} \to \delta^1_{\delta^1}
\]

is the function that sends \([h]^{T}\) to \([F_y \circ h^Q_{\pi}]_{\mu_Q}\). Note that \(F_y \circ h^Q_{\pi}\) has signature \(\text{sign}(Q, (d, q, P))\), is essentially discontinuous, and has uniform cofinality \(\text{ucf}(Q, (d, q, P))\). Of course, \(F^T_B\) is meaningful only when \(T\) is \(\Pi^1_2\)-wellfounded.
Assuming $\Pi_3^1$-determinacy, the $\mathbb{L}[T_3]$-measure $\mu^Y$ will be defined, and $[F]^Y \rightarrow [F \mu^T]$ will represent an element in $\mathbb{L}[j^Y(T_3)]$ modulo $\mu^Y$. Such kind of results related to level-3 ultrapowers are parallel to Section 4.6. They will be handled in Section 6.

Suppose $(\bar{Q}, (d, q, P)) = (Q_i, (d_i, q_i, P_i))_{1 \leq i \leq k}$ is a potential partial level $\leq 2$ tower and $B = (y, \pi) \in \text{desc}(Y, T, (Q_k, (d, q, P)))$. Define $\text{lh}(B) = k$. $B \upharpoonright 0$ is the constant $(Y, T, *)$-description. Suppose $y = (y, X, (e, x, W))$, $0 < \bar{k} < k$. Then

$$B \upharpoonright \bar{k} \in \text{desc}(Y, T, (Q_k, (d_i, q_i, P_i))_{1 \leq i \leq k})$$

defined by the following: letting $l$ be the least such that $\pi(e_l, x_l) \notin \text{desc}(T, Q_k, *)$, $C \in \text{desc}(T, Q_k, *)$ be such that $C \triangleleft_{2, \bar{k}} \pi(e_l, x_l)$, then

1. if $C \neq \pi(e_l, x_l)$, then $B \upharpoonright \bar{k} = (y \upharpoonright \bar{l}^- (-1), \bar{\pi})$, where $\bar{\pi}$ and $\pi$ agree on $\text{Y}_{\text{tree}}(y \upharpoonright \bar{l})$, $\bar{\pi}(e_l, x_l) = C$;

2. if $C = \pi(e_l, x_l)$, then $B \upharpoonright \bar{k} = (y \upharpoonright \bar{l}, \pi \upharpoonright \text{Y}_{\text{tree}}(y \upharpoonright \bar{l}))$.

Define

$$B \triangleleft B'$$

iff $B = B' \upharpoonright \bar{k}$ for some $\bar{k} < \text{lh}(B')$. Define $\triangleleft^{Y, T} = \triangleleft \upharpoonright \text{desc}(Y, T, *)$. As a corollary to Lemma 4.58, $\triangleleft^{Y, T}$ inherits the following continuity property.

**Lemma 4.63.** Suppose $Y$ is a level-3 tree, $T$ is a level $\leq 2$ tree, $Q$ is a level $\leq 2$ proper subtree of $Q'$. Suppose $B = (y, \pi) \in \text{desc}(Y, T, Q)$ and $B' = (y', \pi') \in \text{desc}(Y, T, Q')$, $B \triangleleft^{Y, T} B'$. Suppose $E \in \mu^T_*$ is a club, $\eta \in E'$ iff $\eta \in E$ and $E \cap \eta$ has order type $\eta$. Then for any $F \in (\delta^T_3)^{Y, T}$, for any $h' \in \omega_1^{T, T}$, for any $\bar{\beta} \in [E']^{Q, T}$,

$$F_{\bar{\gamma}} \circ h_{\pi}(\bar{\beta}) = \sup\{F_{\bar{\gamma}'} \circ h_{\pi}(\bar{\gamma}) : \bar{\gamma} \in [E']^{Q, T}, \bar{\gamma} \text{ extends } \bar{\beta}\}.$$  

Hence, the signature and approximation sequence of $F_{\bar{\gamma}} \circ h_{\pi}(\bar{\beta})$ are proper initial segments of those of $F_{\bar{\gamma}'} \circ h_{\pi}(\bar{\beta})$, respectively.

Given a $(Y, T, *)$-description $B = (y, \pi)$, define

$$\langle B \rangle = \pi \oplus y.$$  

Define

$$B \prec B'$$

iff $\langle B \rangle <_{BK} \langle B' \rangle$, the ordering on coordinates in $\text{desc}(T, Q, *)$ for some $T, Q$ again according to $\prec$. The constant $(Y, T, *)$-description $B_0$ is the $\prec$-greatest, and we have $\langle B_0 \rangle = \emptyset$. Define $\prec^{Y, T} = \prec \upharpoonright \text{desc}(Y, T, *)$. As a corollary to Lemma 4.51, $\prec^{Y, T}$ inherits the following ordering property on $F_{\mu}^T$.
Lemma 4.64. Suppose \((Q_i, (d_i, q_i, P_i))_{1 \leq i \leq m}\) is a partial level \(\leq 2\) tower, \(B \in \text{desc}(Y, T, Q_k), B' \in \text{desc}(Y, T, Q_k'), k \leq m, k' \leq m, B \prec_{Y,T} B'.\) Then for any \(F \in (\bar{d}_3)^{Y\uparrow}, \) for any \(\bar{\beta} \in [\omega_1]^{T\uparrow}, j^{Q_\tau,Q_m} \circ F^T_B(\bar{\beta}) < j^{Q_{\bar{\tau}},Q_m} \circ F^T_{B'}(\bar{\beta}).\)

Definition 4.65. Suppose \(R, Y\) are level-3 trees, \(T\) is a level \(\leq 2\) tree. Suppose \(\rho : \text{dom}(R) \cup \{\emptyset\} \rightarrow \text{desc}(Y, T, *)\) is a function. \(\rho\) factors \((R, Y, T)\) iff

1. \(\rho(\emptyset)\) is the constant \((Y, T, \ast)\)-description.
2. For any \(r \in \text{dom}(R), \rho(r) \in \text{desc}(Y, T, R[r]).\)
3. For any \(r^-(a), r^-(b) \in \text{dom}(R),\) if \(a \prec_{BK} b\) and \(R_{\text{tree}}(r^-(a)) = R_{\text{tree}}(r^-(b))\) then \(\rho(r^-(a)) < \rho(r^-(b)).\)
4. For any \(r \in \text{dom}(R), \rho(r^-) \prec_{Y,T} \rho(r).\)

If \(Y\) is a level-3 tree, then

\[
\text{id}_{Y,*} \ 	ext{factors} \ (Y, Y, Q^0) \ 	ext{where} \ \text{id}_{Y,*}(y) = ((y, X, (e, x, W)), \text{id}_{e,x,W}) \ 	ext{for} \ Y[y] = (X, (e, x, W)).
\]

For level-3 trees \(R, Y,\) we say that \(\rho : \text{dom}(R) \rightarrow \text{dom}(Y)\) factors \((R, Y)\) iff

1. If \(r \in \text{dom}(R)\) then \(R(r) = Y(\rho(r)).\)
2. If \(r, r' \in \text{dom}(R)\) and \(r \subseteq r',\) then \(\rho(r) \subseteq \rho(r').\)
3. If \(R_{\text{tree}}(r^-(a)) = R_{\text{tree}}(r^-(b))\) and \(a \prec_{BK} b,\) then \(\rho(r^-(a)) <_{BK} \rho(r^-(b)).\)

If in addition, \(\rho\) is onto \(\text{dom}(Y),\) then \(\rho\) is called a level-3 tree isomorphism between \(R\) and \(Y.\) If \(\rho\) factors \((R, Y)\) and \(\gamma = (\gamma_y)_{y \in \text{dom}(Y)} \in (\bar{d}_3)^{Y\uparrow},\) let \(\gamma_{\rho} = (\gamma_{\rho,r})_{r \in \text{dom}(R)} \in [\delta_3^{R\uparrow}]^{\uparrow}\) where \(\gamma_{\rho,r} = \gamma_{\rho(r)}.\)

If \(\rho\) factors \((R, Y, T)\) and \(F \in (\bar{d}_3^T)^{Y\uparrow},\) let

\[
F^T_{\rho} : [\omega_1]^{T\uparrow} \rightarrow [\delta_3^{R\uparrow}]
\]

be the function that sends \(\xi\) to \((F^T_{\rho,r}(\xi))_{r \in \text{dom}(R)}\). The fact that \(F^T_{\rho}(\xi) \in [\delta_3^{R\uparrow}]^{\uparrow}\) follows from Lemmas 4.64, 4.63.

Suppose \(Y\) is a level-3 tree, \(T\) is a level \(\leq 2\) tree. A representation of \(Y \otimes T\) is a pair \((R, \rho)\) such that

1. \(R\) is a level-3 tree;
2. $\rho$ factors $(R,Y,T)$;
3. $\text{ran}(\rho) = \text{desc}(Y,T,*);$ 
4. If $R_{\text{tree}}(r^{-}(a)) = R_{\text{tree}}(r^{-}(b))$, then $a <_{BK} b$ iff $\pi(r^{-}(a)) \sim^{Y,T} \pi(r^{-}(b))$.

Representations of $Y \otimes T$ are clearly mutually isomorphic. As before, we shall regard 

$$Y \otimes T$$

itself as a “level-$3$ tree” whose domain is the set of non-constant $(Y,T,*)$-descriptions and sends $B \in \text{desc}(Y,T,(Q,(d_i,q_i,P_i)_{1 \leq i \leq k}))$ to $(Q,(d_k,q_k,P_k))$. If $Q$ is a level $\leq 2$ tree, then $(Y \otimes T) \otimes Q$ is a “level-$3$ tree” whose domain consists of non-constant $(Y \otimes T,Q,*)$-descriptions. There is a natural isomorphism

$$\iota_{Y,T,Q}$$

between “level-$3$ trees” $(Y \otimes T) \otimes Q$ and $Y \otimes (T \otimes Q)$, defined as follows:

1. If $A = ((B,Z,(d,z,N)),(\psi)) \in \text{desc}(Y \otimes T,Q,U)$, $B = (y,\pi) \in \text{desc}(Y,T,(Z,(d,z,N)))$, $y = (y,X,(e,x,W))$, then $\iota_{Y,T,Q}(A) = (y,\iota_{T,Q,U}^{-1} \circ (T \otimes \psi) \circ \pi)$.

2. If $A = ((B^\rightarrow(-1),Z^+,\overline{(d,z,N)}),(\psi)) \in \text{desc}(Y \otimes T,Q,U)$, $\overline{(d,z,N)} = (d_i,z_i,N_i)_{1 \leq i \leq l}$, $B = (y,\pi) \in \text{desc}(Y,T,(Z,(d,z,N)))$, $y = (y,X,(e_i,x_i,W_i)_{1 \leq i \leq k})$, then 

   (a) if $y$ is of discontinuous type, then $\iota_{Y,T,Q}(A) = (y^\rightarrow(-1),\psi \ast_0 \pi)$, where $\psi \ast_0 \pi$ factors $(X^+,T \otimes Q,U)$, $X^+$ is a completion of $(X,(e_i,x_i))$, $\psi \ast_0 \pi$ extends $\iota_{T,Q,U}^{-1} \circ (T \otimes \psi) \circ \pi$, $\psi \ast_0 \pi(e_k,x_k) = \iota_{T,Q,U}^{-1}(2,t_0,\tau)$, $t_0 = ((-1),\{0\},\{(0)\},\tau(0)) = \psi(d_i,z_i)$;

   (b) if $y$ is of continuous type, then $\iota_{Y,T,Q}(A) = (y,\psi \ast_1 \pi)$, where $\psi \ast_1 \pi$ factors $(X,T \otimes Q,U)$, $\psi \ast_1 \pi$ extends $\iota_{T,Q,U}^{-1} \circ (T \otimes \psi) \circ \pi | (\text{dom}(X) \backslash \{(e_k,x_k)\})$, $\psi \ast_1 \pi(e_k,x_k) = \iota_{T,Q,U}^{-1}(2,t^\rightarrow(-1),\tau^+)$, where $\pi(e_k,x_k) = (e_k,x_k) = (t,S,(s_i)_{1 \leq m})$, $\tau^+$ extends $\tau$, $\tau^+(s_m) = \psi(d_i,z_i)$.

$\iota_{Y,T,Q}$ justifies the associativity of the $\otimes$ operator acting on level $(3,\leq 2,\leq 2)$ trees.

The identity map $\text{id}_{Y \otimes T}$ factors $(Y \otimes T,Y,T)$. $\rho$ factors $(R,Y,T)$ iff $\rho$ factors $(R,Y \otimes T)$. If $y \in \text{dom}(Y)$, $y = (y,X,(e,x,W)) \in \text{desc}(Y)$,

$$Y \otimes_y T$$
is the level-3 subtree of $Y \otimes T$ whose domain is $\text{dom}(Y \otimes Q^0)$ plus all the $(Y, T, \ast)$-descriptions of the form $(y, \tau)$. If $\pi$ factors level $\leq 2$ trees $(T, Q)$, then

$$Y \otimes \pi$$

factors $(Y \otimes T, Y \otimes Q)$, where $Y \otimes \pi(y, \psi) = (y, (\pi \otimes U) \circ \psi)$ for $(y, \psi) \in \text{desc}(Y, T, U)$.

If $\rho$ factors finite trees $(R, Y, T)$, then $\rho$ induces

$$\tilde{\rho}^T : \text{desc}^{**}(R) \rightarrow \text{desc}^{**}(Y)$$
as follows:

1. If $A = (\emptyset, \emptyset, \emptyset)$, then $\tilde{\rho}^T(A) = A$.

2. If $A = (r, \psi, U)$, $r = (r, Q, (d, q, P))$ is of discontinuous type, $\rho(r) = (y, \pi)$, then $\tilde{\rho}^T(A) = (y, (T \otimes \psi) \circ \pi)$.

3. If $A = (r, \psi, U)$, $r = (r, Q, (d, q, P))$ is of continuous type, $(d, q, P) = (d_i, q_i, P_i)_{1 \leq i \leq k}$, $\rho(r^-) = (y, \pi)$, $y = (y, X, (e_i, x_i, W_i)_{1 \leq i \leq k})$
   
   (a) if $y$ is of discontinuous type, then $\tilde{\rho}^T(A) = (y^-(-1), \psi \ast_0 \pi)$, where $\psi \ast_0 \pi$ factors $(X^+, T \otimes U)$, $\psi \ast_0 \pi$ extends $(T \otimes \psi) \circ \pi$, $\psi \ast_0 \pi(e_k, x_k) = (2, t_0, \tau)$, $t_0 = ((-1), \{(0)\})$, $\tau((0)) = (\psi(d_i, p))$;
   
   (b) if $y$ is of continuous type, then $\tilde{\rho}^T(A) = (y, \psi \ast_1 \pi)$, where $\psi \ast_1 \pi$ factors $(X, T \otimes U)$, $\psi \ast_1 \pi$ extends $(T \otimes \psi) \circ \pi | \text{dom}(X) \setminus \{(e_k, x_k)\}$, $\psi \ast_1 \pi(e_k, x_k) = (2, t^>(-1), \tau^+)$, where $\pi(e_k, x_k) = (2, t, \tau)$, $t = (t, S, (s_i)_{i \leq m})$, $\tau^+ \text{ extends } \tau$, $\tau^+(s_m) = (\psi(d_i, p))$.

$A \prec^*_R A'$ iff $\tilde{\rho}^T(A) \prec^*_R \tilde{\rho}^T(A')$; $A \sim^*_R A'$ iff $\tilde{\rho}^T(A) \sim^*_R \tilde{\rho}^T(A')$. A purely combinatorial argument shows that if $R = Y \otimes T$, then for any $B \in \text{desc}^{**}(Y)$ there is $A \in \text{desc}^{**}(R)$ such that $\tilde{\rho}^T(A) \sim^*_R B$.

**Lemma 4.66.** Suppose $Q$ is a finite level $\leq 2$ tree, $W$ is a finite level-1 tree, $\theta: [\omega_1]^Q_\gamma \rightarrow j^W(\omega_1)$ is a function in $\mathbb{L}_\delta[\chi_\theta]$. Suppose $\text{cf}^\omega(\theta) = \text{seed}^Q_{W^*}$, $D = (d, q, \sigma) \in \text{desc}(Q, W)$.

1. The uniform cofinality of $\theta$ is $\text{ucf}^W_{\gamma^*}(D)$.

2. $\text{ucf}_1(D) = -1$ iff $\text{cf}^\omega(\theta(\xi)) = \omega$ for $\mu^Q$-a.e. $\xi$.

3. Fix $w \in W$. Then $\text{ucf}_1(D) = w$ iff $\text{cf}^\omega(\theta(\xi)) = \text{seed}^W_w$ for $\mu^Q$-a.e. $\xi$.  

94
Proof. Let $g \in \mathbb{L}$ be a strictly increasing function from seed$^{Q,W}_{D}$ to $[\theta]_{\mu,Q}$ cofinally. Find $G \in \mathbb{L}$ such that $[G]_{\mu,Q} = g$. We have $[\theta]_{\mu,Q} = \sup [G]_{\mu,Q}^{\omega}$ seed$^{Q,W}_{D}$. By L"os, for $\mu$-a.e. $\xi$, $\theta(\xi) = \sup G(\xi)^{\omega}(\sigma^{W}(\xi))$. (Recall our convention that $\theta^{W} = j^{W}$..) This shows part 1. Also, for $\mu$-a.e. $\xi$, $\text{cf}^{\omega}(\theta(\xi)) = \text{cf}^{\omega}(\sigma^{W}(\xi))$, which equals to $\omega$ when $\text{ucf}_{1}(D) = -1$, equals to seed$^{W}_{\text{ucf}_{1}(D)}$ otherwise. This shows parts 2-3.

Lemma 4.67. Suppose $R, Y$ are level-3 trees, $\theta : \text{rep}(R) \rightarrow \text{rep}(Y)$ is continuous and order preserving, $\theta \in \mathbb{L}_{\delta_{3}}[T_{2}]$. Then there exists a triple

$$(T, \rho, \tilde{\delta})$$

such that $T$ is a level $\leq 2$ tree, $\rho$ factors $(R, Y, T)$, $\tilde{\delta}$ respects $T$, and

$$\forall F \in (\delta_{3})^{T} \uparrow F^{\rho}_{\rho}(\tilde{\delta}) = [F \circ \theta]^{R}.$$

Proof. For $r \in \text{dom}(R)$, let $R(r) = (Q_{r}, (d_{r}, q_{r}, P_{r})).$ For $q \in \text{dom}(Q_{r})$, let $Q_{r}(q) = (P_{r,q}, P_{r,q}^{r}).$ Then, when $d_{r} = 2$, $P_{r}$ is the completion of $(P_{r,q}^{r}, P_{r,q}^{r}).$

Let $E \in \mu_{t}, y_{r} = (y_{r}, X_{r}, (e_{r}, x_{r}, W_{r})) \in \text{desc}(Y)$ and $\theta_{r} \in \mathbb{L}_{\delta_{3}}[T_{2}]$ be such that for any $\tilde{\beta} \in [E]^{\omega_{T}^{\uparrow}}, \theta_{r}(\tilde{\beta}) \in [\omega_{1}]^{x}_{r}$. Such that $\theta_{r}(\tilde{\beta} \oplus_{R} r) = \theta_{r}(\tilde{\beta} \oplus_{Y} y_{r})$. Let $(e_{r}, x_{r}, W_{r}) = (e_{r,i}, x_{r,i}, W_{r,i})_{1 \leq i \leq \ell_{\text{lh}}(x_{r})}$. For $x \in \text{dom}(Q_{r})$, let $X_{r}(x) = (W_{r,x}, w_{r,x}).$ Thus, when $d_{r,i} = 2$, $W_{r,x}$ is the completion of $(W_{r,x}, w_{r,x}).$

Let $[\theta]_{\mu,Q_{r}} = \gamma_{r} = \gamma_{r,x}(e,x) \in \text{dom}(X_{r}), \theta_{r}(\tilde{\beta}) = (\theta_{r,x}(\tilde{\beta}))(e,x) \in \text{dom}(X_{r})$. So $\gamma_{r,x} = [\theta_{r,x}]_{\mu,Q_{r}}$. For $e \in \{1, 2\}$, let

$$B_{e}^{r} = \{x \in \text{dom}(X_{r}) : \gamma_{r,x} < \omega_{1}\}.$$

So $B_{1}^{r}$ is closed under $\chi_{X_{r}}^{r}$ and $B_{2}^{r} = \emptyset$. For $x \in X_{r} \setminus B_{e}^{r}$, let $(S_{r,x}^{e}, S_{r,x}^{e})$ be the potential partial level $\leq 1$ tower induced by $\gamma_{r,x}, \gamma_{r,x} = (\gamma_{r,x,i}^{e})_{i \leq \ell_{\text{lh}}(x_{r})}, \gamma_{r,x}^{e} = \gamma^{e}_{r,x,\text{lh}(x_{r})-1},$ let $(\text{seed}^{Q,W}_{D_{r,x}})_{i < v_{r,x}}^{e}$ be the signature of $\gamma_{r,x}$, let $(\delta_{r,x,i})_{i \leq v_{r,x}}^{e}$ be the approximation sequence of $\gamma_{r,x}$, and let $\text{cf}^{\omega}(\gamma_{r,x}) = \text{seed}^{Q,W}_{D_{r,x}}^{e}$ if $\text{cf}^{\omega}(\gamma_{r,x}) > \omega$. The existence of $(q_{r,x,i}^{e})_{i < v_{r,x}}^{e}$ and $q_{r,x}^{e}$ follows from Lemma 4.32.

Let $D_{r,x,i}^{e} = (e_{r,x,i}, q_{r,x,i}^{e}, \gamma_{r,x,i}^{e}), D_{r,x}^{e} = (\gamma_{r,x}^{e}, q_{r,x}^{e}, \sigma_{r,x}^{e})$. Let

$$\tau^{e}_{r,x}$$

factor $(S_{r,x}^{e}, Q_{r}, *)$, where $\tau^{e}_{r,x}(s_{r,x,i}^{e}) = D_{r,x,i}^{e}$ for $i < v_{r,x}^{e}$. Let

$$D_{r}^{e} = \{x \in X_{r} \setminus B_{r}^{e} : \gamma_{r,x}^{e} \text{ is essentially continuous}\},$$

$$E_{r}^{e} = \text{dom}(X_{r}) \setminus (B_{r}^{e} \cup D_{r}^{e}).$$

95
Thus, \( v_{r,x}^c = \text{card}(S_{r,x}^c) \). For \( x \in D_r^c \), \( v_{r,x}^c = \text{lh}(s_{r,x}^c) \); for \( x \in E_r^c \), \( v_{r,x}^c = \text{lh}(s_{r,x}^c) - 1 \).

Put \( \text{ucf}(R[y]) = (d_r^r, q_r^r) \), \( \text{ucf}(Y[y_r]) = (e_r^r, x_r^r) \), if \( e_r^r = 2 \) then put \( x_r^r = (x_r^r, W_r^r, \bar{w}_r^r) \).

By order preservation and continuity of \( \theta \), we can see that for \( r \in \text{dom}(R) \),

1. if \( y_r \) is of continuous type, then \( ^c \theta_{r,x} \) has uniform cofinality \( \text{ucf}(R[y]) \);

2. if \( y_r \) is of continuous type and \( \text{lh}(r) = 1 \lor (d_r, q_r) \) does not appear in the contraction of \((\text{sign}_2 Q_r(\theta_{r,x}))\) for any \((e, x) \in \text{dom}(X_r) \setminus \{(e_r, x_r)\}\), then \( ^c \theta_{r,x} \) is essentially discontinuous;

3. if \( y_r \) is of discontinuous type,
   
   (a) if \( d_r^r = 0 \), then \( e_r^r = 0 \);
   
   (b) if \( d_r^r = 1 \), then \( e_r^r = 1 \), \( ^c \theta_{r,x} \) has uniform cofinality \( (1, q_r^r) \), and thus by Lemma \[4.66\] \( D_r^1_{r,x} = (1, q_r^r, \emptyset) \);

   (c) if \( d_r^r = 2 \) and \( q_r^r \in \text{desc}(2Q_r) \), then \( e_r^r = 2 \) and \( x_r^r \in \text{desc}(2X_r) \), \( ^c \theta_{r,x} \) has uniform cofinality \( (2, q_r^r) \), and thus by Lemma \[4.66\] \( \text{ucf}_2^{W_r^r}(D_r^2_{r,x}) = (2, q_r^r) \);

   (d) if \( d_r^r = 2 \) and \( q_r^r \notin \text{desc}(2Q_r) \), then \( e_r^r = 2 \) and \( x_r^r \in \text{desc}(2X_r) \), \( j_{W_r,x}^r ; W_r^r(\theta_{r,x}^r) \) has uniform cofinality \( (2, q_r^r) \), and thus by Lemma \[4.66\] \( \text{ucf}_2^{W_r^r}(D_r^2_{r,x}) = (2, q_r^r) \).

Claim 4.68. Suppose \( r \in \text{dom}(R) \), \( x, x' \in \text{dom}(2X_r) \), \( x = (x')^- \). Suppose the contraction of \((\text{sign}_1(D_{r,x}^2))_{i < v_{r,x}^2} \) is \((w_{r,x})_{i < \text{lh}(x)} \). Then

1. For any \( i < v_{r,x}^2 \), \( \delta_{r,x}^2 = \delta_{r,x'}^2 

2. \( (D_{r,x}^2, \delta_{r,x}^2, \delta_{r,x'}^2)_{i < v_{r,x}^2} \) is a proper initial segment of \((D_{r,x'}^2, \delta_{r,x'}^2)_{i < v_{r,x}'} \). Hence, \( S_{r,x}^2 \) is a proper subtree of \( S_{r,x'}^2 \) and \( s_{r,x}^2 \) is an initial segment of \( s_{r,x'}^2 \).

3. \( \text{sign}_1(D_{r,x}^2, v_{r,x}^2) = w_{r,x} \). In particular, the contraction of \((\text{sign}_1(D_{r,x'}^2))_{i < v_{r,x'}^2} \) is \((w_{r,x})_{i < \text{lh}(x)} \).

4. \( D_{r,x}^2 < q_{r,w,x} \ D_{r,x'}^2 \).

5. If \( x \in D_r^2 \cup E_r^2 \), \( x^- (c), x^- (d) \in \text{dom}(2X) \), \( c < BK d \), then \( \delta_{r,x}^2 < \delta_{r,x'}^2 \).

6. If \( x \in D_r^2 \cup E_r^2 \), \( [h]_{\mu, q_{r,x}} = \delta_{r,x}^2 \), then for any \( g \in E_{Q_r^+} \), \([h \circ g_{r,x}^Q]_{\mu, w_{r,x}} = \theta_r([g]_{Q_r^+}) \).
Proof. By Lemma 4.34, 
\[ j^{Q_r}(j^{W_{r,x},W_{r,x'} \mid j^{W_{r,x}}(\omega_1 + 1)}) = j^{Q_r \otimes W_{r,x},Q_r \otimes W_{r,x'} \mid j^{Q_r \otimes W_{r,x}}(\omega_1 + 1)} \] and \( j^{Q_r}(j^{W_{r,x},W_{r,x'} \mid j^{W_{r,x}}(\omega_1 + 1)}) = j^{Q_r \otimes W_{r,x},Q_r \otimes W_{r,x'} \mid j^{Q_r \otimes W_{r,x}}(\omega_1 + 1)}. \) Since \( \theta_r \) takes values in \([\omega_1]^X\) on a \( \mu^{Q_r} \)-measure one set, for \( \mu^{Q_r} \)-a.e. \( \xi \), we have
\[
j^{W_{r,x},W_{r,x'}}(\theta_r(x)(\xi)) < \theta_r(x)(\xi) < j^{W_{r,x},W_{r,x'}}(\theta_r(x)(\xi))
\]
and
\[
\text{cf}^{W_{r,x}}(\theta_r(x)(\xi)) = \text{seed}_{W_{r,x}}^r.
\]
Hence by Loš,
\[
j^{Q_r \otimes W_{r,x},Q_r \otimes W_{r,x'}}(\theta_r(x)) < \theta_r(x) < j^{Q_r \otimes W_{r,x},Q_r \otimes W_{r,x'}}(\theta_r(x))
\]
and by Lemma 4.66,
\[
\text{ucf}_1(D_{r,x}) = w_{r,x}^r.
\]
We are in a position to apply Lemma 3.14 with
\[
A = \{ l: \exists D \in \text{desc}(Q_r, W_{r,x}) \} u_t = \text{seed}_{D}^{Q_r,W_{r,x}'} \}
\]
leading to parts 1-4. Part 5 also follows from Lemma 3.14 using the fact that \( \gamma_{r,x}^-(c) < \gamma_{r,x}^-(d) \). We now prove part 6. Note that \( \tau_{r,x}^2 \) factors \( (S_{r,x}, Q_r \otimes W_{r,x}) \) and in fact, \( (\tau_{r,x}^2)^{Q_r \otimes W_{r,x}}(\delta_{r,x,v_{r,x}}^2) = \gamma_{r,x}^2 \). Suppose we are given \( h \) with
\[
[h]_{Q_r \otimes W_{r,x}} = \delta_{r,x,v_{r,x}}^2. \]
Define \( h* \) on \( [E]^{Q_r} \) by \( h*>([g]^{Q_r}) = [h \circ g^{Q_r,W_{r,x}}]_{W_{r,x}} \). By Loš, it suffices to show that \( [h*]_{Q_r \otimes W_{r,x}} = \gamma_{r,x}^2 \). But this follows from Lemma 4.32. This finishes the proof of Claim 4.68.

In parallel to Claim 4.55, we have

Claim 4.69. Suppose \( r, r' \in \text{dom}(R) \), \( r = (r')^- \), \( y_r \) is of continuous type, and the contraction of \( ((\text{sign}_2(D_{r,x,v_{r,x}}^{r,r'}))_{1 \leq j < \text{lh}(x_r)}) \) is \( ((d_{r,f}, q_{r,f}))_{1 \leq i < \text{lh}(r)} \). Then

1. \( y_r = y_{r'} \).
2. For any \((e, x) \in \text{dom}(X_r) \setminus \{(x_r, r_r)\}\), \( \gamma_{r,x} = \gamma_{r',x} \).
3. \( (D_{r,x,v_{r,x}}^{r,r'}, \delta_{r,x,v_{r,x}}^{r,r'})_{1 < v_{r,x}^r} \) is a proper initial segment of \( (D_{r',x,v_{r',x}}^{r',r'} , \delta_{r',x,v_{r',x}}^{r',r'})_{1 < v_{r',x}^r} \). Hence \( S_{r,x,v_{r,x}} \) is a proper subtree of \( S_{r',x,v_{r',x}} \), and \( S_{r,x,v_{r,x}} \) is an initial segment of \( S_{r,x,v_{r',x}} \).
4. The level-2 signature of \( r_{x,v_{r,x}} \) is \((1, q_r)\) if \( d_r = 1 \), \((2, q_{r,f})\) \(1 \leq i < \text{lh}(q_r)\) if \( d_r = 2 \). In particular, the contraction of \( ((\text{sign}_2(D_{r',x,v_{r',x}}^{r',r'}))_{1 \leq j < \text{lh}(x_r)}) \) is \( ((d_{r,f}, q_{r,f}))_{1 \leq i < \text{lh}(r)} \).

97
Proof. By order preservation and continuity of $\theta$, $y_r = y_{r'}$ and for $\mu^{Q_r}$-a.e. $\bar{\beta}$,
1. for any $(e, x) \in \text{dom}(X_r) \setminus \{(e_r, x_r)\}$, if $\bar{\beta}$ extends $\bar{\beta}'$ then $\theta_{r,x}(\bar{\beta}) = \theta_{r',x}(\bar{\beta}')$;
2. $e\theta_{r,x}(\bar{\beta}) = \sup\{e\theta_{r',x}(\bar{\beta}') : \bar{\beta}' \text{ extends } \bar{\beta}\}$.

Thus, $\gamma_{r,x} = \gamma_{r',x}$ for any $(e, x) \in \text{dom}(X_r) \setminus \{(e_r, x_r)\}$, and $j_{\sup}^{Q_r, Q_{r'}}(\gamma_{r,x}) \leq j_{\sup}^{Q_r, Q_{r'}}(\gamma_{r',x})$. As $t_x = t_{x'}$ is of continuous type and $(d_r, q_r)$ does not appear in $\text{sign}(\theta_{r',x})$ for any $(e, x) \in \text{dom}(X_r) \setminus \{(e_r, x_r)\}$, $\bar{\beta}_{r',x}$ is essentially discontinuous, giving $j_{\sup}^{Q_r, Q_{r'}}(\gamma_{r,x}) \neq \gamma_{r',x}$. With the help of Lemma 4.32 again, we can find level-1 trees $M_r, M_{r'}$ such that $M_r$ is a subtree of $M_{r'}$ and $j_{\sup}^{M_r, M_{r'}}(\gamma_{r,x}) < \gamma_{r',x} < j_{\sup}^{M_r, M_{r'}}(\gamma_{r,x})$. The claim then follows from Lemma 3.13.

In parallel to Claim 4.56, we have

Claim 4.70. Suppose $r, r' \in \text{dom}(R)$, $r = (r')^-$, $y_r$ is of discontinuous type, and the contraction of $(\text{sign}^2(D_r^{e_r, j}))_{1 \leq j < \text{lh}(x_{r'})} (2, q_{r'})_{1 \leq i \leq \text{lh}(q_{r'})}$ is $((d_{r}, q_{r}))_{1 \leq i \leq \text{lh}(q_{r})}$. Put $x_r^* = x_{r'}^*$ if $\text{ucf}(R(r)) \notin \text{desc}(Q_r)$, $x_r^* = x_{r'}^* \setminus (x_r(\text{lh}(x_r) - 1))^{-}$ if $\text{ucf}(R(r)) \in \text{desc}(Q_r)$. Then

1. $y_r \subsetneq y_{r'}$.
2. For any $(e, x) \in \text{dom}(X_r)$, $\gamma_{r,x} = \gamma_{r',x}$.
3. $(D_{r,x,r', j}^e, \delta_{r,x,r', j}^e)_{1 \leq j < \text{lh}(x_{r'})}$ is a proper initial segment of $(D_{r',x',r', j}^e, \delta_{r',x',r', j}^e)_{1 \leq j < \text{lh}(x_{r'})}$. The signature and approximation sequence of $\gamma_{r,x}$ are proper initial segments of those of $\gamma_{r',x}$. Hence $S_{r,x,r'}$ is a proper subtree of $S_{r',x',r'}$, and $s_{r,x,r'}$ is an initial segment of $s_{r',x',r'}$.
4. The level-2 signature of $\tau_{r,x}(s_{r,x})$ is $((1, q_r))$ if $d_r = 1$, $((2, q_r))_{1 \leq i \leq \text{lh}(q_r)}$ if $d_r = 2$. In particular, the contraction of $(\text{sign}(2)(D_{r,x,r', j}^e))_{1 \leq j < \text{lh}(x_{r'})}$ is $((d_{r}, q_{r}))_{1 \leq i \leq \text{lh}(q_{r})}$.

Proof. By order preservation and continuity of $\theta$, $y_r \subsetneq y_{r'}$ and for $\mu^{Q_r}$-a.e. $\bar{\beta}$,
1. for any $(e, x) \in \text{dom}(X_r)$, if $\bar{\beta}$ extends $\bar{\beta}'$ then $\theta_{r,x}(\bar{\beta}) = \theta_{r',x}(\bar{\beta}')$;
2. if $\text{ucf}(R(r)) \notin \text{desc}(Q_r)$ then $j_{X_r^{e_r, X_r^*}}(\theta_{r,x}(\bar{\beta})) = \sup\{e\theta_{r',x}(\bar{\beta}') : \bar{\beta}' \text{ extends } \bar{\beta}\}$, where $X_r^* = Y_{\text{tree}}(y_{r'} \setminus \text{lh}(y_{r}) + 1)$;
3. if $\text{ucf}(R(r)) \in \text{desc}(Q_r)$ then $e\theta_{r,x}(\bar{\beta}) = \sup\{e\theta_{r',x}(\bar{\beta}') : \bar{\beta}' \text{ extends } \bar{\beta}\}$.
The rest is similar to the proof of Claim 4.69.

Let
\[
\phi^1 : \{ \gamma_{r,x} : r \in \text{dom}(R), x \in B_r^1 \} \rightarrow Z^1
\]
be a bijection such that $Z^1$ is a level-1 tree and $v < v' \iff \phi^1(v) < \phi^1(v')$.
Let
\[
\phi^2 : \{(D_{r,x,i}, \delta_{r,x,i})_{i<1} : r \in \text{dom}(R), e \in \{1, 2\}, x \in D_r^e \cup E_r^e, l < \text{lh}(s_{r,x}) \} \rightarrow Z^2 \cup \{\emptyset\}
\]
be a bijection such that $Z^2$ is a level-1 tree and $v \subseteq v' \iff \phi^2(v) \subseteq \phi^2(v')$, $v <_{BK} v'$ iff $\phi^2(v) <_{BK} \phi^2(v)$, where the ordering of subcoordinates $D_{r,x,i}$ is according to $\prec$. Let
\[
T = \langle T_1, T_2 \rangle
\]
where $T_1 = Z^1$, $T_2$ is a level-2 tree, $\text{dom}(T) = Z^2$,
\[
T_2[\phi^2((D_{r,x,i}, \delta_{r,x,i})_{i<\text{lh}(s_{r,x})})^{-1}(-1)] = (S_{x,r}, \vec{s}_{x,r}) \text{ for } x \in D_r^e,
\]
\[
T_2[\phi^2((D_{r,x,i}, \delta_{r,x,i})_{i<\text{lh}(s_{r,x})})^{-1}] = (S_{x,r}, \vec{s}_{x,r}) \text{ for } x \in E_r^e.
\]
Let
\[
\delta = (\delta_t)_{(c,t) \in \text{dom}(T)}
\]
where $\delta_t = (\phi^1)^{-1}(t)$, $\delta_{\emptyset} = \omega_1$, $\delta_{s_{r,x}} = \delta_{r,x,i}$ where $t = \phi^2((D_{r,x,i}, \delta_{r,x,i})_{i<1})$. For $r \in \text{dom}(R)$, let
\[
\rho(r) = (y_r, \pi_r)
\]
where $\pi_r$ factors $(X_r, T, Q_r)$, defined as follows:
\[
\pi_r(e, x) = \begin{cases} 
(1, \phi^1(\gamma_{e,x}), \emptyset) & \text{if } x \in B_r^e, \\
(2, (\phi^2((D_{r,x,i}, \delta_{r,x,i})_{i<\text{lh}(s_{r,x})})^{-1}(-1), S_{x,r}, \vec{s}_{x,r}), \tau_{r,x}) & \text{if } x \in D_r^e, \\
(2, (\phi^2((D_{r,x,i}, \delta_{r,x,i})_{i<\text{lh}(s_{r,x})})^{-1}, S_{r,x}, \vec{s}_{r,x}), \tau_{r,x}) & \text{if } x \in E_r^e.
\end{cases}
\]
It is easy to check that $(T, \rho, \delta)$ works for the lemma.

Put $[\emptyset]_R = \text{o.t.}(\prec_R)$. For $r = (r, Q, (d, q, \vec{P})) \in \text{desc}^*(R)$, put
\[
[r]_R = [\vec{\beta} \mapsto [\vec{\beta} + R r]_{<R}]_{dR}.
\]
If $r \in \text{desc}(R)$ is of discontinuous type, put $[r]_R = [r]_R$. Note that if $\rho$ factors $\Pi^1_3$-wellfounded trees $(R, Y)$, then $[r]_R \leq [\rho(r)]_Y$ for any $r \in \text{dom}(R)$.
We say that $\rho$ minimally factors $(R, Y)$ iff $\rho$ factors $(R, Y)$, $R, Y$ are both $\Pi^1_3$-wellfounded and $[r]_R = [\rho(r)]_Y$ for any $r \in \text{dom}(R)$. In particular, if $Y$ is $\Pi^1_3$-wellfounded and $T$ is $\Pi^1_2$-wellfounded, then id$_Y$, minimally factors $(Y, Y \otimes T)$. In the assumption of Lemma 4.67 if $R, Y$ are $\Pi^1_3$-wellfounded and ran($\theta$) is a $\prec_Y$-initial segment of rep($Y$), its proof constructs $\rho$ which minimally factors $(R, Y \otimes T)$. This entails the comparison theorem between $\Pi^1_3$-wellfounded trees.
Theorem 4.71. Suppose $R, Y$ are $\Pi^1_3$-wellfounded level-3 trees and $\emptyset_R \leq \emptyset_Y$. Then there exists $(T, \rho)$ such that $T$ is $\Pi^1_2$-wellfounded and $\rho$ minimally factors $(R, Y \otimes T)$. Furthermore, if $\emptyset_R < \emptyset_Y$, we further obtain $B \in \text{dom}(Y \otimes T)$ such that $\text{lh}(B) = 1$ and $\emptyset_R = [B]_{Y \otimes T}$.

4.9 Representations of ordinals in $\delta^1_3$

We introduce a coding system for ordinals in $\delta^1_3$ which is the higher level analog of WO. The coding system is guided by Corollary 2.12. Identifying $u_\omega$ with $(V_\omega \cup u_\omega)^{<\omega}$, we shall assume $X$ is a $\Delta^1_3$ subset of $\mathbb{R} \times (V_\omega \cup u_\omega)^{<\omega}$ so that the map $v \mapsto X_v$ is a surjection from $\mathbb{R}$ onto $\mathcal{P}((V_\omega \cup u_\omega)^{<\omega})$.

For a finite level-3 tree $R$ and a tuple $\vec{\beta} \oplus_R t \in \text{rep}(R)$, put

$$v \in \text{LO}^R_{\vec{\beta} \oplus_R t}$$

iff for each $\vec{\gamma} \oplus_R s \leq_R \vec{\beta} \oplus_R t$,

$$(X_v)_{\vec{\gamma} \oplus_R s} = \text{def} \{ (\xi, \eta) : (v, \vec{\gamma} \oplus_R s, \xi, \eta) \in X_v \}$$

is a linear ordering on $u_\omega$. Put

$$v \in \text{LO}^R$$

iff $v \in \text{LO}^R_{\vec{\beta} \oplus_R t}$ for all $\vec{\beta} \oplus_R t \in \text{rep}(R)$. The relations “$v \in \text{LO}^R_{\vec{\beta} \oplus_R t}$” and “$v \in \text{LO}^R$” are $\Delta^1_3$. Put

$$v \in \text{WO}^R_{\vec{\beta} \oplus_R t}$$

iff for each $\vec{\gamma} \oplus_R s \leq_R \vec{\beta} \oplus_R t$, $(X_v)_{\vec{\gamma} \oplus_R s}$ is a wellordering on $u_\omega$, and the map $\vec{\gamma} \oplus_R s \mapsto \text{o.t.}((X_v)_{\vec{\gamma} \oplus_R s})$ is continuous, order preserving for $\vec{\gamma} \oplus_R s \leq_R \vec{\beta} \oplus_R t$. Put

$$v \in \text{WO}^R$$

iff $v \in \text{WO}^R_{\vec{\beta} \oplus_R t}$ for all $\vec{\beta} \oplus_R t \in \text{rep}(R)$. The relations “$v \in \text{WO}^R_{\vec{\beta} \oplus_R t}$” and “$v \in \text{WO}^R$” are $\Pi^1_3$. If $(X_v)_{\vec{\beta} \oplus_R t}$ is a wellordering on $u_\omega$, its order type is denoted by $\|v\|_{\vec{\beta} \oplus_R t}$. A member $v \in \text{WO}^R$ codes a tuple of ordinals $[v]^R$ that respects $R$:

$$[v]^R = [\vec{\beta} \oplus_R t \mapsto \|v\|_{\vec{\beta} \oplus_R t}]^R.$$
Observe the simple fact that for any finite level-1 tree $W$, for any $\vec{\alpha} = (\alpha_w)_{w \in W}$ respecting $W$, there is a $\Pi_1^1$-wellfounded level-1 tree $W'$ extending $W$ such that $\alpha_w = \| (w) \|_{< \omega^{W'}}$ for any $w \in W$. Intuitively, $W'$ “represents” $\vec{\alpha}$ in the sense that $\vec{\alpha}$ extends to a tuple $\vec{\alpha}'$ respecting $W'$ and if $\vec{\beta}$ respects $W'$, then $\forall w \in W \alpha_w \leq \beta_w$. It is implicitly used in proving that $0^\#$ is the unique wellfounded remarkable EM blueprint. Likewise, its higher level analog will be an ingredient in the level-3 EM blueprint formulation of $0^{3\#}$. The goal of the remaining of this section is to prove Lemma 4.79, which states that every $\vec{\gamma}$ respecting a finite level-3 tree $R$ is “representable”. Lemma 4.79 will essentially be a strengthening of [13, Theorem 5.3].

The next lemma is an easy corollary of Lemma 4.18. In its statement, $(T, \vec{\gamma})$ is the “amalgamation” of $(Q, \vec{\beta})$ and $(Q', \vec{\beta}')$.

**Lemma 4.72.** Suppose $Q, Q'$ are level $\leq 2$ trees, $\vec{\beta} = (d_3/(d, q) \in \text{dom}(Q) \text{ respects } Q, \vec{\beta}' = (d_3'/(d, q) \in \text{dom}(Q') \text{ respects } Q'$. Then there are $2 \leq 2$ tree $T$, a tuple $\vec{\gamma} = (d_3/(d, t) \in \text{dom}(T)$ and maps $\pi, \pi'$ factoring $(Q, T), (Q', T)$ respectively such that $\text{dom}(T) = \text{ran}(\pi) \cup \text{ran}(\pi')$, $d_3/(d, q) = d_3/(d, q)$ for any $(d, q) \in \text{dom}(Q), d_3'/(d, q) = d_3'/(d, q)$ for any $(d, q) \in \text{dom}(Q')$.

Amalgamation of level-3 trees is similar, using Lemma 4.46 instead.

**Lemma 4.73.** Suppose $R, R'$ are level-3 trees, $\vec{\gamma} = (\gamma_r)_{r \in \text{dom}(R)} \text{ respects } R, \vec{\gamma}' = (\gamma_r')_{r \in \text{dom}(R')} \text{ respects } R'$. Then there are a level-3 tree $Y$, a tuple $\vec{\delta} = (\delta_y)_{y \in \text{dom}(Y)}$ and maps $\rho, \rho'$ factoring $(R, Y), (R', Y)$ respectively such that $\text{dom}(Y) = \text{ran}(\rho) \cup \text{ran}(\rho')$, $\delta_y(r) = \gamma_r$ for any $r \in \text{dom}(R), \delta_y'(r) = \gamma_r'$ for any $r \in \text{dom}(R')$.

**Lemma 4.74.** For any $\alpha \in \omega^{< \omega}$, $\{ [2, (a)]_Q : Q \text{ is a } \Pi_1^1 \text{-wellfounded level } \leq 2 \text{ tree}, (a) \in \text{dom}(Q) \}$ is a cofinal subset of $u_2$.

**Proof.** Note that $u_2 = \delta_3^1$ is the sup of ranks of $\Sigma_2^1$ wellfounded relations on $\mathbb{R}$. Given $<^*$, a $\Sigma^1_2$ wellfounded on $\mathbb{R}$, we need to find a $\Pi^1_2$-wellfounded level $\leq 2$ tree $Q$ such that $\text{rank}(<^*) \leq [2, (0)]_Q$. This suffices for the Lemma by rearranging the nodes in a level $\leq 2$ tree in a suitable way. Put $x <^* x' \iff \exists y \ x \oplus x' \oplus y \in A$, where $A$ is $\Pi^1_1$. Let $(P_x)_{x \in \omega^{< \omega}}$ be a regular level-1 system such that $P_{x \oplus x' \oplus y}$ is $\Pi^1_1$-wellfounded iff $x \oplus x' \oplus y \in A$. Fix an effective bijection $\phi : \omega^{< \omega} \leftrightarrow (\omega^{< \omega})^{< \omega}$. If $(W_n)_{n<\omega}$ is a sequence of nonempty level-1 trees, their join is $\oplus_{n<\omega} W_n = \{ (n) \cdot w : w \in W_n \}$. Let $Q^*$ be an infinite level-2 tree whose domain is $\{ (0) \cdot q : q \in (\omega^{< \omega})^{< \omega} \}$, and for any real $v$, $\cup_{n<\omega} Q^*_{\text{tree}}\,(\{0\} \cdot \phi(v \uparrow n)) = \oplus_{n<\omega} P_{(v_{2n+2} \oplus (v_{2n} \oplus v)_{2n+1}}$. Then $Q^*$ is $\Pi^1_2$-wellfounded. Let $Q = (\emptyset, Q^*)$. The proof of Kunen-Martin shows that $\text{rank}(<^*) \leq [2, (0)]_Q$. □
Lemma 4.75. Suppose $Q$ is a $\Pi^1_2$-wellfounded level $\leq 2$ tree, $q^* \in \text{dom}(Q)$. $P^*$ is the completion of $Q(q^*)$. Then $\{[2,q']_Q : Q'$ is $\Pi^1_2$-wellfounded, $\{2,q^*\} \models 2Q'_{\text{tree}}(q') = P^*\}$ is a cofinal subset of $\{2,q^*\}_{Q}$. 

Proof. If $q^* = \emptyset$, we are reduced to Lemma 4.74. Suppose now $q^* \neq \emptyset$. Put $\pi Q(q^*) = (P^-,p^*)$, so $P^*$ is the completion of $(P^-,p^*)$.

Let $p^{**} = \text{pred}_{\leq P^*}(p^*)$. By remarkability of the level-1 sharps, letting $f(\beta) = [\overline{\alpha} \mapsto \Vert(2,\overline{\alpha} \upharpoonright P^* \text{-} p) \cup g(\alpha_{p^*}) \cup q^*) (-(1))\Vert_{\leq \mu^P}$ for $\beta = [g]_{\mu L} < u_2$, then $\sup f'' u_2 = j^{P^*}(\{2,q^*\}_Q)$. Fix $\beta = [g]_{\mu L} < u_2$, and we try to find $Q', q'$ such that $\pi Q'[\{q^*\}] = \pi Q[q^*]$, $\pi Q'_{\text{tree}}(q') = P^*$, and $f(\beta) < \{2,q^*\}_Q$. Let $U$ be a $\Pi^1_2$-wellfounded level $\leq 2$ tree obtained by Lemma 4.74 such that $\beta < \{2,((0))\}_U$. Let $(X,\pi)$ be a representation of $Q \otimes U$, and let $\theta : \text{rep}(X) \rightarrow \text{rep}(Q)$ be the order preserving bijection. Let $C = (2,\{q,\pi \}) \in \text{desc}(Q,\{q\},*)$, where $q = \{q^*) (-(1)), P^*, \bar{p}^*\}$, $\pi$ extends $\text{id}_{\Pi^1_2}$, $\pi (\tau(t)) = (2,((0)),(0)), \sigma)$, $\sigma(0)) = \pi^*-\text{a.e.}$. Then for $\mu^P$-a.e. $\alpha^*, \theta(2,\overline{\alpha} \otimes 2x \ x) = (2,\overline{\alpha} \upharpoonright (g(\alpha_{p^*}) \cup q^*) (-(1))$. Therefore, $\{2,q\}_Q = f(\{2,((0))\}_U) > f(\beta)$. $\square$

Suppose $Q$ is a level $\leq 2$ tree and $\overline{\epsilon} = (\overline{\epsilon}_t)_{(d,t) \in \text{dom}(Q)}$ is a tuple of ordinals indexed by $\text{dom}(Q)$. We say that $\overline{\epsilon}$ is represented by $Q'$ if $Q'$ is a subtree of $Q'$. $Q'$ is $\Pi^1_2$-wellfounded and $\overline{\epsilon} = (\{d,t\}_Q')_{(d,t) \in \text{dom}(Q')}$. 

Lemma 4.76. Suppose $Q$ is a finite level $\leq 2$ tree and $\overline{\beta} = (\overline{\beta}_q)_{(d,q) \in \text{dom}(Q)}$ respects $Q$. Then $\overline{\beta}$ is represented by some level $\leq 2$ tree $Q'$. 

Proof. By rearranging the nodes in $\text{dom}(Q')$ in a suitable way, it suffices to produce a level $\leq 2$ tree $Q'$ and a map $\pi$ factoring $(Q, Q')$ such that for any $(d,q) \in \text{dom}(Q)$, $\overline{\beta}_q = [\pi(d,q)]_{Q}$. By a repeated application of Lemma 4.72 it suffices to show that for any $(d^*,q^*-\text{a.e.}) \in \text{dom}(Q)$,

1. if $d^* = 1$, then there is a $\Pi^1_2$-wellfounded level $\leq 2$ tree $Q'$ and $q^* \in Q'$ such that $\overline{\beta}_{q^*} = [1,q^*]_{Q'}$. 
2. if $d^* = 2$ and $P^* = Q'_{\text{tree}}(q^*-\text{a.e.})$, then there is a $\Pi^1_2$-wellfounded level $\leq 2$ tree $Q'$ and $q^* \in \text{dom}(Q')$ such that $\overline{\beta}_{q^*-\text{a.e.}} = [2,q^*]_{Q'}$. 

The case $d^* = 1$ is obvious. We assume now $d^* = 2$.

Lemma 4.75 gives us a $\Pi^1_2$-wellfounded level $\leq 2$ tree $T$ and $t \in \text{dom}(T)$ such that $[2,t]_T \geq \overline{\beta}_{q^*-\text{a.e.}}$, $2T[t^-] \geq 2Q[q^*]$, $2T'_{\text{tree}}(t) = P^*$. Minimizing $[2,t]_T$, we may further assume that for any $\Pi^1_2$-wellfounded $T'$ and any $t'$ such that $[2,t]_{T'} \geq \overline{\beta}_{q^*-\text{a.e.}}$, $2T'[t'^-] \geq 2Q[q^*]$, $2T'_{\text{tree}}'(t') = P^*$, we have
Therefore, \( J \beta < \omega \).

If \( T \{ t, - \} \) has a \(<_{BK}\)-maximum \( t' \), then \( [2, t]_T = [2, t']_T + \omega \). So \( \beta < \omega \), contradicting the minimization assumption.

If \( T \{ t, - \} \) has \(<_{BK}\)-limit order type, then \( [2, t]_T = \sup \{ [2, t']_T : t' \in T \{ t, - \} \} \), so there is \( t' \) satisfying \( \beta < [2, t']_T < [2, t]_T \), contradiction again.

Case 2: \( cf_T(2, t) = 1 \).

For \( \beta = \omega_1 \), put \( f(\beta) = \{ (\bar{\alpha} : \beta < \omega_1) : \| (2, \bar{\alpha}^\rightarrow (\beta) \odot_T t^\rightarrow (-1)) \|_{<_T} \} \). Then \( [2, t]_T = \sup \{ f(\beta) : \beta < \omega_1 \} \). For each limit \( \beta < \omega_1 \), we shall find a \( \Pi^1_3\)-wellfounded and a node \( t' \) such that \( [2, t']_T = f(\beta) \), \( 2^T \{ t' \} = 2 \), \( \tau_{tree}(t') = P^* \), contradicting to the minimization assumption. Fix a limit ordinal \( \beta < \omega_1 \). Let \( U \) be a \( \Pi^1_3\)-wellfounded level \( \leq 2 \) tree such that \( [1, (0)]_U = \beta \). Let \( (X, \pi) \) be a representation of \( T \odot U \) and let \( \theta : \text{rep}(X) \to \text{rep}(T) \) be the order preserving bijection. Let \( C = (2, t, \tau) \in \text{desc}(T, U, *) \), \( t = \tau^\rightarrow (-1), S, \bar{s}, \tau \) extends \( \text{id}_{s, S}, \tau (S_{\text{lh}(\bar{s}) - 1}) = (1, 0, 0) \). Let \( (2, x) = \pi^{-1}(C) \). Then for \( \mu^\text{rep}\)-a.e. \( \bar{\alpha}, \theta(2, \bar{\alpha} \odot_{2X} x) = (2, \bar{\alpha}^\rightarrow (\beta) \odot_T t^\rightarrow (-1)) \).

Therefore, \( [2, x]_X = f(\beta) \). \( (X, x) \) plays the role of the desired \( (T', t') \).

Case 3: \( cf_T(2, t) = 2 \).

Let \( p^* \) be the \(<_{BK}\)-\( P^* \cup \{ p^* \}\)-predecessor of \( p^* \). For \( \beta = \omega_1 < u_2 \), put \( f(\beta) = \{ (\bar{\alpha} \mapsto \| (2, \bar{\alpha}^\rightarrow g(\alpha_{p^*}) \odot_T t^\rightarrow (-1)) \|_{<_T} \} \). Then \( [2, t]_T = \sup \{ f(\beta) : \beta < u_2 \} \). For each limit \( \omega_1 < \beta < u_2 \), we shall find a \( \Pi^1_3\)-wellfounded \( T' \) and a node \( t' \) such that \( [2, t']_T = f(\beta) \), \( 2^T \{ t' \} = 2 \), \( \tau_{tree}(t') = P^* \). Fix a limit ordinal \( \omega_1 < \beta < u_2 \). By Case 1 and Case 2 of this lemma applied to \( (2, (0)) \) in place of \( (p^*, q^\rightarrow (a)) \), we can find a \( \Pi^1_3\)-wellfounded level \( \leq 2 \) tree \( U \) such that \( [2, (0)]_U = \beta \). Let \( (X, \pi) \) be a representation of \( T \odot U \) and let \( \theta : \text{rep}(X) \to \text{rep}(T) \) be the order preserving bijection. Let \( C = (2, t, \tau), t = \tau^\rightarrow (-1), S, \bar{s}, \tau \) extends \( \text{id}_{s, S}, \tau (S_{\text{lh}(\bar{s}) - 1}) = (1, (0), (0), (0), (0), \sigma) \), \( \sigma((0)) = p^* \). Let \( (2, x) = \pi^{-1}(C) \). \( (X, x) \) plays the role of the desired \( (T', t') \).

The level-3 version of Lemmas 4.74-4.76 are similarly proved.

**Lemma 4.77.** For any \( a \in \omega^\omega \), \( \{([a])_R : R \text{ is a } \Pi^1_3\text{-wellfounded level-3 tree, } (a) \in \text{dom}(R) \} \) is a cofinal subset of \( \delta^1_3 \).

**Proof.** It is possible to imitate the proof of Lemma 4.73. We give an alternative proof using the prewellordering property of the pointclass \( \Pi^1_3 \). Let \( G \) be a good universal \( \Pi^1_3\)-set and let \( (R_x)_{x \in \omega^\omega} \) be an effective level-3 system satisfying \( x \in G \) iff \( R_x = \text{def } \cup_{a \in \omega} R_{x \rightarrow a} \) is \( \Pi^1_3\)-wellfounded. \( G \) is equipped with the \( \Pi^1_3\)-norm \( \varphi(x) = [0]_{R_x} \), the complexity from Theorem 2.1. By
Suppose Moschovakis [36, 4C.14], o.t.(ran(φ)) = δ₁³. The rest of the proof is simple.

**Lemma 4.78.** Suppose R is a Π₁³-wellfounded level-3 tree, r* ∈ dom(R), Q* is a completion of R(r*). Then \([r^*]_{R'} : R' is Π₁³-wellfounded, R'[(r')^−] = R[r^*], R_{\text{tree}}(r') = Q^*\) is a cofinal subset of \(j^{R_{\text{tree}}(r^*)}Q^*([r^*]_R)\).

**Proof.** Put \(R(r^*) = (Q^−, (d^*, q^*, P^*))\), \(R[r^*] = (Q^−, (d, q, P))\).

Case 1: cf(\(R(r^*)\)) = 1.

By Lemma 4.31, letting \(f(ξ) = [β ↦ \|\vec{β}^−(ξ) ⊕_R r^∗\|_{r^∗}]_{μQ^−}\) for \(ξ < ω_1\), then \(sup f''ω_1 = j^{Q^−Q^∗}([r^*]_R)\). Fix \(β < ω_1\), and we try to find \(R' \) and \(r'\) such that \(R'[(r')^−] = R[r^*], R'(r') = Q^*, \) and \(f(β) < [r^*]_R\). Let \(U\) be a Π₁³-wellfounded level ≤ 2 tree such that \(β < [1, 0]_U\). Let \((Z, ρ)\) be a representation of \(R ⊗ U\), and let \(θ : \text{rep}(Z) → \text{rep}(U)\) be the order preserving bijection. Let \(B = (r, π) ∈ \text{desc}(R, U, *)\), where \(r = (r^−(−1), Q^∗, (d, q, P^*))\), \(π\) extends id₄, \(π(d^*, q^*) = (1, 0, θ)\). Let \(z = ρ−1(B)\). Similarly to Case 1 of the proof of Lemma 4.75, \((Z, z)\) plays the role of the desired \((R', r')\).

Case 2: cf(\(R(r^*)\)) = 2.

Put \(E = (e, z, \text{id}_{P^*}) = \text{ucf}^−(R(r^*))\). By Lemma 4.31, letting \(f(ξ) = [β ↦ \|\vec{β}^−(j^{P^∗}(g)(β₂)) ⊕_R r^∗\|_{r^∗}]_{μQ^−}\) for \(ξ = [g]_{μ₁} < u₂\), then \(sup f''μ₂ = j^{Q^−Q^∗}([r^*]_R)\). Fix \(β < u₂\) and we try to find \(R', r'\) as in Case 1. Let \(U\) be a Π₁³-wellfounded level ≤ 2 tree such that \(β < [2, 0]_U\), obtained by Lemma 4.74. Let \((Z, ρ, θ)\) be as in Case 1. Let \(B = (r, π) ∈ \text{desc}(R, U, *)\), where \(r = (r^−(−1), Q^∗, (d, q, P^*))\), \(π\) extends id₄, \(π(d^*, q^*) = (2, (0), ((0)), (0)), τ)\), \(τ(0) = E\). Let \(z = ρ−1(B)\). \((Z, z)\) plays the role of the desired \((R', r')\).

**Lemma 4.79.** Suppose \(R\) is a finite level-3 tree and \(γ = (γ_r)_{r ∈ \text{dom}(R)}\) respects \(R\). Then there is a Π₁³-wellfounded level-3 tree \(R'\) such that \(R ⊆ R'\) and for any \(r ∈ \text{dom}(R)\), \(γ_r = [r]_{R'}\).

**Proof.** It suffices to produce a level-3 tree \(R'\) and a map \(ρ\) factoring \((R, R')\) such that for any \(r ∈ \text{dom}(R)\), \(γ_r = [r]_{R'}\). By Lemma 4.73, it suffices to show that for any \(r^∗(a) ∈ \text{dom}(R)\), letting \(Q^* = R_{\text{tree}}[r^∗(a)]\), there is a Π₁³-wellfounded level-3 tree \(R'\) and \(r' ∈ \text{dom}(R')\) such that \(γ_r = [r']_{R'}, R'[(r')^−] = R[r^∗], R_{\text{tree}}(r') = Q^∗\).

Lemma 4.75 gives us a Π₁³-wellfounded level-3 tree \(Y\) and \(y ∈ \text{dom}(Y)\) such that \([y]_Y ≥ γ_r^∗(a), Y[y^−] = R[r^∗], Y_{\text{tree}}(y) = Q^∗\). Minimizing \([y]_Y\), we may further assume that for any Π₁³-wellfounded \(Y'\) and any \(y'\) such that \([y']_{Y'} ≥ γ_r^∗(a), Y'[(y')^−] = R[r^∗], Y_{\text{tree}}'(y') = Q^∗\), we have \([y']_{Y'} ≥ [r^∗(a)]_Y\). We claim that \([y]_Y = γ_r^∗(a)\). Suppose otherwise. Put \(Y(y) = (Q^∗, (d^∗, q^∗, P^*))\).
Case 1: cf($Y(y)$) = 0.
Argue as in Case 1 in the proof of Lemma 4.76 to obtain a contradiction.
Case 2: cf($Y(y)$) = 1.
For $\beta < \omega$, put $f(\beta) = [\alpha \mapsto \|\tilde{\alpha}^{-}(\beta) \oplus \beta, y^{-}(\beta)\|_{\delta}]_{y^{\omega}}$. So $[\delta]_{\delta} = \sup\{f(\beta) : \beta < \omega\}$. For each limit ordinal $\beta < \omega$, we shall find a $\Pi_{3}^{1}$-wellfounded $Y'$ and a node $y' \in \text{dom}(Y')$ such that $[\alpha]_{\delta} = f(\beta)$, contradicting the minimization assumption. Fix a limit ordinal $\beta < \omega$. Let $U$ be a $\Pi_{3}^{1}$-wellfounded level $\leq 2$ tree such that $[1, (0)]_{U} = \beta$. Let $(Z, \rho)$ be a representation of $Y \otimes U$ and let $\theta : \text{rep}(Z) \to \text{rep}(Y)$ be the order preserving bijection. Let $B = (y, \pi) \in \text{desc}(Y, U, *)$, $y = (y^{-}(\delta), X, (e, x, W))$, $\pi$ extends $\text{id}_{z} = \pi(\text{lh}(\delta), x_{\text{lh}(\delta)}) = (1, (0), \varnothing)$. Let $z = \rho^{-1}(B)$. Similarly to Case 2 of the proof of Lemma 4.76, $(Z, z)$ constitutes a counterexample.

Case 3: cf($Y(y)$) = 2.
Let $E = (e, z, \text{id}_{\rho^{\omega}}) = \text{ucf}^{-1}(Y(y))$. For $\beta = [g]_{\mu} < u_{2}$, put $f(\beta) = [\alpha \mapsto \|\tilde{\alpha}^{-} f^{\omega}(\beta)(\alpha_{2}) \oplus \beta, y^{-}(\beta)\|_{\delta}]_{y^{\omega}}$. So $[\delta]_{\delta} = \sup\{f(\beta) : \beta < u_{2}\}$. For each limit ordinal $\omega_{1} < \beta < u_{2}$, we shall find a $\Pi_{3}^{1}$-wellfounded $Y'$ and a node $y' \in \text{dom}(Y')$ such that $[\delta]_{y'} = f(\beta)$. Let $U$ be a $\Pi_{3}^{1}$-wellfounded level $\leq 2$ tree such that $[2, ((0))]_{U} = \beta$. Let $Z, \rho, \theta$ be as in Case 2. Let $B = (y, \pi)$, where $y = (y^{-}(\delta), X, (e, x, W))$, $\pi$ extends $\text{id}_{z} = \pi(\text{lh}(\delta), x_{\text{lh}(\delta)}) = (2, ((0), \{0\}, \{0\}), \tau)$, $\tau((0)) = E$. Let $z = \rho^{-1}(B)$. Similarly to Case 3 of the proof of Lemma 4.76, $(Z, z)$ constitutes a counterexample. □

5 The lightface level-3 sharp
Sections 5.1–5.3 defines a $\Pi_{1}^{1}$ singleton $0^{3 \#}$ which is many-one equivalent to $M_{\#}^{2}$, under boldface $\Pi_{3}^{1}$-determinacy. The assumption of $\Pi_{3}^{1}$-determinacy is very likely not optimal. Section 5.4 formulates the existence of $0^{3 \#}$ as a purely syntactical large cardinal axiom based on the weaker assumption of $\Delta_{1}^{1}$-determinacy.
Recall that $\mathbb{L}[T_{3}] = \bigcup_{x \in \mathbb{R}} \mathbb{L}[T_{3}, x]$, $\mathbb{L}_{\Delta_{1}^{1}}[T_{3}] = \bigcup_{x \in \mathbb{R}} \mathbb{L}_{\Delta_{1}^{1}}[T_{3}, x]$. By Steel, $\mathbb{L}_{\Delta_{1}^{1}}[T_{3}] = \mathbb{V}_{\Delta_{1}^{1}} \cap \mathbb{L}[T_{3}]$.

Lemma 5.1. Assume $\Pi_{3}^{1}$-determinacy.

1. $\mathbb{L}_{\Delta_{1}^{1}}[T_{2}] = \mathbb{L}_{\Delta_{1}^{1}}[T_{3}]$.

2. Every subset of $\Delta_{1}^{1}$ in $\mathbb{L}_{\Delta_{1}^{1}}[T_{3}]$ is definable over $M_{2, \infty}(x)$ from $\{x\}$ for some $x \in \mathbb{R}$.

Proof. 1. $T_{2}$ is a $\Delta_{1}^{1}$ subset of $u_{\omega}$, and of course a $\Sigma_{1}^{1}$ subset of $\delta_{3}^{1}$, and hence $T_{2} \in \mathbb{L}[T_{3}]$ by Becker-Kechris [3]. This gives the $\subseteq$ inclusion.
If \( A \in L_{\delta_3^1}[T_3, x] \), by Theorem 2.18 there must be \( \xi < \delta_3^1 \) such that \( A \in L_{\xi}[T_3, \xi, x] \). Pick \( y \geq_T x \) such that \( \kappa_3^\delta > \xi \). Then \( A \in L_{\kappa_3^\delta}[T_2, y] \) by Lemma 4.37. This gives the \( \supseteq \) inclusion.

2. By Theorem 2.18, every subset of \( \delta_3^1 \) in \( L_{\delta_3^1}[T_3] \) is in \( M_{2,\infty}^2(x) \) for some \( x \in \mathbb{R} \). If \( A \subseteq \delta_3^1 \) is definable over \( M_{2,\infty}^2(x) \) from \( \{ \gamma, x \} \), \( \gamma < \delta_3^1 \), letting \( y \geq_T M_{2,\infty}^2(x) \) such that \( \gamma \) is definable over \( L_{\kappa_3^\delta}[T_2, y] \), then \( A \) is definable over \( M_{2,\infty}^2(y) \) from \( \{ y \} \).

Caution that Lemma 5.1 does not give a real \( x \) for which \( T_3 \in L[T_2, x] \). \( L_{\kappa_3^\delta}[T_2, x] \) computes a proper initial segment of \( T_3 \), and by varying \( x \), these proper initial segments are cofinal in \( T_3 \). However, there is not a single \( x \) with \( T_3 \in L[T_2, x] \).

### 5.1 Level-3 boundedness

Recall in Corollary 2.10 that the rank of a \( \Sigma_3^1(<u_\omega, x) \) wellfounded relation is bounded by \( \kappa_3^\delta \). We would like to strengthen this fact by allowing a suitable code for an arbitrary ordinal in \( \delta_3^1 \). The strengthening is based on an inner model theoretic characterization of \( u_\omega \) in \( L[T_3, x] \). We say that

\[ \delta \text{ is an L-Woodin cardinal} \]

iff \( L(V_\delta) \models \delta \) is Woodin.

**Theorem 5.2** (Woodin, 41, Theorem 3.21). Assume \( \Pi_3^1 \)-determinacy. Let \( \kappa = u_\omega \). For \( x \in \mathbb{R} \), \( M_{2,\infty}^2(x) \models \kappa \) is the least L-Woodin cardinal.

**Corollary 5.3** (Level-3 boundedness). Assume \( \Pi_3^1 \)-determinacy. Suppose \( x \in \mathbb{R} \), \( \mathcal{N} \in \mathcal{F}_{2,x} \), \( \eta \) is a cardinal and strong cutpoint of \( \mathcal{N} \), \( \xi = \pi_{\mathcal{N},\infty}(\eta) \). Suppose \( g \) is Coll(\( \omega, \eta \))-generic over \( \mathcal{N} \), \( r \in \mathbb{R} \cap \mathcal{N}[g] \). Let \( \lambda \) be the least L-Woodin cardinal in \( M_{2,\infty}(x) \) above \( \xi \). Suppose \( G \) is a \( \Pi_3^1(r, < u_\omega) \) set equipped with a regular \( \Pi_3^1(r, < u_\omega) \) norm \( \varphi \). Suppose \( A \) is a \( \Sigma_3^1(r, < u_\omega) \) subset of \( G \). Then

\[
\sup \{ \varphi(y) : y \in A \} < (\lambda^+)_{M_{2,\infty}}(x).
\]

**Proof.** Put \( x = 0 \) for simplicity. Put

\[ \mathcal{G}_{2}^{\mathcal{N},\eta} = \{ \mathcal{P} \in \mathcal{F}_2 : \mathcal{P} \text{ is a nondropping iterate of } \mathcal{N} \text{ above } \eta \} \].

\( \mathcal{G}_{2}^{\mathcal{N},\eta} \) is a subsystem of \( \mathcal{F}_2 \). Let \( M_{2,\infty}^{\mathcal{N},\eta} \) be the direct limit of \( \mathcal{G}_{2}^{\mathcal{N},\eta} \). The inclusion map of direct systems induces an embedding between direct limits

\[ \pi_{x}^{\mathcal{N},\eta} : M_{2,\infty}^{\mathcal{N},\eta} \rightarrow M_{2,\infty}^\# \].

106
Let $r_g \in \mathbb{R}$ be the real coding $(g, N|\eta)$. Every mouse $\mathcal{P} \in \mathcal{G}_2^{N, \eta}$ corresponds to an $r_g$-mouse $\mathcal{P}[g] \in \mathcal{F}_{2, r_g}$ (converted into an $r_g$-mouse in the obvious way, cf. [42]). So in the direct limit,

$$M_{2, \infty}^{N, \eta}[g] = M_{2, \infty}^{\#}(r_g).$$

By Corollary 2.10,

$$\sup\{\varphi(y) : y \in A\} < \kappa^\prime,$$

which in turn is smaller than the successor of $u_\omega$ in $M_{2, \infty}^{\#}(r_g)$, as $\{T_2, r\} \in M_{2, \infty}^{\#}(r_g)$. By Theorem 5.2, $u_\omega$ is the least L-Woodin cardinal of $M_{2, \infty}^{\#}(r_g)$, hence the least L-Woodin cardinal of $M_{2, \infty}^{N, \eta}$ above $\eta$. By elementarity, $\pi_{\mathcal{P}_z, \infty}(u_\omega) = \lambda$. So $\pi_{\mathcal{P}_z, \infty}(\kappa^\prime) < (\lambda^+)_{M_{2, \infty}}$. This finishes the proof. 

We also need to code ordinals in $\delta_3^1$ by direct limits of iterations of $\Pi_3^1$-iterable mice. Suppose $x \in \mathbb{R}$ and $z$ codes a $\Pi_3^1$-iterable $x$-mouse $\mathcal{P}_z$. Then

$$\pi_{\mathcal{P}_z, \infty} : \mathcal{P}_z \to (\mathcal{P}_z)_\infty$$

is the direct limit map of all the nondropping iterates of $\mathcal{P}_z$. $o((\mathcal{P}_z)_\infty)$ is the length of a $\Delta_1^1(z)$-prewellordering, namely the one induced by iterations of $\mathcal{P}_z$. By Corollary 2.15, $\pi_{\mathcal{P}_z, \infty}$ and $(\mathcal{P}_z)_\infty$ are both in $L_{\kappa_3^1}(T_2, M_1^{\#}(z))$ and $\Delta_1$-definable over $L_{\kappa_3^1}(T_2, M_1^{\#}(z))$ from $\{T_2, M_1^{\#}(z)\}$.

### 5.2 Putative level-3 indiscernibles

The higher level analog of the type of $L$ with $n$ indiscernibles is the type of $M_{2, \infty}$ realized by an appropriate $[F]^R$, where $F \in (\delta_3^1)^{RT}$. Such functions $F$ are coded by subsets of $u_\omega$ in $L_{\delta_3^1}[T_2]$. The coding system is provided by Corollary 2.12.

$L = \{\in\}$ is the language of set theory. For a level-3 tree $R$, $L^R$ is the expansion of $L$ which consists of additional constant symbols $c_r$ for each $r \in \text{dom}(R)$. For a level-3 tree $R$ and a tuple of ordinals $\bar{\gamma} = (\gamma_r)_{r \in \text{dom}(R)}$, the $L$-structure $M_{2, \infty}$ expands to the $L^R$-structure

$$(M_{2, \infty}; \bar{\gamma})$$

whose constant $c_r$ is interpreted as $\gamma_r$.

**Definition 5.4.** $C \subseteq \delta_3^1$ is said to be firm iff every member of $C$ is additively closed, the set $\{\xi : \xi = \text{o.t.}(C \cap \xi)\}$ has order type $\delta_3^1$ and $C \cap \xi \in L_{\delta_3^1}[T_2]$ for all $\xi < \delta_3^1$. 

107
**Definition 5.5.** \( C \subseteq \delta_3^1 \) is called a set of *potential level-3 indiscernibles* for \( M_{2,\infty} \) if for any level-3 tree \( R \), for any \( F,G \in C^{\mathbb{R}_T} \cap L_{\delta_1^1}^{[T_2]} \),

\[
(M_{2,\infty}; [F]^R) \equiv (M_{2,\infty}; [G]^R).
\]

A firm set of potential level-3 indiscernibles for \( M_{2,\infty} \) is the higher level analog of a set of order indiscernibles for \( L \). Note that the successor elements of \( C \) don’t really play a part in computing \( [F]^R = ([F]_{\mu^R_{\text{tree}(r)}})_{r \in \text{dom}(R)} \), as the relevant ultrapowers \( \mu^R_{\text{tree}(r)} \) concentrate on tuples of limit ordinals, hence the prefix “potential”.

**Lemma 5.6.** Assume \( \Pi_3^1 \)-determinacy. Then there is a firm set of potential level-3 indiscernibles for \( M_{2,\infty} \).

**Proof.** Suppose \( R \) is a finite level-3 tree. Let \( \varphi \) be an \( L^R \)-sentence. Consider the game \( G^R; \varphi \) where I produces reals \( v, x, c \) and a natural number \( p \), II produces reals \( v', x', c' \) and a natural number \( p' \). The payoff is decided according to the following priority list:

1. I and II must take turns to ensure that \( v \in WO^R \) and \( v' \in WO^R \). If one of them fails to do so, and \( w \in \text{rep}(R) \) is \( <^R \)-least for which \( v \notin WO^R \) \( \lor \) \( v' \notin WO^R \), then I loses iff \( v \notin WO^R \), and II loses iff \( v \in WO^R \).

2. If 1 is satisfied, put \( \vec{\gamma} = (\gamma_r)_{r \in \text{dom}(R)} \), where \( \gamma_r = \max([v]^R_r, [v']^R_r) \). I must ensure
   (a) \( x \) codes a 2-small premouse \( P_x \) which satisfies “I am closed under the \( M_4^R \)-operator”;
   (b) \( c \) codes a strictly increasing, cofinal-in-\( (P_x) \) sequence of ordinals \( (c_n)_{n < \omega} \) relative to \( x \) such that each \( c_n \) is a cardinal cutpoint of \( P_x \);
   (c) \( P_x|c_1 \) is a \( \Pi_3^1 \)-iterable mouse;
   (d) \( p \) codes a tuple of ordinals \( \vec{\alpha} = (\alpha_r)_{r \in \text{dom}(R)} \) in \( P_x|c_0 \) relative to \( x \);
   (e) For each \( r \in \text{dom}(R) \), \( \pi_{P_x|c_0}(\alpha_r) = \gamma_r \);
   (f) \( (P_x; \vec{\alpha}) \models \varphi \).

Otherwise he loses.

3. If 1-2 are satisfied, II must ensure 2(a)-(f) with \( (x, c, (c_n)_{n < \omega}, p, \vec{\alpha}, \varphi) \) replaced by \( (x', c', (c'_n)_{n < \omega}, p', \vec{\alpha}', \neg \varphi) \), otherwise he loses.
4. If 1-3 are satisfied, I and II must take turns to ensure for all \(2 \leq n < \omega\),

(a) \(\mathcal{P}_x|c_n\) is a \(\Pi^1_3\)-iterable mouse and \(\mathcal{P}_x'|c_{n-1} <_{D^J} \mathcal{P}_x|c_n\);

(b) \(\mathcal{P}_x|c_n\) is a \(\Pi^1_3\)-iterable mouse and \(\mathcal{P}_x|c_n <_{D^J} \mathcal{P}_x'|c_n\).

If one of them fails to do so, and \(n\) is least for which (a) or (b) fails at \(n\), then I loses iff (a) fails at \(n\), and II loses iff (a) holds at \(n\).

5. It is impossible that both players obey all the rules, due to a successful comparison between \(\mathcal{P}_x\) and \(\mathcal{P}_x\). The definition of \(G^{R|\varphi}\) is finished.

The payoff of \(G^{R|\varphi}\) has complexity \((\emptyset)_R + \omega)\)-\(\Pi^1_3\) for both players. The nontrivial part about the complexity is that 2(e) is \(\Delta^1_3\), as follows. According to rules 2(a)-(c), \(\mathcal{P}_x|c_1\) is \(\Pi^1_3\)-iterable and closed under the (genuine) \(M^R_1\)-operator, \(c_0 < c_1\), and therefore \(M^R_1(\mathcal{P}_x|c_0)\) is canonically coded in \(x\). \(\pi_{\mathcal{P}_x|c_0}^\omega(\alpha_s)\) is the length of a \(\Delta^1_3(\mathcal{P}_x|c_0)\) prewellordering, induced by iterations. By Corollary 2.15, \(\pi_{\mathcal{P}_x|c_0}(\alpha_s)\) is \(\Delta^1_1\)-definable over \(L_{\alpha_3}[T_2, x]\) from \(\{T_2, x\}\). \(\bar{\alpha}\) is clearly \(\Delta^1_1\)-definable over \(L_{\alpha_3}[T_2, v]\) from \(\{T_2, v\}\). So 2(e) is expressed into a \(\Delta^1_1\) statement over \(L_{\alpha_3}[v, x]\) from \(\{T_2, v, x, c\}\), or equivalently, \(\Delta^1_3(v, x, c)\) by Theorem 2.1.

Hence \(G^{R|\varphi}\) is defined. Suppose for definiteness II has a winning strategy \(\sigma\) in \(G^{R|\varphi}\). Let \(C\) be the set of \(L\)-Woodin cardinal cutpoints of \(M^{\omega}_{\omega, \omega}(\sigma)\) and their limits. We show that

\[
\forall F \in C^{R^+} (M_{\omega, \omega})^R \models [F]^R = \neg\varphi
\]

Suppose towards a contradiction that \(F \in C^{R^+}\) but \((M_{\omega, \omega})^R \models [F]^R = \varphi\). As \(\delta^1_3\) is inaccessible in \(M^R_{\omega, \omega}\), there is a club \(D \subseteq M^R_{\omega, \omega}\) in \(\delta^1_3\) so that \(M^R_{\omega, \omega}|\lambda \prec M^R_{\omega, \omega}\) for any \(\lambda \in D\). There is thus a continuous, order preserving \(G: \omega + 1 \to C \setminus \sup ran(F)\) for which \((M^R_{\omega, \omega}|\alpha^R; [F]^R) = \varphi\). Pick \(\mathcal{P} \in \mathcal{F}_2\) and ordinals \((c_n)_{n<\omega}, (\alpha_r)_{r \in \text{dom}(R)}\) in \(\mathcal{P}\) such that \(\pi_{\mathcal{P}, \omega}(c_n) = \mathcal{G}(n)\) for any \(n < \omega\) and \(\pi_{\mathcal{P}, \omega}^\omega(\alpha_r) = [F]^R_r\) for any \(r \in \text{dom}(R)\). Thus, \((\mathcal{P}|\sup_{n<\omega} c_n; \bar{\alpha}) \models \varphi\). Let Player I play \((v, x, c, p)\), where \(v \in WO^{R^+}\), \([v]^R_w = F(w)\) for any \(w \in \text{rep}(R)\), \(x\) codes \(\mathcal{P}|\sup_{n<\omega} c_n\), \(c\) codes \((c_n)_{n<\omega}\) relative to \(x\), \(p\) codes \((\alpha_r)_{r \in \text{dom}(R)}\). The response according to \(\sigma\) is denoted by \((v', x', c', p') = (v, x, c, p) * \sigma\). We shall derive a contraction by showing neither player breaks the rules, using \(\Sigma^1_3\)-boundedness.

As \(\sigma\) is a winning strategy, Player II is not the first person to break the rules. So \(v \in WO^{R^+}\) implies \(v' \in WO^{R^+}\). For each \(w \in \text{rep}(R)\) which is either the \(<^R\)-minimum or a \(<^R\)-successor, if \(N \in \mathcal{F}_2, \eta \in N, \pi_{N, \omega}(\eta) = F(w), g\) is Coll(\(\omega, \eta\))-generic over \(N\), \(r_g \in R\) being the real coding \((g, N|\eta)\), then \((v', x', c', p')\) belongs to the set

\[
A_w = \{(\bar{v}, \bar{x}, \bar{c}, \bar{p}) : \bar{v} \in WO^R_{w} | \xi\}
\]

109
which is $\Sigma^1_4(M^\#_1(r_g), < u_\omega)$ by Corollary 2.15 and Theorem 2.1. Since $\sigma$ is a winning strategy, $A_w$ is a subset of $B_w = \left\{ (\vec{v}', \vec{x}', \vec{c}', \vec{p}') : \vec{v}' \in \text{WO}^{R^+}_w \right\}$.

$B_w$ is a $\Pi^1_3(< u_\omega)$ set, equipped with the $\Pi^1_3(< u_\omega)$ prewellordering $(\vec{v}', \vec{x}', \vec{c}', \vec{p}') \mapsto \|\vec{v}'\|_w$. By Corollary 5.3, $\|v'\|_w < \min(C \setminus (F(w) + 1))$. By continuity, if $w$ has $< R$-limit order type, then $\|v'\|_w \leq \|v\|_w$. Consequently, for $r \in \text{dom}(R)$, $[v']_r \leq [v]_R$, so if $\vec{v}$ is defined from $v, v'$ as in Rule 2, then $\vec{v} = [v]_R$.

By our choice of $F$ and $G$, Rule 2 is satisfied. Let $P_x, (c_n)_{n < \omega}, \vec{a}, P'_x, (c'_n)_{n < \omega}, \vec{a}'$ be defined as in Rules 2 and 3. For each $1 \leq n < \omega$, using the $\Pi^3_1$-iterable mice, a similar boundedness argument shows that $\|P_x | c'_n\| < DJ < \min(C \setminus (G(n) + 1))$, and hence $P_x | c'_n < DJ P_x | c_{n+1}$. So Rule 4 is satisfied. This is impossible. \hfill $\square$

**Definition 5.7.** Assume $\Pi^1_3$-determinacy. Let $C$ be a firm set of potential level-3 indiscernibles for $M^-_{2,\infty}$. Then

$0^{3\#}$

is a map sending a level-3 tree $R$ to the complete consistent $L^R$-theory $0^{3\#}(R)$, where $^R \varphi \in 0^{3\#}(R)$ iff $\varphi$ is an $L^R$-formula and for all $\vec{v} \in [C]^{R^+}$,

$(M^-_{2,\infty}; \vec{v}) \models \varphi$.

$0^{3\#}$ is the higher level analog of $0^\#$. Each individual $0^{3\#}(R)$ is the higher level analog of the $n$-type that is realized in $L$ by $n$ indiscernibles. As with the level-1 sharps, we shall give a $\Pi^4_1$ axiomatization of $0^{3\#}$ in Section 5.4.

The proof of Lemma 5.6 shows

**Lemma 5.8.** Assume $\Pi^1_3$-determinacy. For a finite level-3 tree $R$, $0^{3\#}(R)$ is a $\sigma([0]_R + \omega)$-$\Pi^1_3$ real.

In fact, the complexity of $0^{3\#}(R)$ relies only on $[0]_R$.

**Lemma 5.9.** If $Q$ is a finite level $\leq 2$ tree, then $j^Q(M^-_{2,\infty}), j^Q \upharpoonright M^-_{2,\infty}$ are definable over $M^-_{2,\infty}$, uniformly in $Q$. If $X$ is another finite level $\leq 2$ trees and $\pi$ factors $(X, Q)$, then $\pi^Q \upharpoonright j^X(M^-_{2,\infty})$ is definable over $M^-_{2,\infty}$, uniformly in $(\pi, X, Q)$.

**Proof.** By Theorem 2.18, $j^Q(M^-_{2,\infty}) = L[j^Q(T_3)]$, and every $\Sigma^1_1$ subset of $\delta^1_3$ is definable over $M^-_{2,\infty}$. It suffices to show that $j^Q(T_3), j^Q \upharpoonright \delta^1_3, \pi^Q \upharpoonright \delta^1_3$ are all $\Sigma^4_1$ in the codes.
Let $G$ be a good universal $\Pi^1_3$ set and let $\varphi : G \to \delta^1_3$ be a regular $\Pi^1_3$ norm. Then $x \in G \land y \in G \land j^Q(\varphi(x)) = \varphi(y)$ iff there exists $z \in \mathbb{R}$ such that $z = M^\#_1(x, y)$ and $L_{\kappa_3}[T_2, z] \models j^Q(\varphi(x)) = \varphi(y)$. Here, the statement $j^Q(\varphi(x)) = \varphi(y)$ is $\Delta_1$ over $L_{\kappa_3}[T_2, z]$ from $\{T_2, z\}$ by Corollary 2.15 and Lemma 4.37. Similarly, using Lemma 4.35, $\pi^Q \upharpoonright \kappa_3^3$ and $j^Q(T_3) \upharpoonright \kappa_3^3$ are $\Delta_1$-definable over $L_{\kappa_3^3}[T_2, x]$ from $\{T_2, x\}$, uniformly in $x$. So $\pi^Q | \delta^1_3$ and $j^Q(T_3)$ are $\Sigma^1_4$ using similar arguments.

Based on Theorem 2.18 and $\Sigma^1_4$-absoluteness of iterates of $M^\#$, a $\Sigma^1_3$ set $A \subseteq \delta^1_3$ has the following alternative definition: $\alpha \in A$ iff $M_{2\infty}^\langle \alpha \rangle$ satisfies that for any $\xi > \alpha$ cardinal cutpoint, in the Coll($\omega, \xi$)-generic extension, $\pi_{K|\xi, \infty}(\alpha) \in A$.

We introduce the following informal symbols arising from the proof of Lemma 5.9 that will occur in $L$-formulas or $L^R$-formulas for a level-3 tree $R$:

1. If $Q$ is a finite level $\leq 2$ tree, $j^Q$ is the informal symbol so that $j^Q(a) = b$ iff for any $\xi$ cardinal cutpoint such that $\{a, b\} \subseteq K|\xi$, the Coll($\omega, \xi$)-generic extension satisfies $j^Q(\pi_{K|\xi, \infty}(a)) = \pi_{K|\xi, \infty}(b)$.

2. If $\pi$ factors finite level $\leq 2$ trees $(X, T)$, $\pi^T$ is the informal symbol so that $\pi^T(a) = b$ iff for any $\xi$ cardinal cutpoint such that $\{a, b\} \subseteq K|\xi$, the Coll($\omega, \xi$)-generic extension satisfies $\pi^T(\pi_{K|\xi, \infty}(a)) = \pi_{K|\xi, \infty}(b)$.

3. If $Q$ is a level $\leq 2$ subtree of $Q'$, $Q'$ is finite, then $j^{Q,Q'}$ is an informal symbol, where $id_Q$ factors $(Q, Q')$, $id_Q(d, q) = (d, q)$.

4. Corresponding to items 1-3, $j^{Q, \sup}, \pi^{Q, \sup}$, $j^{Q, \sup}$ stand for functions on ordinals that send $\alpha$ to $\sup(j^Q(\alpha)), \sup(\pi^T(\alpha)), \sup(j^{Q, Q'}(\alpha))$ respectively.

5. $S^3_3$ is the informal symbol such that $((\emptyset, \emptyset) \in S^3_3$ and $((R_i)_i \subseteq n, (\alpha_i)_i \subseteq n) \in S^3_3$ if $R$ is a finite regular level-3 tower and letting $t_i \subseteq \text{dom}(R_{i+1}) \setminus \text{dom}(R_i)$, then $r_k = (t_1)^{-1} \rightarrow \alpha_k < j^{(R_n)}(\text{tree}(t_k), (R_n)_{\text{tree}(t_k)}(\alpha_k))$.

6. For $1 \leq n \leq \omega$, $u_n$ is the symbol so that for any $\xi > u_n$ cardinal cutpoint, the Coll($\omega, \xi$)-generic extension satisfies $\pi_{K|\xi, \infty}(u_n) = u_n$.

7. Suppose $T$ is a finite level $\leq 2$ tree. If $D \in \text{desc}(T, U)$, $||D||_{\omega, T, U} = n$, then $\text{seed}^T_D = u_{n+1}$. If $(1, t) \subseteq \text{dom}(T)$, then $\text{seed}^T_{(1,t)} = \text{seed}^T_{(1,t,0)}$. If $(2, t) \subseteq \text{dom}(T)$, and $\text{tree}(t) = (S, s)$, then $\text{seed}^T_{(2,t)} = \text{seed}^T_{(2,t,S), \text{id}_S}$. $\text{seed}^T = (\text{seed}^T_{(d,t)})_{(d,t) \in \text{dom}(T)}$.  

111
8. If $k$ is a definable class function and $W$ is a definable class, then $k(W) = \bigcup \{ k(W \cap V_\alpha) : \alpha \in \text{Ord} \}$.

9. If $X, T, T'$ are finite level $\leq 2$ trees, $T$ is a subtree of $T'$, $a \in j^X(V)$, $d \in \{ 1, 2 \}$, then

(a) $B^T_{X,a} = \{ \pi^{T,Q}(a) : Q \text{ finite level } \leq 2 \text{ tree, } \pi \text{ factors } (X, T \otimes Q) \}$;

(b) $H^T_{X,a}$ is the transitive collapse of the Skolem hull of $B^T_{X,a} \cup \text{ran}(j^T)$ in $j^T(V)$ and $\phi^T_{X,a} : H^T_{X,a} \rightarrow j^T(V)$ is the associated elementary embedding;

(c) $j^T_{X,a} = (\phi^T_{X,a})^{-1} \circ j^T$;

(d) $j^{T,T'}_{X,a} = (\phi^{T'}_{X,a})^{-1} \circ j^{T,T'} \circ \phi^T_{X,a}$;

(e) $B^T_{1,a} = B^T_{Q^0,a} \cup B^T_{Q^1,a}$, $B^T_{2,a} = B^T_{Q^0,a} \cup B^T_{Q^2,a} \cup B^T_{Q^1,a}$;

(f) $H^T_{d,a}$ is the transitive collapse of the Skolem hull of $B^T_{d,a} \cup \text{ran}(j^T)$ in $j^T(V)$ and $\phi^T_{d,a} : H^T_{d,a} \rightarrow j^T(V)$ is the associated elementary embedding;

(g) $j^T_{d,a} = (\phi^T_{d,a})^{-1} \circ j^T$;

(h) $j^{T,T'}_{d,a} = (\phi^{T'}_{d,a})^{-1} \circ j^{T,T'} \circ \phi^T_{d,a}$.

10. Suppose $R$ is a level-3 tree.

(a) If $r = (r, Q, (d, q, P)) \in \text{desc}^*(R)$, $c_r$ is the informal $L^R$-symbol whose interpretation is

\[
c_r = \begin{cases} 
  j^{R_{\text{tree}}(r^-),Q}(c_r^-) & \text{if } r \in \text{desc}(R) \text{ of continuous type}, \\
  c_r & \text{if } r \in \text{desc}(R) \text{ of discontinuous type}, \\
  j^{R_{\text{tree}}(r),Q}(c_r) & \text{if } r \notin \text{desc}(R).
\end{cases}
\]

(b) If $T, U$ are finite level $\leq 2$ trees and $B = (r, \pi) \in \text{desc}(R, T, U)$, $r \neq 0$, then $c_B$ is the informal $L^R$-symbol which stands for $\pi^{T,U}(c_r)$.

(c) If $A = (r, \pi, T) \in \text{desc}^{**}(R)$, $r \neq 0$, then $c_A$ is the informal $L^R$-symbol which stands for $\pi^{T}(c_r)$.

Put $L^Z = \{ \langle, \rangle \}$; for a level-3 tree $R$, put $L^R_Z = L^R \cup \{ x \}$, where $x$ is a constant symbol. All of the above informal symbols work in $L^Z$ or $L^R_Z$, which are intended to be interpreted in $M^Z_{\infty}(x)$ for $x \in \mathbb{R}$. In particular, for $x \in \mathbb{R}$,

\[
(L[S_x])^{M^Z_{\infty}(x)} = M^Z_{\infty}.
\]
Lemma 5.10. Assume $\Pi_3^1$-determinacy. Suppose $R, Y$ are finite level-3 trees, $T$ is a finite level $\leq 2$ tree, $\rho$ factors $(R, Y, T)$. Then

$$\gamma \varphi(c_{r_1}, \ldots, c_{r_n}) \in 0^3#(R)$$

iff

$$\gamma_j T(V) \models \varphi(c^T_{\rho(r_1)}, \ldots, c^T_{\rho(r_n)}) \in 0^3#(Y).$$

Proof. Put $\rho(r) = (y_r, \pi_r)$. Put $R_{\text{tree}}(r) = Q_r$. Suppose $\varphi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, $r_1, \ldots, r_n \in \text{dom}(R)$, and

$$\gamma \varphi(c_{r_1}, \ldots, c_{r_n}) \in 0^3#(R).$$

Let $C$ be a firm set of potential level-3 indiscernibles. Suppose $F \in C^{Y^\uparrow}$. By Lemma 4.46, $F^T_\rho$ is a function from $\omega_1^{T^\uparrow}$ to $[C]^{R^\uparrow}$. Recall that $F^T_\rho(\delta) = (F^T_\rho(y_r))_{r \in \text{dom}(R)}$. Hence for any $\delta \in \omega_1^{T^\uparrow}$,

$$M_{2,\infty}^{\uparrow} \models \varphi(F^T_{\rho(r_1)}(\delta), \ldots, F^T_{\rho(r_n)}(\delta)).$$

For each $r \in \text{dom}(R)$, by definition of $\pi^T_{r, Q_r}$, $\pi^T_{r, Q_r}([F]^Y_{y_r}) = [F^T_{\rho(r)}]_{y_r}$. Hence by Loś,

$$J^T(M_{2,\infty}^{\uparrow}) \models \varphi(\pi^T_{r_1, Q_{r_1}}([F]^Y_{y_1}), \ldots, \pi^T_{r_n, Q_{r_n}}([F]^Y_{y_n})).$$

Finally, by Lemma 4.60 for $y = (y, X) \in \text{desc}(Y)$, if $y$ is of discontinuous type then $[F]^Y_y = [F]^Y_{y^\downarrow}$; if $y$ is of continuous type then $[F]^Y_y = j_{\text{sup}}^{Y_{\text{tree}}(y^\downarrow), X}([F]^Y_{y^\downarrow})$. Hence,

$$\gamma_j T(V) \models \varphi(c^T_{\rho(r_1)}, \ldots, c^T_{\rho(r_n)}) \in 0^3#(Y).$$

As a corollary to Lemma 5.10, Lemma 5.9 and Theorem 4.71, we obtain:

Corollary 5.11. Assume $\Pi_3^1$-determinacy. Suppose $R$ and $Y$ are finite level-3 trees and $\sq [0]_R = \sq [0]_Y$. Then $0^3#(R) \equiv_m 0^3#(Y)$.

5.3 The equivalence of $x^3#$ and $M_2^\#(x)$

For the other direction of the reduction, we want to compute $\mathcal{C}(u_{\omega}, \Pi^1_3)$ truth using $0^3#$ as an oracle.

Lemma 5.12. Assume $\Pi^1_3$-determinacy. For a finite level-3 tree $R$, the universal $\mathcal{C}([0]_R, \Pi^1_3)$ real is many-one reducible to $0^3#(R)$, uniformly in $R$. 113
Proof. Let \( B \subseteq \{0\}_R \times \mathbb{R} \) be \( \Pi^1_1 \). Let \( \theta \) be a \( \Sigma_1 \) formula such that

\[
(\xi, x) \in B \iff L_{\kappa_3}[T_2, x] \models \theta(\xi, x).
\]

\( G \) is the game with output \( \text{Diff} B \). We need to decide the winner of \( G \) from \( 0^{\beta\#}(R) \). \( B \) is equipped with the \( \Pi^1_3 \)-norm

\[
\psi(\xi, x) = \text{the least } \alpha < \kappa_3^\beta \text{ such that } L_\alpha[T_2, x] \models \theta(\xi, x).
\]

If \( E \in \mu_\ell \) is a club, let \( \rho^E : \{0\}_R \to \text{rep}(R) | E \) be the order preserving bijection. For \( \bar{\gamma} \) respecting \( R \), let \( \bar{\theta}(\bar{\gamma}) \) be the following formula:

There exist \( H \in (\delta_3^1)^{R\uparrow} \) and a strategy \( \tau \) for Player I such that \( [H]^R = \bar{\gamma} \) and for any club \( E \in \mu_\ell \), if \( x \) is an infinite run according to \( \tau \), then for any even \( \alpha < [0]_R \), \( \forall \beta < \alpha((\beta, x) \in B \land \psi(\beta, x) < H(\rho^E(\beta + 1))) \) implies \( (\alpha, x) \in B \land \psi(\alpha, x) < H(\rho^E(\alpha + 1)) \), and there is \( \alpha < [0]_R \) such that \( (\alpha, x) \notin B \).

Let \( \bar{\theta}^I(\bar{\gamma}) \) be the following formula:

There exist \( K \in (\delta_3^1)^{R\uparrow} \) and a strategy \( \sigma \) for Player II such that \( [K]^R = \bar{\gamma} \) and for any club \( E \in \mu_\ell \), if \( x \) is an infinite run according to \( \sigma \), then for any odd \( \alpha < [0]_R \), \( \forall \beta < \alpha((\beta, x) \in B \land \psi(\beta, x) < K(\rho^E(\beta + 1))) \) implies \( (\alpha, x) \in B \land \psi(\alpha, x) < K(\rho^E(\alpha + 1)) \).

Let \( C \) be a firm set of level-3 indiscernibles for \( M_{2,\infty}^- \). Suppose firstly Player I has a winning strategy \( \tau \) in \( G \). Let \( D \) be the subset of \( C \) consisting of \( L \)-Woodin cardinals in \( M_{2,\infty}(\sigma) \) and their limits. By Corollary 5.3, if \( x \) is a consistent run according to \( \sigma \), then \( (0, x) \in B \land \psi(0, x) < \min(D) \), for any odd \( \alpha < [0]_R \), \( (\alpha, x) \in B \) implies \( (\alpha + 1, x) \in B \land \psi(\alpha + 1, x) < \min(D \setminus \{\psi(\alpha, x) + 1\}) \), and there is \( \alpha < [0]_R \) such that \( (\alpha, x) \notin B \). Let \( H \in D^{R\uparrow} \). Then \( (H, \tau) \) witnesses \( \bar{\theta}^I([H]^R) \). Let \( \mathcal{P} \in \mathcal{F}_2 \) and \( \bar{\eta} \in \mathcal{P} \) such that \( \pi_{\mathcal{P}, \infty}(\bar{\eta}) = [H]^R \). Let \( \xi_{\bar{\eta}} \) be the least successor cardinal cutpoint of \( \mathcal{P} \) above \( \max(\bar{\eta}) \) and let \( g \) be Coll(\( \omega, \xi \))-generic over \( \mathcal{P} \). Let \( r_{g, \bar{\eta}} \) be the real coding \((g, \bar{\eta}) \). Then \( \bar{\theta}^I([H]^R) \) is equivalent to a \( \Sigma^1_4(r_{g, \bar{\eta}}) \) statement \( \bar{\theta}^I(r_{g, \bar{\eta}}) \), hence true in \( \mathcal{P}[g] \). Hence,

\[
\mathcal{P}^{\text{Coll}(\omega, \xi_{\bar{\eta}})} \models \bar{\theta}^I(r_{g, \bar{\eta}})
\]

By elementarity,

\[
(M_{2,\infty})^{\text{Coll}(\omega, \xi_{\bar{\eta}})} \models \bar{\theta}^I(r_{g, [H]^R}^\uparrow).
\]

By Lemma 5.6, for any \( \bar{\gamma} \in [C]^{R\uparrow} \),

\[
(M_{2,\infty})^{\text{Coll}(\omega, \xi_{\bar{\gamma}})} \models \bar{\theta}^I(r_{g, \bar{\gamma}}).
\]
By a symmetrical argument, if Player II has a winning strategy in $G$, then for any $\vec{\gamma} \in [C]^{|C|}$, 

$$ (M_{2,\infty})^{\text{Coll}(\omega, \xi_\gamma)} \models \bar{\theta}^I(\hat{r}_{g, \vec{\gamma}}). $$

Finally, there does not exist $\vec{\gamma}$ such that 

$$ (M_{2,\infty})^{\text{Coll}(\omega, \xi_\gamma)} \models \bar{\theta}^I(\hat{r}_{g, \vec{\gamma}}) \land \bar{\theta}^I(\hat{r}_{g, \vec{\gamma}}). $$

Otherwise, by absoluteness, $\theta^I(\vec{\gamma})$ holds. Let $(H, \tau)$ witness $\theta^I(\vec{\gamma})$ and let $(K, \sigma)$ witness $\theta^I(\vec{\gamma})$. Let $E \in \mu_L$ be a club such that $H \restriction (\text{rep}(R) \restriction E) = K \restriction (\text{rep}(R) \restriction E)$. Let $x$ be the infinite run according to both $\tau$ and $\sigma$. Then inductively we can see that for any $\alpha < J \emptyset K R$, $(\alpha, x) \in B \land \psi(\alpha, x) < H(\rho^E(\alpha + 1))$, but there is $\alpha < [0]^R$ such that $(\alpha, x) \notin B$, which is impossible.

In conclusion, Player I has a winning strategy in $B$ iff for any $\vec{\gamma} \in [C]^{|C|}$, 

$$ (M_{2,\infty})^{\text{Coll}(\omega, \xi_\gamma)} \models \bar{\theta}^I(\hat{r}_{g, \vec{\gamma}}). $$

For a real $x$, $x^{3^#}$ is the obvious relativization of $0^{3^#}$. Combining Lemmas 5.8 and 5.12, Theorem 4.5 and Neeman [37,38], we obtain the equivalence of $x^{3^#}$ and $M^2_2(x)$.

**Theorem 5.13.** Assume $\Pi^1_3$-determinacy. For $x \in \mathbb{R}$, $x^{3^#}$ is many-one equivalent to $M^2_2(x)$, the many-one reduction being independent of $x$.

By Theorem 5.13 and Moschovakis third periodicity, the winner of the game in the proof of Lemma 5.6 has a winning strategy recursive in $0^{3^#}$. Hence, the set of $L$-Woodin cardinals in $M_{2,\infty}(0^{3^#})$ and their limits form a firm set of potential level-3 indiscernibles for $M_{2,\infty}$.

### 5.4 Syntactical properties of $0^{3^#}$

Suppose $\mathcal{M}, \mathcal{N}$ are countable $\Pi^1_3$-iterable mice. A map $\pi : \mathcal{M} \to \mathcal{N}$ is **essentially an iteration map** iff there are $\mathcal{P}$ and iteration maps $\psi_\mathcal{M} : \mathcal{M} \to \mathcal{P}$, $\psi_\mathcal{N} : \mathcal{N} \to \mathcal{P}$ such that $\psi_\mathcal{M} = \psi_\mathcal{N} \circ \pi$. For $\alpha \in \mathcal{M}$, $\beta \in \mathcal{N}$, say that $(\mathcal{M}, \alpha) <_{DJ} (\mathcal{N}, \beta)$ iff either $\mathcal{M} <_{DJ} \mathcal{N}$ or there exist $\mathcal{P}$ and iteration maps $\psi_\mathcal{M} : \mathcal{M} \to \mathcal{P}$, $\psi_\mathcal{N} : \mathcal{N} \to \mathcal{P}$ such that $\psi_\mathcal{M}(\alpha) < \psi_\mathcal{N}(\beta)$.

**Definition 5.14 (Level-3 EM blueprint).** A **pre-level-3 EM blueprint** is a function $\Gamma$ sending any finite level-3 tree $Y$ to a complete consistent $\mathcal{L}^Y$-theory $\Gamma(Y)$ which contains all of the following additional axioms:

1. ZFC + there is no inner model with two Woodin cardinals + $V = K$ + there is no strong cardinal + $V$ is closed under the $M^1_1$-operator.

2. Suppose $X, T, Q, Z$ are finite level $\leq 2$ trees, $\pi$ factors $(X, T)$, $\psi$ factors $(T, Z)$. 

115
(a) $j^T : V \to j^T(V)$ is $\mathcal{L}$-elementary. $j^{Q_0}$ is the identity map on $V$.

(b) $\pi^T : j^X(V) \to j^T(V)$ is $\mathcal{L}$-elementary. $j^{Q_0,T} = j^T$. $j^\mathcal{T}$ is the identity map on $j^T(V)$.

(c) $(\psi \circ \pi)^Z = \psi^Z \circ \pi^T$.

(d) $j^T \circ j^Q = j^{T \otimes Q}$.

(e) $j^Q(\pi^T) = (Q \otimes \pi)^{Q \otimes T}$.

(f) $\pi^T \upharpoonright j^X \otimes Q(V) = (\pi \otimes Q)^{T \otimes Q}$.

3. If $\xi$ is a cardinal and strong cutpoint, then $V^{\text{Coll}(\omega, \xi)}$ satisfies the following: If $U$ is a $\Pi^1_3$-wellfounded level $\leq 2$ tree, then $K|\xi$ and $(j^U)_K(K|\xi)$ are countable $\Pi^1_3$-iterable mice and $(j^U)_K(K|\xi)$ is essentially an iteration map from $K|\xi$ to $(j^U)_K(K|\xi)$. Here $(j^U)_K$ stands for the direct limit map of $(j^{Z,Z'}_K)_K$ for $Z,Z'$ finite subtrees of $U$, $Z$ a finite subtree of $Z'$.

4. For any $y \in \text{dom}(Y)$, “$c_y \in \text{Ord}$” is an axiom.

5. If $y \not\sim^Y y'$, then “$c_y < c_{y'}$” is an axiom; if $y \sim^Y y'$, then “$c_y = c_{y'}$” is an axiom.

A level-3 EM blueprint is a pre-level-3 EM blueprint satisfying the coherency property: if $R,Y,T$ are finite, $\rho$ factors $(R,Y,T)$, then for each $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n)$, for each $r_1, \ldots, r_n \in \text{dom}(R)$,

$$\Gamma^R(\varphi(c_{r_1}, \ldots, c_{r_n})) \in \Gamma(R)$$

iff

$$\Gamma^T(V) \models \varphi(c^T_{\rho(r_1)}, \ldots, c^T_{\rho(r_n)}) \in \Gamma(Y).$$

In particular, if $\Gamma$ is a level-3 EM blueprint, $\rho_0$ factors $(R,Y)$, then $\text{id}_{Y^*} \circ \rho_0$ factors $(R,Y,Q^0)$, so by coherency, $\Gamma_\varphi(c_{r_1}, \ldots) \in \Gamma(R)$ iff $\Gamma_\varphi(c_{\rho_0(r_1)}, \ldots) \in \Gamma(Y)$. This degenerates to the usual indiscernability of the (level-1) EM blueprint.

Lemma 5.15. Assume $\Pi^1_3$-determinacy. Then $0^3#$ is a level-3 EM blueprint.

Proof. We verify Axioms 1-5 in Definition 5.14. Axiom 1 follows from Theorem 2.18. Axiom 2 follows from Lemma 4.53. Axioms 4-5 follow from Lemma 4.61.
Axiom 3 is shown as follows. Let $\xi$ be a cardinal strong cutpoint of $M_{2,\infty}$. Let $\mathcal{P} \in \mathcal{F}_2$ and $\eta \in \mathcal{P}$ such that $\pi_{\mathcal{P},\infty}(\eta) = \xi$. Let $g$ be Coll($\omega, \eta$)-generic over $\mathcal{P}$. Suppose $T$ is a $\Pi_2$-wellfounded tree in $\mathcal{P}[g]$. The direct limit of $j^T$ is wellfounded by Proposition 4.19. We need to show that in $\mathcal{P}[g]$, $(j^T)^\mathcal{P}(\mathcal{P}|\eta) : \mathcal{P}|\eta \rightarrow (j^T)^\mathcal{P}(\mathcal{P}|\eta)$ is essentially an iteration map, where $(j^T)^\mathcal{P}$ is the direct limit map of $(j^T)^\mathcal{P}$ for finite subtrees $T'$ of $T$. Since $\mathcal{P}[g]$ is $\Sigma^1_4$-correct, we need to show the same fact in $V$.

Note that $M_{2,\infty}$ is definable over $M_{2,\infty}(g)$. In fact, $M_{2,\infty} = (L[S_3])^{M_{2,\infty}(g)}$. Let $Q \in \mathcal{F}_{2,\infty}$ and $\nu$ so that $\pi_{Q,\infty}(\nu) = \xi$. The maps from $\mathcal{P}$ to $\pi_{Q,\infty}(M_{2,\infty}|\xi)$ and from $\pi_{Q,\infty}^{-1}(M_{2,\infty}|\xi)$ to $L_\xi[S_3]$ plus Dodd-Jensen implies that $\mathcal{P}|\eta \sim_{DJ} \pi_{Q,\infty}^{-1}(M_{2,\infty}|\xi)$. By $\Sigma^1_4$-correctness of set-generic extensions of $Q$, $Q$ thinks that “$\mathcal{P}|\eta \sim_{DJ} L_\nu[S_3]$”. By elementarity, $(j^T(V))^Q$ thinks “$\mathcal{P}|\eta \sim_{DJ} j^T(L_\nu[S_3])$”.

We claim that $(j^T(V))^Q$ is also $\Sigma^1_3$-correct in set-generic extensions. To see this, it suffices to show $p[(j^T(S))^{Q}] \subseteq p[S_3]$. We know that $(j^T(S))^Q$ embeds into $j^T(S_3)$, so $p[(j^T(S))^Q] \subseteq p[j^T(S_3)]$. But $x \in p[j^T(S_3)]$ implies $x \in p[S_3]$ by absoluteness of wellfoundedness and elementarity of $j^T$ acting on $L[S_3, x]$. Hence, in reality we have $\mathcal{P}|\eta \sim_{DJ} j^T(L_\nu[S_3]))^Q$. But $(j^T)^\mathcal{P}(\mathcal{P}|\eta)$ embeds into $(j^T(L_\nu[S_3]))^Q$, implying that $(j^T)^\mathcal{P}(\mathcal{P}|\eta) \leq_{DJ} \mathcal{P}|\eta$.

Of course, $\mathcal{P}|\eta \leq_{DJ} j^T)^\mathcal{P}(\mathcal{P}|\eta)$. So $\mathcal{P} \sim_{DJ} (j^T)^\mathcal{P}(\mathcal{P}|\eta)$. A similar argument shows that for any $\alpha \in \mathcal{P}$, $(\mathcal{P}, \alpha) \sim_{DJ} ((j^T)^\mathcal{P}(\mathcal{P}|\eta), (j^T)^\mathcal{P}(\alpha))$. This finishes verifying Axiom 3 of Definition 5.14.

Finally, the coherency property of $0^{\sharp\sharp}$ is a consequence of Lemma 5.10.

We say that the upward closure of $A \subseteq (\omega^{<\omega})^{<\omega}$ is

$$\{r \in (\omega^{<\omega})^{<\omega} : \exists a \in A(r \subseteq a)\}.$$ 

The upward closure does not apply to subcoordinates of $a \in A$. For instance, $b \subseteq a (\text{lh}(a) - 1)$ does not imply that $a \sim b$ is in the upward closure of $A$.

For a level-3 tree $R$ and nodes $s_1, \ldots, s_n, s'_1, \ldots, s'_n$ in $\text{dom}(R)$,

$\vec{s'}$ is an $R$-shift of $\vec{s}$

iff there are a level-3 tree $S$ and maps $\rho, \rho'$ factoring $(S, R)$ such that $\text{ran}(\rho)$ is the upward closure of $\vec{s}$, $\text{ran}(\rho')$ is the upward closure of $\vec{s'}$, and $\rho^{-1}(s_i) = (\rho')^{-1}(s'_i)$ for any $i$.

**Lemma 5.16** (Level-3 indiscernibility). Suppose $\Gamma$ is a level-3 EM blueprint. Suppose $R$ is a level-3 tree and $s_1, \ldots, s_n, s'_1, \ldots, s'_n$ are nodes in $\text{dom}(R)$. 

117
Suppose that $\vec{s}'$ is a shift of $\vec{s}$ with respect to $R$. Then for each formula $\varphi$, $\Gamma(R)$ contains the formula

$$\varphi(c_{s_1},\ldots,c_{s_n}) \iff \varphi(c_{s_1}',\ldots,c_{s_n}').$$

Proof. Let $S$ be a level-3 tree and $\rho, \rho'$ both factor $(S,R)$ such that $\text{ran}(\rho)$ is upward closure of $\{s_1,\ldots,s_n\}$, $\text{ran}(\rho')$ is the upward closure of $\{s_1',\ldots,s_n'\}$. Let $\rho^{-1}(s_i) = t_i = (\rho')^{-1}(s_i')$. Applying coherency of $\Gamma$ to $\rho, \rho'$,

$$\Gamma(\varphi(c_{s_1},\ldots,c_{s_n}) \in \Gamma(R) \iff \Gamma(\varphi(c_{s_1}',\ldots,c_{s_n}') \in \Gamma(S)$$

$$\iff \Gamma(\varphi(c_{t_1},\ldots,c_{t_n}) \in \Gamma(R).$$

Of course, there is extra information in the coherency property beyond Lemma 5.16.

As with the usual treatment of $0^\#$, a level-3 EM blueprint $\Gamma$ admits an $L$-Skolemized conservative extension. That means, since ZFC + $V = K$ is a part of the axioms, so is “there is a $\Sigma^L_1$-definable wellordering of the universe”. Thus, to each $L$-formula $\varphi(v,w_1,\ldots,w_n)$ we may attach a definable $L$-Skolem term $\tau_{\varphi}(w_1,\ldots,w_n)$ so that the formula $\forall w_1 \ldots w_n (\exists v \varphi(v,w_1,\ldots,w_n) \rightarrow \varphi(\tau_{\varphi}(w_1,\ldots,w_n),w_1,\ldots,w_n))$ belongs to $\Gamma(R)$, for any $R$.

If $Y$ is an infinite level-3 tree, put

$$\Gamma(Y) = \bigcup \{\Gamma(R) : R \text{ is a finite level-3 subtree of } Y\}.$$ 

By coherency, $\Gamma(R) \subseteq \Gamma(R')$ whenever $R \subseteq R'$ are finite. Hence by compactness, $\Gamma(Y)$ is a complete consistent $L_Y$-theory. The usual argument of EM models with order indiscernibles carries over to obtain a unique up to isomorphism $L_Y$-structure

$$\mathcal{M}_{\Gamma,Y} = (M; \leq^M, \alpha^M : t \in \text{dom}(Y)).$$

such that $\mathcal{M}_{\Gamma,Y}$ is $L$-Skolem generated by $\{\alpha^M : t \in \text{dom}(Y)\}$, and

$$\mathcal{M}_{\Gamma,Y} \models \Gamma(Y).$$

$\mathcal{M}_{\Gamma,Y}$ is called the EM model associated to $\Gamma$ and $Y$. When $\leq^{M_{\Gamma,Y}}$ is well-founded, $\mathcal{M}_{\Gamma,Y}$ is identified with its transitive collapse. Since $\mathcal{M}_{\Gamma,Y}$ is a model of $V = K$, the extender sequence on $K^{M_{\Gamma,Y}}$ is definable over $\mathcal{M}_{\Gamma,Y}$, this allows us to sometimes treat $\mathcal{M}_{\Gamma,Y}$ as a structure in the language of premice.

If $L^*$ is a first-order language expanding $L$, $\mathcal{N}$ is an $L^*$-structure satisfying axioms [13] in Definition 5.14, we make the following notations:
1. If $T$ is a finite level $\leq 2$ tree, then $j^T_N = (j^T_N)^N$, $\mathcal{N}^T = (\pi^T(j^T(V)))^N$ is an $\mathcal{L}^*$-structure so that $j^T_N : \mathcal{N} \to \mathcal{N}^T$ is $\mathcal{L}^*$-elementary.

2. If $\pi$ factors finite level $\leq 2$ trees $(X, T)$, then $\pi^T_N = (\pi^T_N)^N$. If $T, T'$ are finite level $\leq 2$ trees, $T$ is a subtree of $T'$, then $j^T_{N,T'} = (j^T_{N,T'})^N$.

3. If $T$ is a level $\leq 2$ tree, then $\mathcal{N}^T$ is the direct limit of $(\mathcal{N}^{T' \times T''}, j^T_{N,T',T''} : T', T'' \text{ finite subtrees of } T, T' \text{ a finite subtree of } T'')$ and $j^T_N : \mathcal{N} \to \mathcal{N}^T$ is the direct limit map; if $T'$ is a finite subtree of $T$, then $j^T_{N,T'} : \mathcal{N}^{T'} \to \mathcal{N}^T$ is the tail of the direct limit map. The wellfounded part of $\mathcal{N}^T$ is always assumed to be transitive.

4. If $\pi$ factors level $\leq 2$ trees $(X, T)$, then $\pi^T_N : \mathcal{N}^X \to \mathcal{N}^T$ is the factor map between direct limits.

5. If $X$ is a finite level $\leq 2$ tree, $a \in \mathcal{N}^X$, $d \in \{1, 2\}$

(a) if $T, T'$ are finite level $\leq 2$ trees, $T$ is a subtree of $T'$, then $\mathcal{N}^T_{X,a} = (H^T_{X,a})^N$, $j^T_{X,a,N} = (j^T_{X,a})^N$, $\phi^T_{X,a,N} = (\phi^T_{X,a})^N$, $j^{T,T'}_{N,T} = (j^{T,T'}_{N,T})^N$, $\mathcal{N}^T_{d,a} = (H^T_{d,a})^N$, $j^T_{d,a,N} = (j^T_{d,a})^N$, $\phi^T_{d,a,N} = (\phi^T_{d,a})^N$;

(b) if $T$ is a level $\leq 2$ tree, then $\mathcal{N}^T_{X,a}$ is the natural direct limit, $j^T_{X,a,N} : \mathcal{N} \to \mathcal{N}^T_{X,a}$ is the direct limit map, $\phi^T_{X,a,N} : \mathcal{N}^T_{X,a} \to \mathcal{N}^T$ is the natural factoring map between direct limits; if $T'$ is a finite subtree of $T$, then $j^T_{X,a,N} : \mathcal{N}^T_{X,a} \to \mathcal{N}^T_{X,a}$ is the tail of the direct limit map; similarly define $\mathcal{N}^T_{d,a}, j^T_{d,a,N}, \phi^T_{d,a,N}$.

If $\Gamma$ is a level-3 EM blueprint and $R$ is $\Pi^1_{\omega}$-wellfounded, we make further notations:

1. If $T$ is a level $\leq 2$ tree, then $\mathcal{M}^T_{\Gamma,Y} = (\mathcal{M}^\Gamma_{\Gamma,Y})^T$, $j^T_{\Gamma,Y} = j^\Gamma_{\mathcal{M},Y}$.

2. If $T$ is a finite subtree of $T'$, then $j^{T,T'}_{\Gamma,Y} = j^\Gamma_{\mathcal{M},Y}$.

3. If $\pi$ factors $(X, T)$, then $\pi^T_{\Gamma,Y} = \pi^\Gamma_{\mathcal{M},Y}$.

4. If $T$ is a finite subtree of $T'$, $y \in \text{dom}(Y)$, $X = Y_{\text{tree}}(y)$, $d \in \{1, 2\}$, then $c_{\Gamma,Y,y} = (c_y)^{\mathcal{M},Y}$, $\mathcal{M}^T_{\Gamma,Y,y} = (\mathcal{M}^\Gamma_{\Gamma,Y})^T_{X,c_{\Gamma,Y,y}}$, $j^T_{\Gamma,Y,y} = j^\Gamma_{X,c_{\Gamma,Y,y},\mathcal{M},Y}$, $\phi^T_{\Gamma,Y,y} = \phi^T_{X,c_{\Gamma,Y,y},\mathcal{M},Y}$, $j^{T,T'}_{\Gamma,Y,y} = j^{\Gamma}_{X,c_{\Gamma,Y,y},\mathcal{M},Y}$, $\mathcal{M}^T_{\Gamma,R,d} = (\mathcal{M}^\Gamma_{\Gamma,R,d})^T_{d_{\Gamma,R,d}(0)}$, $j^T_{\Gamma,R,d} = j^\Gamma_{d_{\Gamma,R,d}(0),\mathcal{M},R}$, $\phi^T_{\Gamma,R,d} = \phi^T_{d_{\Gamma,R,d}(0),\mathcal{M},R}$, $j^{T,T'}_{\Gamma,R,d} = j^{\Gamma}_{d_{\Gamma,R,d}(0),\mathcal{M},R}$.
5. If $B \in \text{desc}(Y, T', *)$ and $T'$ is a finite subtree of $T$, then $c^T_{\Gamma, Y, B} = j_{\Gamma, Y}^{T', T} (c^T_B)^{M_{\Gamma, Y}}$.

By coherency, if $\rho$ factors $(R, Y, T)$, then $\rho$ induces an elementary embedding

$$\rho_{\Gamma, Y}^{Y, T}: M_{\Gamma, R} \rightarrow M_{\Gamma, Y}^T$$

where

$$\rho_{\Gamma, Y}^{Y, T}(\tau^{\cdot R, Y}(c_{\Gamma, R, r_1}, \ldots)) = \tau^{M_{\Gamma, Y}}(c_{\Gamma, Y, \rho(r_1)}, \ldots).$$

If $\rho$ factors $(R, Y)$, then $\rho$ induces

$$\rho_{\Gamma}^{Y}: M_{\Gamma, R} \rightarrow M_{\Gamma, Y}$$

where $\rho_{\Gamma}^{Y}(\tau^{\cdot R, Y}(c_{\Gamma, R, r_1}, \ldots)) = \tau^{M_{\Gamma, Y}}(c_{\Gamma, Y, \rho(r_1)}, \ldots)$.

Recall that wellfoundedness of a (level-1) EM blueprint is a $\Pi^1_2$ condition, stating that for every countable ordinal $\alpha$, the EM model generated by order indiscernibles of order type $\alpha$ is wellfounded. Its higher level analog is called iterability, which is a $\Pi^1_3$ condition.

**Definition 5.17.** Let $\Gamma$ be a level-3 EM blueprint. $\Gamma$ is iterable iff for any $\Pi^1_3$-wellfounded level-3 tree $Y$, $M_{\Gamma, Y}$ is a $\Pi^1_3$-iterable mouse.

**Lemma 5.18.** Assume $\Pi^1_3$-determinacy. Then $0^{\beta\#}$ is iterable.

*Proof.* Let $Y$ be any $\Pi^1_3$-wellfounded level-3 tree. Let $F \in [C]^Y$, where $C$ is a firm set of potential level-3 indiscernibles for $M_{2, \infty}$. Then $M_{0^{\beta\#}, Y}$ elementarily embeds into $M_{2, \infty}$, the map being generated by $c_{0^{\beta\#}, Y, s} \mapsto [F]_s$. Therefore, $M_{0^{\beta\#}, Y}$ is iterable. \(\square\)

**Lemma 5.19.** Assume $\Delta^1_2$-determinacy.


   (a) If $T$ is a $\Pi^1_3$-wellfounded level $\leq 2$ tree, then $N^T$ is a $\Pi^1_3$-iterable mouse and $j_N^T: N \rightarrow N^T$ is essentially an iteration map.

   (b) If $\psi$ minimally factors level $\leq 2$ trees $(T, X)$, then $\psi^X_N: N^T \rightarrow N^X$ is essentially an iteration map.

2. Suppose $\Gamma$ is an iterable level-3 EM blueprint and $Y$ is a $\Pi^1_3$-wellfounded level-3 tree. If $\psi$ minimally factors level-3 trees $(Y, R)$ and $[0]_Y = [0]_R$, then $\psi^R_{\Gamma}: M_{\Gamma, Y} \rightarrow M_{\Gamma, R}$ is essentially an iteration map.

120
Proof. 1(a). By Axiom \([\text{Unboundedness}]\) in Definition 5.14, there are cofinally many cardinal strong cutpoints in \(N\). \(j_N^\tau\) is cofinal in \(N^T\) by definition and a direct limit argument. By Dodd-Jensen, it suffices to show that for any cardinal strong cutpoint \(\xi\) of \(N\), \(j_N^\tau(N[\xi])\) is \(\Pi_3^1\)-iterable and \(j_N^\tau| (N[\xi])\) is essentially an iteration map from \(N[\xi] \to N^T| j_N^\tau(N[\xi])\). Fix such \(\xi\). Let \(g\) be \(\text{Coll}(\omega, \xi)\)-generic over \(N\). The statement “for any \(\Pi_3^1\)-wellfounded level \(\leq 2\) tree \(T'\), \(j_N^\tau(T')\) is a \(\Pi^1_3\)-iterable and \(j_N^\tau(T')\) is essentially an iteration map from \(N[\xi]\) to \(j_N^\tau(N[\xi])\)” is \(\Pi_3^1\) in a real \(z \in N[g]\) coding \((N[\xi], j_N^\tau(T')| (N[\xi]))\) finite level \(\leq 2\). This statement is true in \(N[g]\) by Level \(\leq 2\) ultrapower invariance axiom in Definition 5.14. It suffices to show \(N[g] \prec_{\Sigma^3_1} V\). But by Axiom \([\text{Unboundedness}]\) in Definition 5.14, \(N \models \text{“I am closed under the } M^\#_1\text{-operator”}\). Since \(N\) is a \(\Pi^1_3\)-iterable mouse, the \(M^\#_1\)-operators are correctly computed in \(N\). Using genericity iterations [51], \(M^\#_1\)-operators figure out \(\Sigma^3_1\)-truth. Hence, \(N[g] \prec_{\Sigma^3_1} V\).

1(b). By Theorem 4.57, there is a \(\Pi^1_2\)-wellfounded \(Q\) and \(\pi\) minimally factoring \((X, T \otimes Q)\). So \(id_{T^*} = \pi \circ \psi\). By Axiom \([\text{Unboundedness}]\) in Definition 5.14, \(j_N^{T^*} = \pi_{N^T}^T \circ \psi_N^X\), which is essentially an iteration from \(N^T\) to \(N^T \otimes Q\) by part 1(a). By Dodd-Jensen, \(\psi_N^X\) is essentially an iteration map.

2. By Theorem 4.71, there is a \(\Pi^1_3\)-wellfounded \(T\) and \(\rho\) minimally factoring \((R, Y \otimes T)\). So \(id_{T^*} = \rho \circ \psi\). By Axiom \([\text{Unboundedness}]\) in Definition 5.14 and part 1, \(j_N^{T^*} = \rho_T^Y \circ \psi_T^R\) is essentially an iteration \(M_{T^*, Y} \to M_{T^*, Y}\). By Dodd-Jensen, \(\psi_T^R\) is essentially an iteration map. \(\Box\)

We start to introduce the remarkable property of a level-3 EM blueprint. For \(r, s \in \omega^{<\omega}\), define \(r <_0 s\) iff \(r(0) <_{\text{BK}} s(0)\), \(r \leq^R s\) iff \(r(0) \leq_{\text{BK}} s(0)\). If \(\vec{r} = (r_i)_{1 \leq i \leq n}\) is a tuple of nodes in \(\omega^{<\omega}\), define \(\vec{r} <_0 s\) iff \(r_i <_0 s\) for any \(i\). Similarly define \(\vec{r} \leq s\), \(\vec{r} < s\), etc.

**Definition 5.20** (Unboundedness). A level-3 EM blueprint \(\Gamma\) is **unbounded** if for any level-3 tree \(R\), if \(\tau\) is an \(\mathcal{L}\)-Skolem term, \(\{t, r_1, \ldots, r_m\} \subseteq \text{dom}(R), \vec{r} <_0 t\), then \(\Gamma(R)\) contains the formula

\[
\tau(c_{r_1}, \ldots, c_{r_m}) \in \text{Ord} \Rightarrow \tau(c_{r_1}, \ldots, c_{r_m}) < c_t.
\]

**Lemma 5.21.** Assume \(\Pi^1_3\)-determinacy. Then \(0^{3\#}\) is unbounded.

**Proof.** Let \(C\) be a firm club of potential level-3 indiscernibles for \(M^{\omega, \infty}_2\). Let \(\eta \in D\) iff \(C \cap \eta\) has order type \(u_\omega\xi\) for some ordinal \(\xi\).

We may further assume that \(\text{dom}(R)\) is the upward closure of \(\vec{r} \cup \{t\}\) and \(R^\tau =_{\text{DEF}} R| (\text{the upward closure of } \vec{r})\) is a level-3 subtree of \(R\). The reason is because we can find level-3 trees \(S^-\), \(S\), \(\rho\) factoring \((S', R), \rho\) factoring \((S, R)\) so that \(S^-\) is a subtree of \(S\), \(\rho^\tau = \rho| S^-\), \(\text{ran}(\rho^-)\) is the upward...
closure of $\vec{r}$, $\text{ran}(\rho)$ is the upward closure of $\vec{r} \cup \{t\}$. We then work with $S$ and $\rho^{-1}(\vec{r}, t)$ instead, and finally apply the coherency of $0^3\#$.

Suppose $\Gamma(R)$ contains the formula \(\tau(\zeta_1, \ldots, \zeta_m) \in \text{Ord}\). Then for any $\vec{r} \in [C]^{R^t}$,
\[
\tau^{M_{2,\infty}(x)}(\gamma_{r_1}, \ldots, \gamma_m) < \delta^3_3.
\]
Our assumption $\vec{r} <_0 t$ ensures the existence of $\vec{r} \in [D]^{R^t}$ extending $\vec{r}$ such that
\[
\tau^{M_{2,\infty}}(\gamma_{r_1}, \ldots, \gamma_m) < \delta_t.
\]
Hence, $\Gamma(R)$ contains the formula \(\tau(c_{r_1}, \ldots, c_{r_m}) < c_t\).

\[\square\]

**Definition 5.22** (Weak remarkability). A level-3 EM blueprint $\Gamma$ is weakly remarkable iff $\Gamma$ is unbounded and for any level-3 tree $R$, if $\tau$ is an $\mathcal{L}$-Skolem term, $\vec{r} \subseteq \vec{s} \subseteq \vec{s}' \subseteq \{t\} \subseteq \text{dom}(R)$, $\vec{r} <_0 t \leq 0 \vec{s} \leq 0 \vec{s}'$, $\vec{s}'$ is an $R$-shift of $\vec{s}$, $\text{lh}(t) = 1$, then $\Gamma(R)$ contains the formula
\[
\tau(c_{r_1}, \ldots, c_{r_m}, c_{s_1}, \ldots, c_{s_n}) \leq c_t \rightarrow \tau(c_{r_1}, \ldots, c_{r_m}, c_{s_1}, \ldots, c_{s_n}) = \tau(c_{r_1}, \ldots, c_{r_m}, c_{s'_1}, \ldots, c_{s'_n}).
\]

**Lemma 5.23.** Assume $\Pi^1_3$-determinacy. Then $0^3\#$ is weakly remarkable.

**Proof.** Again, we may assume that $\text{dom}(R)$ is the upward closure of $\vec{r} \cup \vec{s} \cup \vec{s}' \cup \{t\}$.

Suppose $0^3\#(R)$ contains the formula \(\tau(c_{r_1}, \ldots, c_{r_m}) \in \text{Ord}\). We need to show that $0^3\#(R)$ contains the formula \(\tau(c_{r_1}, \ldots, c_{r_m}) = \tau(c_{r_1}, \ldots, c_{r'_m})\).

By Axiom [in Definition 5.14] we may further assume that $t$ is in the upward closure of $\vec{s}$. Let $S$ be a level-3 tree and $\rho, \rho'$ both factor $(S, R)$ such that $\text{ran}(\rho)$ is the upward closure of $\vec{r} \cup \vec{s}$, $\text{ran}(\rho')$ is the upward closure of $\vec{r} \cup \vec{s}'$. Put $\rho^{-1}(r_i, s_j, t) = (r'_i, s'_j, t) = (\rho')^{-1}(r'_i, s'_j, t')$.

Let $C$ be a firm set of potential level-3 indiscernibles for $M_{2,\infty}$. Let $C = \bigcup_{\xi < \delta^1_3} C_\xi$ be a disjoint partition of $C$ such that for any $\xi < \delta^1_3$, $\text{o.t.}(C_\xi) = u_\omega$, and for any $\xi < \eta < \delta^1_3$, any member of $C_\xi$ is smaller than any member of $C_\eta$. Let $D$ be a club in $\delta^1_3$ where $\nu \in D$ iff $\sup_{\xi < \nu} C_\xi = \nu$. As $C$ is firm, $D$ has order type $\delta^1_3$.

If $X, Y$ are subsets of ordinals, define $X \sqsubseteq Y$ iff $X \subseteq Y$ and $X = Y \cap \alpha$ for some $\alpha$. For each $0 < \xi < \delta^1_3$, let $F^\xi \in D_{\text{st}}^S$ so that $(F^\xi)^{\nu}(\text{rep}(U) \subseteq C_0, (F^\xi)^{\nu}(\text{rep}(S) \setminus \text{rep}(U)) \subseteq C_\xi$. Define $\gamma^\xi = (\gamma^\xi_x)_{x \in \text{dom}(S)} = [F^\xi]^S$. Define $\epsilon^\xi = \tau^{M_{2,\infty}}(\gamma^\xi_{r_1}, \ldots, \gamma^\xi_{r_m}).$

Hence,
\[\epsilon^\xi < \text{min}(C_1).\]
For $0 < \eta < \xi < \delta_3^1$, define $\vec{\gamma}^\eta_\xi = (\gamma^\eta_\xi)_y \in \text{dom}(R)$ where $\gamma^\eta_\xi = \gamma^\eta_\xi x$ and $\gamma^\eta_\rho(x) = \gamma^\eta_\xi$ for any $x \in \text{dom}(S)$. By Lemma 4.46, $\gamma^\eta_\xi \in [D]^{R^1}$. Suppose towards a contradiction.

Case 1: $0_3^\#(R)$ contains the formula “$\tau(c_{r_1}, \ldots, c_{s_1}) > \tau(c_{r_1}, \ldots, c_{s_1})$”.

Then $\vec{\gamma}^\xi_\eta$ witnesses that $\epsilon_\eta > \epsilon_\xi$ whenever $0 < \eta < \xi < \delta_3^1$. This is an infinite descending chain of ordinals.

Case 2: $0_3^\#(R)$ contains the formula “$\tau(c_{r_1}, \ldots, c_{s_1}) < \tau(c_{r_1}, \ldots, c_{s_1})$”.

Then $\epsilon_\eta < \epsilon_\xi$ whenever $0 < \eta < \xi < \delta_3^1$, contradicting to $\epsilon_\xi < \min(C_1)$.

If $R$ is a level-3 tree, $t \in \text{dom}(R)$, $\text{lh}(t) = 1$, let

$$R \upharpoonright t = R \upharpoonright \{r \in \text{dom}(R) : r <_0 t\}.$$

**Lemma 5.24.** Suppose $\Gamma$ is a weakly remarkable level-3 EM blueprint. Suppose $R$ is a level-3 tree, $t \in \text{dom}(R)$, $\text{lh}(t) = 1$.

1. If $\tau$ is an $L$-Skolem term, $\vec{r} \cup \vec{s} \cup \vec{s'} \upharpoonright \{t\} \subseteq \text{dom}(R)$, $\vec{r} <_0 t \leq \vec{s} <_0 \vec{s'}$, $\vec{s'}$ is an $R$-shift of $\vec{s}$, then $\Gamma(R)$ contains the formula

$$\tau(c_{r_1}, \ldots, c_{r_m}, c_{s_1}, \ldots, c_{s_n}) < c_t \rightarrow \tau(c_{r_1}, \ldots, c_{r_m}, c_{s_1}, \ldots, c_{s_n}) = \tau(c_{r_1}, \ldots, c_{r_m}, c_{s'_1}, \ldots, c_{s'_n}).$$

2. $\Gamma(R)$ contains the scheme “$K|c_t < V$”. In particular, $\Gamma(R)$ contains the formula “$c_t$ is inaccessible and there are cofinally many cardinal strong cutpoints below $c_t$.”

**Proof.** 1. Assume without loss of generality that $\text{dom}(R)$ is the upward closure of $\vec{r} \cup \vec{s} \cup \vec{s'} \upharpoonright \{t\}$. Suppose $\Gamma(R)$ contains the formula “$\tau(c_{r_1}, \ldots, c_{s_1}, \ldots) < c_t$”. Expand $R$ to the level-3 tree $S$ where $\text{dom}(S)$ is the upward closure of $\text{dom}(R) \cup \{s'_i : 1 \leq i \leq n\}$, each $s'_i \notin \text{dom}(R)$, $\vec{s'}$ is an $R$-shift of $\vec{s}$, $\vec{s} <_0 \vec{s'}$. By coherency and weak remarkability, $\Gamma(S)$ contains the formula $\tau(c_{r_1}, \ldots, c_{s_1}, \ldots) = \tau(c_{r_1}, \ldots, c_{s'_1}, \ldots)$. But $\vec{r} <_0 \vec{s} <_0 \vec{s'}$ is a shift of $\vec{r} <_0 \vec{s'} <_0 \vec{s'}$. By indiscernability, $\Gamma(S)$ contains the formula $\tau(c_{r_1}, \ldots, c_{s_1}, \ldots) = \tau(c_{r_1}, \ldots, c_{s'_1}, \ldots)$. Hence, $\Gamma(S)$ contains the formula $\tau(c_{r_1}, \ldots, c_{s_1}, \ldots) = \tau(c_{r_1}, \ldots, c_{s'_1}, \ldots)$.

2. Put $\vec{N} = \mathcal{M}_{T,R}$. By coherency of $\Gamma$, we may assume that $A = \{s \in \text{dom}(R) : s <_0 t\}$ has $<_{BK}$-limit order type. By Tarski’s criterion, we need to show that if $w = \tau^\vec{N}(z_1, \ldots, z_k) \in \text{Ord}$, $z_1, \ldots, z_k < c_t^\vec{N}$, then $w < c_t^\vec{N}$. To save notations, let $k = 1$, $z_1 = \sigma^\vec{N}(c_{r_1}^N, \ldots, c_{s_1}^N, \ldots) < c_t^N$, $\vec{r} <_0 t \leq \vec{s}$. 

123
Pick $t^*$ of length 1 such that $\vec{r} <_0 t^* <_0 t$. Build a level-3 tree $S$ that extends $R$ in which there are nodes $t^*, \vec{s} \in \text{dom}(S)$ such that $\vec{r} <_0 (t^*) \sim \vec{s} <_0 r^*$ and $(t^*) \sim \vec{s}$ is an $S$-shift of $(t) \sim \vec{s}$. Put $\mathcal{P} = \mathcal{M}_{\Gamma, S}$. By weakly remarkability,

$$
\sigma^\mathcal{P}(c_{r_1}^\mathcal{P}, \ldots, c_{s_1}^\mathcal{P}) = \sigma^\mathcal{P}(c_{r_1}^\mathcal{P}, \ldots, c_{s_1}^\mathcal{P}, \ldots).
$$

By unboundedness of $\Gamma$,

$$
\tau^\mathcal{P}(\sigma^\mathcal{P}(c_{r_1}^\mathcal{P}, \ldots, c_{s_1}^\mathcal{P}, \ldots)) < c_\mathcal{P}^\mathcal{P}.
$$

By coherency of $\Gamma$, $w < c_\mathcal{P}^\mathcal{P}$. By Axiom 5 in Definition 5.14, $c_\mathcal{P}^\mathcal{P} \in \mathcal{C}$.

A level-3 tree $R$ is said to be universal above $t$ iff $t \in \text{dom}(R)$, $\text{lh}(t) = 1$, and for any level-3 tree $S$, if $S \upharpoonright t$ is isomorphic to $R \upharpoonright t$ via $\pi$ and $\text{dom}(S) \setminus \text{dom}(S \upharpoonright t)$ is finite, then there is a map $\rho$ factoring $(S, R)$ that extends $\pi$. Clearly, for any $R$, there is $(R', t)$ such that $R' \upharpoonright t$ is isomorphic to $R$ and $R'$ is universal above $t$. If $R$ is $\Pi^1_3$-wellfounded, we may further demand that $R'$ is $\Pi^1_3$-wellfounded.

**Lemma 5.25.** Suppose $\Gamma$ is a weakly remarkable level-3 EM blueprint, $R$ is universal above $t$, $R'$ is universal above $t'$, $R \upharpoonright t$ is isomorphic to $R' \upharpoonright t'$. Then

$$(K|_{c_\mathcal{P}^\mathcal{P}})^{\mathcal{M}_{\Gamma, R}} \cong (K|_{c_\mathcal{P}^\mathcal{P}})^{\mathcal{M}_{\Gamma, R'}}.$$

**Proof.** To begin with, we build an isomorphism $\psi : (c_\mathcal{P}^\mathcal{P})^{\mathcal{M}_{\Gamma, R}} \to (c_{\mathcal{P}'})^{\mathcal{M}_{\Gamma, R'}}$ which preserves membership relationships in the respective EM models. Given $a \in \mathcal{M}_{\Gamma, R}$ such that $\mathcal{M}_{\Gamma, R} \models a < c_\mathcal{P}$, find a Skolem term $\tau$ and nodes $\vec{r}, \vec{s}$ such that $\vec{r} <^R t \subseteq^R \vec{s}$ and

$$a = (\tau(c_{r_1}, \ldots, c_{s_1}, \ldots))^{\mathcal{M}_{\Gamma, R}}.$$

Let $S$ be a level-3 tree and $\rho$ factor $(S, R)$ such that $\text{ran}(S)$ is the upward closure of $\text{dom}(R \upharpoonright t) \cup \vec{s} \cup \{t\}$. By universality, pick $\rho'$ factoring $(S, R')$ which extends $\pi$. By coherency of $\Gamma$, $(\tau(c_{\pi(r_1)}, \ldots, c_{\rho'\circ \rho^{-1}(s_1)}, \ldots))^{\mathcal{M}_{\Gamma, R'}} <^{\mathcal{M}_{\Gamma, R'}} c_{\mathcal{P}'}.$ Define

$$\psi(a) = (\tau(c_{\pi(r_1)}, \ldots, c_{\rho'\circ \rho^{-1}(s_1)}, \ldots))^{\mathcal{M}_{\Gamma, R'}}.$$

$\psi$ is well-defined and preserves membership. For this, we firstly show that $\psi(a)$ does not depend on the choice of $\rho'$. Suppose $\rho''$ is another candidate for $\rho'$. Then $\rho'' \circ \rho^{-1}(\vec{s})$ is an $R'$-shift of $\rho' \circ \rho^{-1}(\vec{s})$. By Lemma 5.24, $\mathcal{M}_{\Gamma, R'} \models \tau(c_{\pi(r_1)}, \ldots, c_{\rho''\circ \rho^{-1}(s_1)}, \ldots) = \tau(c_{\pi(r_1)}, \ldots, c_{\rho'\circ \rho^{-1}(s_1)}, \ldots)$. Secondly, the reason why $\psi(a)$ does not depend on the choice of $\tau$ and $\vec{r}, \vec{s}$ is because of coherency of $\Gamma$. In the same spirit, we can show that $\psi$ preserves membership. A completely symmetrical argument gives $\psi' : (c_{\mathcal{P}'})^{\mathcal{M}_{\Gamma, R'}} \to (c_\mathcal{P}^\mathcal{P})^{\mathcal{M}_{\Gamma, R}}$. 124
By Lemma 5.16, \( \psi \circ \psi' \) and \( \psi' \circ \psi \) are both identity functions. So \( \psi \) is an isomorphism between \( (c_\ell)^{\mathcal{M}_{r,r}} \) and \( (c_\ell)'^{\mathcal{M}_{r,r}} \).

\( \mathcal{M}_{r,r} \) is a model of \( V = K \). Working in \( \mathcal{M}_{r,r} \), \( K|c_\ell \) has a canonical wellordering of order type \( \omega c_\ell \), and similarly for \( \mathcal{M}_{r,r}' \). \( \psi \) extends to \( \psi^* \), acting on \( (K|c_\ell)^{\mathcal{M}_{r,r}} \) according to these canonical wellorderings. Using the same argument as before, \( \psi^* \) is an isomorphism from \( (K|c_\ell)^{\mathcal{M}_{r,r}} \) to \( (K|c_\ell)'^{\mathcal{M}_{r,r}} \).

\( \square \)

A level-3 tree \( R \) is universal based on \( Y \) iff there is \( t \in \text{dom}(R) \) such that \( \text{lh}(t) = 1 \), \( R \) is universal above \( t \) and \( R \upharpoonright t \) is isomorphic to \( Y \). Suppose \( \Gamma \) is a weakly remarkable level-3 EM blueprint. For a level-3 tree \( Y \), if \( R \) is universal based on \( Y \), \( t \in \text{dom}(R) \), \( \text{lh}(t) = 1 \), \( R \upharpoonright t \) is isomorphic to \( Y \), put

\[ \mathcal{M}_{r,Y}^* = (K|c_\ell)^{\mathcal{M}_{r,r}}. \]

\( \mathcal{M}_{r,Y}^* \) is well-defined up to an isomorphism. Its wellfounded part is transitivized. By Lemma 5.24, there are cofinally many cardinal strong cutpoints in \( \mathcal{M}_{r,Y}^* \). Similarly, for a level \( \leq 2 \) tree \( T \), define

\[ \mathcal{M}_{s,Y}^{*T} = (K|c_\ell)^{\mathcal{M}_{s,s}^T}. \]

Hence, \( \mathcal{M}_{s,Y}^{*T} = (\mathcal{M}_{r,Y}^*)^T \). If \( \rho \) factors \( (Y,Y') \), \( R' \) is universal above \( R \), then \( \rho_Y^{*Y'} = \rho_Y^R \upharpoonright \mathcal{M}_{r,Y}^* \). If \( \rho \) factors \( (Y,Y',T) \), \( R' \) is universal above \( R \), then \( \rho_T^{*Y',T} = \rho_T^{Y',T} \upharpoonright \mathcal{M}_{r,Y}^* \).

\( \Pi_3 \)-iterable mouse \( P \) is full iff for any strong cutpoint \( \eta \) of \( P \), for any \( \Pi_3 \)-iterable mouse \( Q \) extending \( P|\eta \) which is sound and projects to \( \eta \), \( Q \leq P \).

**Lemma 5.26.** Assume \( \Delta_2 \)-determinacy. Suppose \( \Gamma \) is an iterable, weakly remarkable level-3 EM blueprint.

1. Suppose \( Y,Y' \) are \( \Pi_3 \)-wellfounded level-3 trees. Then \( [0]_Y = [0]_{Y'} \) iff \( \mathcal{M}_{\Gamma,Y} \sim_{D\mathcal{J}} \mathcal{M}_{\Gamma,Y'} ; [0]_Y < [0]_{Y'} \) iff \( \mathcal{M}_{\Gamma,Y} <_{D\mathcal{J}} \mathcal{M}_{\Gamma,Y'} \).
2. Suppose \( Y \) is a \( \Pi_3 \)-wellfounded level-3 tree. Then \( \mathcal{M}_{\Gamma,Y}^* \) is full.
3. Suppose \( Y,Y' \) are \( \Pi_3 \)-wellfounded level-3 trees. Then \( [0]_Y = [0]_{Y'} \) iff \( \mathcal{M}_{\Gamma,Y} \sim_{D\mathcal{J}} \mathcal{M}_{\Gamma,Y'} ; [0]_Y < [0]_{Y'} \) iff \( \mathcal{M}_{\Gamma,Y}^* <_{D\mathcal{J}} \mathcal{M}_{\Gamma,Y'}^* \).

**Proof.** 1. If \( [0]_Y \leq [0]_{Y'} \), by Theorem 4.71, there exist a \( \Pi_3 \)-wellfounded \( Z \) and \( \rho \) minimally factoring \( (Y,Z) \), \( \rho' \) minimally factoring \( (Y',Z) \) so that \( [0]_Y = [0]_Z \). By Lemma 5.19, \( \mathcal{M}_{\Gamma,Y} \leq_{D\mathcal{J}} \mathcal{M}_{\Gamma,Z} \sim_{D\mathcal{J}} \mathcal{M}_{\Gamma,Y'} \).

If \( [0]_Y < [0]_{Y'} \), we further obtain \( t \in \text{dom}(R) \) so that \( \text{lh}(t) = 1 \) and \( [0]_Y = [t]_Z \). By unboundedness of \( \Gamma \), \( \text{ran}(\rho_T^{Y',T}) \subseteq \mathcal{M}_{\Gamma,Z}[c_{\Gamma,Z,t}] \). Hence, \( \rho_T^{Y',T} \) is \( \Sigma_1 \)-elementary from \( \mathcal{M}_{\Gamma,Y} \) into \( \mathcal{M}_{\Gamma,Z}[c_{\Gamma,Z,t}] \). Hence \( \mathcal{M}_{\Gamma,Y} <_{D\mathcal{J}} \mathcal{M}_{\Gamma,Z} \).

125
2. Recall that there are cofinally many cardinal strong cutpoints in \( \mathcal{M}_{\Gamma,Y}^* \). Suppose \( \eta \) is a strong cutpoint of \( \mathcal{M}_{\Gamma,Y}^* \) and \( \mathcal{M}_{\Gamma,Y}^* \big| \eta \subset \mathcal{P} \), \( \mathcal{P} \) is a sound \( \Pi_3^1 \)-iterable mouse, \( \rho_\mathcal{P}(\mathcal{P}) \leq \eta \). Let \( Y' \) be a \( \Pi_3^1 \)-wellfounded level-3 tree such that \( \| \mathcal{P} \|_{DJ} < \emptyset \) and \( Y' \) is universal based on \( Y \). By part 2, \( \mathcal{P} \) is an order preserving function \( \mathcal{M}_{\Gamma,Y}^* \) and \( \eta \) is a strong cutpoint of \( \mathcal{M}_{\Gamma,Y}^* \), the comparison between \( \mathcal{P} \) and \( \mathcal{M}_{\Gamma,Y}^* \) is above \( \eta \). It follows that \( \mathcal{P} \subset \mathcal{M}_{\Gamma,Y}^* \). Hence \( \mathcal{P} \subset \mathcal{M}_{\Gamma,Y}^* \).

3. By parts 1-2 and remarkability of \( \Gamma \). \( \Box \)

Assume \( \Delta^1_2 \)-determinacy. Suppose \( \Gamma \) is an iterable, weakly remarkable level-3 EM blueprint. Suppose \( Y \) is a \( \Pi_3^1 \)-wellfounded level-3 tree.

For \( s \in \text{dom}(Y) \), let

\[
c_{\Gamma,Y, s} = c_{\Gamma,Y}^s \quad \text{and} \quad c_{\Gamma,Y, s, \infty} = \pi_{\mathcal{M}_{\Gamma,Y}}(c_{\Gamma,Y, s}).
\]

In fact, \( c_{\Gamma,Y, s, \infty} \) depends only on \( ([s]_Y, Y_{\text{tree}}(s)) \), shown as follows. Suppose \( Y' \) is another \( \Pi_3^1 \)-wellfounded level-3 tree and \( ([s]_Y, Y_{\text{tree}}(s)) = ([s']_Y, Y'_{\text{tree}}(s')) \). By Lemma 4.47, \( Y'[s] = Y'[s'] \). By Theorem 4.71, we can find \( \Pi_3^1 \)-wellfounded level-3 trees \( R, R' \) which are universal based on \( Y, Y' \) respectively, a \( \Pi_3^1 \)-wellfounded \( Z \) and \( \rho \) minimally factoring \( (R, Z) \), \( \rho' \) minimally factoring \( (R', Z) \). In particular, \( \rho(s) = \rho'(s') \). By Lemma 5.19, \( \rho^* : \mathcal{M}_{\Gamma,R} \rightarrow \mathcal{M}_{\Gamma,Z} \) is essentially an iteration map, sending \( c_{\Gamma,Y, s} \) to \( c_{\Gamma,Z, \rho(s)} \), and similarly on the \( \rho' \)-side. Hence \( c_{\Gamma,Y, s, \infty} = c_{\Gamma,Y', s', \infty} \). We can safely define

\[
c_{\Gamma,Q, \gamma} = c_{\Gamma,Y, s, \infty}
\]

for \( Y_{\text{tree}}(s) = Q \) and \( \gamma = [s]_Y \).

If \( (Q, (d, q, P)) = (Q, (d_i, q_i, P_i)_{1 \leq i \leq k}) \) is a potential partial level \( \leq 2 \) tower, let \( F \in B(Q, (d, q, P)) \) iff \( F : [\omega_1]^Q \rightarrow B \) is an order preserving function and

1. if \( (Q, (d, q, P)) \) is of continuous type, then the signature of \( F \) is \( (d_i, q_i)_{1 \leq i \leq k} \), \( F \) is essentially continuous;

2. if \( (Q, (d, q, P)) \) is of discontinuous type, then the signature of \( F \) is \( (d_i, q_i)_{1 \leq i \leq k} \), \( F \) is essentially discontinuous, \( F \) has uniform cofinality \( \text{ucf}(Q, (d, q, P)) \).

Let \( \gamma \in [B(Q, (d, q, P))]_\mathcal{Q} \) iff \( \gamma = [F]_{\mathcal{Q}} \) for some \( F \in B(Q, (d, q, P)) \). \( \gamma \) is said to respect \( (Q, (d, q, P)) \) iff \( \gamma \in (\delta^*_3)(Q, (d, q, P)) \). \( \gamma \) is said to respect \( Q \) if \( \gamma \) respects...
some potential partial level \( \leq 2 \) tower \((Q, (d', q', P'))\). By Lemma 4.79, \( \gamma \) respects \( Q \) iff there is a \( \Pi_3^1 \)-wellfounded \( Y \) and \( s \) such that \( Y_{\text{tree}}(s) = Q \) and \( \gamma = [s]_Y \). Hence, \( c_{\Gamma, Q, \gamma} \) is defined whenever \( \gamma \) respects \( Q \) and the map \( \gamma \mapsto c_{\Gamma, Q, \gamma} \) is order preserving. Define

\[
c_{\Gamma, \gamma} = c_{\Gamma, Q^0, \gamma},
\]

\( c_{\Gamma, \gamma} \) is defined whenever \( \gamma < \delta_3^1 \) is a limit ordinal. Remarkability will ensure that the map \( \gamma \mapsto c_{\Gamma, \gamma} \) is continuous. Assuming \( \Delta_3^1 \)-determinacy, define

\[
c_{(3)}^{(3)}_{Q, \gamma} = c_{(0)^\#, Q, \gamma}, \quad c_{\gamma}^{(3)} = c_{(0)^\#}, \quad I^{(3)} = \{ c_{(3)}^{(3)}_{Q, \gamma} : \gamma \text{ respects } Q \}.
\]

\( I^{(3)} \) is the higher analog of Silver indiscernibles for \( L \).

**Lemma 5.27.** Assume \( \Pi_3^1 \)-determinacy. Then there is a club \( C \subseteq \delta_3^1 \) such that \( C \in L[T_3, \emptyset^\#] \) and for any potential partial level \( \leq 2 \) tree \((Q, (d, q, P))\), for any \( \gamma \in [C](Q, (d, q, P))^\uparrow \),

\[
\gamma = c_{(3)}^{(3)}_{Q, \gamma}.
\]

**Proof.** Let \( D \) be a firm set of potential level-3 indiscernibles for \( M_{2\infty} \) and let \( \eta \in C \) iff \( \eta \in D \) and \( D \cap \eta \) has order type \( \eta \). \( C \) works for the lemma. \( \square \)

Recall Definition 4.23 for the definition of \( R^d \). An ordinal \( \alpha < \omega_1 \) is \( \omega_1 \)-represented by \( T \) iff \((1, (0)) \in \text{dom}(T) \) and \([1, (0)]_T = \alpha \). \( \alpha < \omega_2 \) is \( \omega_2 \)-represented by \( T \) iff \((2, (0))) \in \text{dom}(T) \) and \([2, (0)]_T = \alpha \).

**Definition 5.28** (Remarkability). A weakly remarkable level-3 EM blueprint \( \Gamma \) is remarkable iff

1. \( \Gamma(R^0) \) contains the axiom “\( c_{((0))} \) is not measurable”.
2. \( \Gamma(R^1) \) contains the following axiom: if \( \xi \) is a cardinal and strong cut-point, \( c = c_{((0))}, b = (\phi^{Q^1}_{1,c})^{-1}(c) \), then \( V^{\text{Coll}(\omega, \xi)} \) satisfies the following:

   (a) If \( \alpha \) is \( \omega_1 \)-represented by both \( T \) and \( T' \), then \((j^{T}_{1,c}(K|\xi), (j^{Q^1, T}_{1,c}(b))) \sim_{D J} (j^{T'}_{1,c}(K|\xi), (j^{Q^1, T'}_{1,c}(b)))\). Here \((j^{U}_{1,c})^K \) stands for the direct limit of \((j^{Z, Z'}_{1,c})^K \) for \( Z, Z' \) finite subtrees of \( U \), \( Z \) a finite subtree of \( Z' \), and \((j^{Q^1, U}_{1,c})^K \) stands for the tail of the direct limit map from \((j^{Q^1}_{1,c})^K(K) \) to \((j^{U}_{1,c})^K(K) \).
(b) Let $F(\alpha) = \pi_{(j_{T,e})^\kappa(K|\xi),\infty}((j_{1,e}^{Q_{T}})K(b))$ for $\alpha$ represented by $T$.

Then $\sup_{\alpha<\omega_1} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

3. $\Gamma(R^2)$ contains the following axiom: if $\xi$ is a cardinal and strong cut-point, $e \in \{0, 1\}$, $c = c(0)$, $b = (\phi_{1,e}^{Q_{2e}})^{-1}(c)$, then $V^{\infty(\omega, \xi)}$ satisfies the following:

(a) If $\alpha$ is $w_2$-represented by both $T$ and $T'$, then $((j_{T,e}^{Q_{2e}})^K(K|\xi), (j_{2,e}^{Q_{2e},T}(b))) \sim_{D J} ((j_{T,e}^{Q_{2e}})^K(K|\xi), (j_{2,e}^{Q_{2e},T'}(b)))$. Here $(j_{U,e}^{Q_{2e}})^K$ stands for the direct limit of $(j_{Z,Z'}^{Q_{2e}})^K$ for $Z, Z'$ finite subtrees of $U$, $Z$ a finite subtree of $Z'$, and $(j_{2,e}^{Q_{2e},U})^K$ stands for the tail of the direct limit map from $(j_{2,e}^{Q_{2e}})^K(K)$ to $(j_{U,e}^{Q_{2e}})^K(K)$.

(b) Let $F(\alpha) = \pi_{(j_{T,e})^\kappa(K|\xi),\infty}((j_{2,e}^{Q_{2e},T}(b)))$ for $\alpha$ represented by $T$.

Then $\sup_{\alpha<\omega_2} F(\alpha) = \pi_{K|\xi,\infty}(c)$.

In the next lemma, we denote $\mathbf{y}^1 = (((0), -1), Q^1, ((1, (0), 0))) \in \text{desc}(R^1)$, $\mathbf{B}^1 = (\mathbf{y}^1, \text{id}_{Q_{2e},*}) \in \text{desc}(R^1, Q^1, Q^0)$, $\mathbf{y}^{2e} = (((0), -1), Q^{2e}, ((2, ((0)), ((0)))))) \in \text{desc}(R^2)$, $\mathbf{B}^{2e} = (\mathbf{y}^{2e}, \text{id}_{Q_{2e},*}) \in \text{desc}(R^2, Q^{2e}, Q^0)$ for $e \in \{1, 2\}$. Note that if $\Gamma$ is a level-3 EM blueprint, $d \in \{0, 1\}$, then $\Gamma(R^1)$ contains the axiom

$$c(0) = c_{\mathbf{y}^1} = c_{\mathbf{B}^1}^{Q^1_{R^1}}$$

and $\Gamma(R^2)$ contains the axiom

$$c(0) = c_{\mathbf{y}^{20}} = c_{\mathbf{y}^{21}} = c_{\mathbf{B}^{20}} = c_{\mathbf{B}^{21}}^{Q^2_{R^2}}$$

**Lemma 5.29.** Suppose $\Gamma$ is a level-3 EM blueprint. Suppose $d \in \{1, 2\}$, $T$ is a finite level $\leq 2$ tree.

1. $M_{T,R,d,*} = M_{T,R^{d\otimes T}, \phi_{T,R^{d,*}}^T} = (\text{id}_{R^{d\otimes T}})^{R^{d, T}}$.

2. If $Q^1$ is a subtree of $T$, then $j_{T,R^1}^{Q^1,T} \circ (\phi_{T,R^1,*}^{Q^1})^{-1}(c_{T,R^1,(0)}) = c_{T,R^{d\otimes T}, \mathbf{B}^1}$.

3. For $e \in \{1, 2\}$, if $Q^{2e}$ is a subtree of $T$, then $j_{T,R^2}^{Q^{2e},T} \circ (\phi_{T,R^2,*}^{Q^{2e}})^{-1}(c_{T,R^2,(0)}) = c_{T,R^{d\otimes T}, \mathbf{B}^{2e}}$.

**Proof.**

1. Put $Y = R^d$, $c = c_{T,Y,(0)}$, $R = Y \otimes T$, $p = \text{id}_R$ factoring $(R,Y,T)$, $\psi = \text{id}_{Y,*}$ factoring $(Y,R)$. We only prove the typical case when $d = 2$. Put $y = (((0)), Q^0, (\bar{e}, x, \bar{W})) \in \text{desc}(Y)$. We have to show that

$$\text{ran}(\phi_{T,Y,*}^T) = \text{ran}(\beta_{T,Y}^T)$$

128
Lemma 5.30. Assume $\Delta^1_3$-determinacy. Suppose $\Gamma$ is an iterable, weakly remarkable level-3 EM blueprint. The following are equivalent:

1. $\Gamma$ is remarkable.
2. The map $\gamma \mapsto c_{\Gamma, \gamma}$ is continuous.
3. There exist $\gamma_0, \gamma_1, \gamma_2$ such that for $d \in \{0, 1, 2\}$, $\mathrm{cf}^{\mathbb{R}_{\alpha \mapsto \gamma}}_{[\alpha]}(\gamma_d) = u_d$ and $c_{\Gamma, \gamma_d} = \{c_{\Gamma, \beta} : \beta < \gamma_d\}$.

In particular, if $\Pi^1_3$-determinacy holds, then $0^{3\#}$ is remarkable, and hence the map $\gamma \mapsto c_{\gamma}^{(3)}$ is continuous.

Proof. 1 $\Rightarrow$ 2: Suppose $\gamma < \delta^1_3$ is a limit of limit ordinals. By Lemma 4.79 there exists a $\Pi^1_3$-wellfounded tree $Y$ such that $\gamma = [((0))]_Y$.

Case 1: $\mathrm{cf}^{\mathbb{R}_{\alpha \mapsto \gamma}}_{[\alpha]}(\gamma) = \omega$.

Then $R^0$ is a subtree of $Y$ and $A = \{a < \omega^\omega : a < B_K ((0)), (a) \in \mathrm{dom}(Y)\}$ has limit order type. By indiscernibility, $c^{*\mathbb{R}_{\alpha \mapsto \gamma}}_{[\alpha]}(\gamma) = \sup_{a \in A} c_{\Gamma, Y, (a)}^{*\mathbb{R}_{\alpha \mapsto \gamma}}$. By weak remarkable, for $a \in A$, $\mathcal{M}_{\Gamma, Y}^{*\mathbb{R}_{\alpha \mapsto \gamma}}(\gamma, (a)) = \mathcal{M}_{\Gamma, Y, (a)}^{*\mathbb{R}_{\alpha \mapsto \gamma}}$. By remarkable, $\pi_{\mathcal{M}_{\Gamma, Y, \infty}}$ is continuous at $c_{\Gamma, Y, (a)}^{*\mathbb{R}_{\alpha \mapsto \gamma}}$. It follows that $c_{\Gamma, \gamma} = \sup_{\beta < \gamma} c_{\Gamma, \beta}$.

Case 2: $\mathrm{cf}^{\mathbb{R}_{\alpha \mapsto \gamma}}_{[\alpha]}(\gamma) = u_d, d \in \{1, 2\}$.

Then $R^d$ is a subtree of $Y$. Let $F(\alpha) = [\mathbb{B}^1]_{Y \otimes T}$ for $\alpha = t_\omega$ represented by $T$. Then $\sup_{\alpha < u_d} F(\alpha) = \gamma$. By remarkable, Lemma 5.29 and absoluteness, $\sup_{\alpha < u_d} c_{\Gamma, F(\alpha)} = c_{\gamma}$.

2 $\Rightarrow$ 3: Trivial.

3 $\Rightarrow$ 1: By Lemma 4.79 there exist $\Pi^1_3$-wellfounded trees $Y^d$ for $d \in \{0, 1, 2\}$ such that $\gamma_d = [((0))]_Y$. Reverse the argument in 1 $\Rightarrow$ 2. \qed
Definition 5.31 (Level ≤ 2 correctness). A level-3 EM blueprint \( \Gamma \) is level \( \leq 2 \) correct iff for each finite level-3 tree \( Y, \) for each \( y \in \text{dom}(Y), \) putting \( X = Y_{\text{tree}}(y), \) \( \Gamma(Y) \) contains the following axiom:

\[ c = c_y, \quad b = (\phi^Y_{X,c})^{-1}(c), \quad \xi > c \text{ is a cardinal and strong cutpoint, then } \]

\[ V^\text{Coll}(\omega,\xi) \text{ satisfies the following:} \]

1. If \( \bar{\alpha} = (\bar{\alpha}_x)_{(x \in \text{dom}(X)} \) is represented by both \( T \) and \( T' \), then \((j^T_{X,c})^{K}(K|\xi), (j^{X,T}_{X,c})^{K}(b)) \sim_{D} (j^{T'}_{X,c})^{K}(K|\xi), (j^{X,T'}_{X,c})^{K}(b)). \) Here \((j^{X}_{X,c})^{K}\) stands for the direct limit of \((j^{Z,Z'}_{X,c})^{K}\) for \( Z, Z' \) finite subtrees of \( U, Z \) a finite subtree of \( Z' \), and \((j^{X,T}_{X,c})^{K}\) stands for the tail of the direct limit map from \((j^{X}_{X,c})^{K}\) to \((j^{Y}_{Y,c})^{K}\).

2. Let \( F(\bar{\alpha}) = \pi(j^{X}_{X,c})^{K}(K|\xi), (j^{X,T}_{X,c})^{K}(b)) \) for \( \bar{\alpha} \) represented by \( T \). Then \( [F]_{\mu,x} = \pi_{K,\xi}(c) \).

Lemma 5.32. Suppose \( \Gamma \) is a level-3 EM blueprint, \( Y \) is a finite level-3 tree, \( T \) is a finite level \( \leq 2 \) tree, \( y \in \text{dom}(Y), \) \( y = (y, X, (e, x, W)) \in \text{desc}(Y), \) \( B = (y, \text{id}_{X,e}) \in \text{desc}(Y, X, Q^0). \) Then

1. \( \mathcal{M}^T_{\Gamma,Y,y} = \mathcal{M}_{\Gamma,Y \otimes_2 T}, \phi^T_{\Gamma,Y,y} = (\text{id}_{Y \otimes_y T})_{Y,T}, \) where \( \text{id}_{Y \otimes_y T} \) factors \( (Y \otimes_y T, Y, T). \)

2. If \( X \) is a subtree of \( T, \) then \( j^{X,T}_{\Gamma,Y} \circ (\phi^X_{Y \otimes_y T})^{-1}(c_{T,Y \otimes_y X,B}) = c_{T,Y \otimes_y X,B}. \)

Proof. 1. Put \( c = c_{T,Y \otimes_y T}, R = Y \otimes_2 T, \psi = \text{id}_{X,e} \) factoring \( (Y, R), \)

\( Y[y] = (x, (e, x, W)). \) We have to show that \( \text{ran}(\phi^T_{\Gamma,Y,y}) = \text{ran}(\rho^T_{\Gamma,Y}). \)

The \( \subseteq \) direction: If \( a \in \mathcal{M}_{\Gamma,Y}, \) then \( j^T_{\Gamma,Y}(a) = \rho^T_{\Gamma,Y} \circ \psi^R_{\Gamma}(a). \) If \( Q \) is finite, \( \pi \) factors \( (X, T \otimes Q), \) then \( \pi' = \text{DEF} \text{id}_{T \otimes Q_x} \circ \pi \) factors \( (X, (T \otimes Q) \otimes X) \) and \( (y, \pi') \in \text{dom}(Y \otimes_2 (T \otimes Q)). \) \( \psi_{X,T,Q}(y, \pi') \) is of the form \( (B, \tau) \) where \( B = (y, \varphi) \in \text{dom}(Y \otimes_y T). \) Hence,

\( (\pi \otimes Q(c_y))^{\mathcal{M}_{\Gamma,Y}} = c_{T,Y,(y, \pi')} = \rho^T_{\Gamma}(c_{T,R,(B, \tau)}). \)

The \( \supseteq \) direction: If \( B \in \text{dom}(Y \otimes Q^0), \) then \( c^T_{\Gamma,Y,B} = j^T_{\Gamma,Y}(c_{T,Y,\psi^{-1}(B)}). \) If \( B = (y, \pi) \in \text{dom}(R), \) then \( c^T_{\Gamma,Y,B} \in \text{ran}(\phi^T_{\Gamma,Y,y}) \) by definition.

2. Set \( X = T \) in part 1. \( \blacksquare \)

It is straightforward to compute that if \( Y, y, B \) are as in the assumption of Lemma 5.32, then

130
1. if \( \bar{\alpha} = (d_{\alpha}(x))_{(d,x) \in \text{dom}(X)} \) is represented by both \( T \) and \( T' \), then \([B]_Y \otimes_T = [B]_T \otimes_{T'}
\)

2. letting \( G(\bar{\alpha}) = [B]_Y \otimes_T \) for \( \bar{\alpha} \) represented by \( T \), then \([y]_Y = [G]_\mu \n\)

From Lemmas 5.32.5.27, and absoluteness, we conclude:

**Lemma 5.33.** Assume \( \Delta^1_3 \)-determinacy. Suppose \( \Gamma \) is an iterable level-3 EM blueprint. Then the following are equivalent.

1. \( \Gamma \) is level \( \leq 2 \) correct.

2. For any potential partial level \( \leq 2 \) tower \( (X, (e, X, W)) \) of continuous type, if \( F \in (\delta^1_3)(X, (e, x, W)) \), then

\[
c_T, X, [F]_\mu, X = [\bar{\alpha} \mapsto c_T, F(\bar{\alpha})]_\mu, X.
\]

3. For any potential partial level \( \leq 2 \) tower \( (X, (e, x, W)) \) of continuous type, there exists \( F \in (\delta^1_3)(X, (e, X, W)) \) satisfying

\[
c_T, X, [F]_\mu, X = [\bar{\alpha} \mapsto c_T, F(\bar{\alpha})]_\mu, X.
\]

In particular, if \( \Pi^1_3 \)-determinacy holds, then \( 0^{\#} \) is level \( \leq 2 \) correct, and hence, if \( F \in (\delta^1_3)(X, (e, X, W)) \), then

\[
c_X, [F]_\mu, X = [\bar{\alpha} \mapsto c_X, F(\bar{\alpha})]_\mu, X.
\]

**Theorem 5.34.** Assume \( \Pi^1_3 \)-determinacy. Then \( 0^{\#} \) is the unique iterable, remarkable, level \( \leq 2 \) correct level-3 EM blueprint.

**Proof.** It remains to show uniqueness. Suppose \( \Gamma, \Gamma' \) are both iterable remarkable level-3 EM blueprints. We carry out a “comparison” between \( \Gamma \) and \( \Gamma' \).

By Corollary 2.15, the function \( \gamma \mapsto (c_T, \gamma, c_T, \gamma) \) is \( \Sigma^1_4(\Gamma, \Gamma') \) in the codes, and hence belongs to \( L[T_3, \Gamma, \Gamma'] \). By Lemma 5.30, there is a club \( C \subseteq L[T_3, \Gamma, \Gamma'] \) such that \( \gamma = c_T, \gamma = c_T, \gamma \) for any \( \gamma \in C \). By Lemma 5.33 if \( \gamma \in [C](\bar{\gamma}, (d, q, P)) \), then \( \gamma = c_T, Q, \gamma = c_T, Q, \gamma \).

Suppose \( R \) is a finite level-3 tree. Let \( \bar{\gamma} \in [C][R] \). By Lemma 4.79, we can find a \( \Pi^1_3 \)-wellfounded \( Y \) extending \( R \) so that \( [0]_Y \in C \) and for any \( r \in \text{dom}(R), \gamma_r = [r]_Y \).

Then \( (M^r_Y) \infty = (M^r_Y) \infty = M^r_\mu, Y \infty = c^{(3)}_{\bar{\gamma}}|_{\bar{\gamma}+\omega} \) and for any \( r \in \text{dom}(R), \pi_M^r, Y, \infty (c^n_{\bar{\gamma}, Y, r} = \pi_M^r, Y, \infty (c^n_{\bar{\gamma}, Y, r}) = \gamma_r \). This ensures that \( (M^r_Y; c^n_{\bar{\gamma}, Y, r})_{r \in \text{dom}(R)} \) is elementarily equivalent to \( (M^r_Y; c^n_{\bar{\gamma}, Y, r})_{r \in \text{dom}(R)} \).

Hence, \( \Gamma(R) = \Gamma'(R) \). \( \square \)
The existence of an iterable, remarkable, level \( \leq 2 \) correct level-3 EM blueprint is a purely syntactical definition of a large cardinal. The minimum background assumption to make sense of it is \( \Delta^1_2 \)-determinacy. However, its existence and uniqueness is proved under boldface \( \Pi^1_3 \)-determinacy. It is unclear if the assumption of boldface \( \Pi^1_3 \)-determinacy can be weakened, at least to \( \Delta^1_3 \)-determinacy+\( \Pi^1_3 \)-determinacy. To draw a complete analogy with the level-1 sharp, one would naturally ask

**Question 5.35.** Assume \( \Delta^1_2 \)-determinacy. Are the following equivalent?

1. There is an iterable, remarkable, level \( \leq 2 \) correct level-3 EM blueprint.
2. There is an \( (\omega_1, \omega_1) \)-iterable \( M^#_2 \).
3. \( \Pi^1_3 \)-determinacy.

**Theorem 5.36.** Assume \( \Delta^1_2 \)-determinacy. If there is an iterable, remarkable, level \( \leq 2 \) correct level-3 EM blueprint, then \( \Pi^1_3 \)-determinacy holds.

**Proof.** Let \( \Gamma \) be an iterable, remarkable, level \( \leq 2 \) correct level-3 EM blueprint. Suppose \( A \subseteq \mathbb{R} \) is \( \Pi^1_3 \) and \( G \) is the game on \( \omega \) with payoff set \( A \). Let \(( R_x \) \( x \in \omega^\omega \)) be an effective regular level-3 system such that \( x \in A \iff R_x = \text{DEF} \cup_{n<\omega} R_{x^n} \) is \( \Pi^1_3 \)-wellfounded. By iterability of \( \Gamma \), \( M^*_{\Gamma,R^0} \) is a \( \Pi^1_3 \)-iterable system. Working in \( M^*_{\Gamma,R^0} \), define the auxiliary game \( H(c^{(i())}) \) where in rounds \( 2n \) and \( 2n+1 \), I plays \( x(2n) \in \omega \), \( \gamma_n \in c^{(i())} \). II plays \( x(2n+1) \). Player I is said to follow the rules at stage \( k \) iff letting \( r_n \in \text{dom}(R_{x^n+1}) \setminus \text{dom}(R_{x^n}) \) for \( n < k \), then for any \( n < m < k \), \( r_n = (r_m)^- \rightarrow \gamma_m < j(R_{x^n+1}) \text{ree}(r_n),(R_{x^n+1}) \text{ree}(r_m)(\gamma_n) \). Players I wins \( H(c^{(i())}) \) iff he follows the rules at every finite stage \( k \). \( H(c^{(i())}) \) is a closed game for Player I, hence determined in \( M^*_{\Gamma,R^0} \).

Case 1: \( M^*_{\Gamma,R^0} \models \text{"a \( \sigma \) is a winning strategy for Player I in } H(c^{(i())})\text{"} \).

Let \( \sigma^* \) be the strategy for Player I in \( G \) obtained by following \( \sigma \) and ignoring the auxiliary moves \( \gamma_n \). If \( x \) is (in \( V \)) a complete run according to \( \sigma^* \), then \( R_x \in p[S_2^{M^*_{\Gamma,R^0}}] \). By \( \Sigma^1_3 \)-correctness of set-generic extensions of \( M^*_{\Gamma,Y} \), \( p[S_2^{M^*_{\Gamma,R^0}}] \subseteq p[S_3] \). Hence \( x \in A \). This shows \( \sigma^* \) is winning for Player I.

Case 2: \( M^*_{\Gamma,R^0} \models \text{"a \( \sigma \) is a winning strategy for Player II in } H(c^{(i())})\text{"} \).

We define a strategy \( \sigma^* \) for II in \( G \) as follows: if \( \text{lh}(s) = 2n + 1 \), then \( \sigma^*(s) = a \) iff the formula

\[
\sigma^*((s(0), c_r), s(1), \ldots, (s(2n), c_r)) = a
\]

belongs to \( \Gamma(R_{x^{n+1}}) \), where \( r_k \in \text{dom}(R_{x^{k+1}}) \setminus \text{dom}(R_{x^k}) \). We claim that \( \sigma^* \) is a winning strategy for Player II. Suppose otherwise and \( x \) is a complete
run according to $\sigma^*$ but $x \in A$. Then $R_x$ is $\Pi^3_3$-wellfounded. Let $\mathcal{N} = \mathcal{M}^*(\Gamma, R^+_x)$, where $R^+_x$ extends $R_x$, $\text{dom}(R^+_x) = \text{dom}(R_x) \cup \{((1))\}$, $R^+_x((1))$ has degree 0. By iterability of $\Gamma$, $\mathcal{N}$ is a $\Pi^3_3$-iterable mouse. By coherency of $\Gamma$, $N = M^*(\Gamma, R^+_x)$, where $R^+_x((1))$ is a complete run according to $\sigma$ which is legal according to the rules of $(H(c((1))))^N$. However, $x \oplus (c_x^N)_{k<\omega}$ is a complete run according to $\sigma$ which is legal according to the rules of $(H(c((1))))^N$. In $V$, the tree of attempts of building a complete run according to $\sigma$ which is legal according to the rules of $(H(c((1))))^N$ is illfounded. By absoluteness of wellfoundedness, $\mathcal{N}$ can see such a complete run. Contradiction.

Theorem 5.36 is a generalization of Martin’s theorem that $0^\# \implies \Pi^1_1$-determinacy. It proves $1 \implies 3$ in Question 5.35. For a real $x$, a level-3 EM blueprint over $x$ is the obvious generalization of Definition 5.14, i.e., a function $\Gamma$ that sends $R$ to $\Gamma(R)$, a complete consistent $L^x,R$-theory containing the additional axioms “$x \in R$” and “$x(i) = j$” when $x(i) = j$. Assume $\Pi^1_1$-determinacy, $x^{3\#}$ is the unique iterable, remarkable, level $\leq 2$ correct level-3 EM blueprint over $x$. Thus, in combination with Neeman [37, 38] and Woodin [43], we reach an affirmative answer to the boldface version of Question 5.35.

Theorem 5.37. Assume $\Delta^1_2$-determinacy. The following are equivalent.

1. For all $x \in \mathbb{R}$, there is an iterable, remarkable, level $\leq 2$ correct level-3 EM blueprint over $x$.

2. For all $x \in \mathbb{R}$, there is an $(\omega_1, \omega_1)$-iterable $M^\#_2(x)$.

3. $\Pi^1_3$-determinacy.

Recall the basic fact that $L$ is the Skolem hull of the class of Silver indiscernibles. We exploit its higher level analog. Clearly, $M^\#_{3\omega}$ is not the Skolem hull of $\{c^{(3)}_{\alpha,X} : \alpha \text{ respects } X\}$, as by remarkability, the Skolem hull contains only countably many ordinals below $c^{(3)}_\omega$. The missing part will be generated by ordinals below $u_\omega$ in a specific way.

Lemma 5.38. Suppose $\mathcal{N}$ is $\Pi^1_3$-iterable and satisfies $0^{3\#}(\emptyset)$. Then for any limit ordinal $\alpha \in \mathcal{N}$,

$$\pi_{\mathcal{N},\infty}(\alpha) = \sup\{\pi_{\mathcal{N}T,\infty}(\beta) : T \text{ is } \Pi^1_3 \text{-wellfounded}, \beta < j^T_{\mathcal{N}}(\alpha)\}.$$ 

Proof. Let the universality of level $\leq 2$ ultrapowers axiom be the following:
If $\alpha$ is a limit ordinal and $\xi > \alpha$ is a cardinal and cutpoint, then $V^{\text{Coll}(\omega, \xi)}$ satisfies $\pi_{K|\xi, \infty}(\alpha) = \sup\{\pi((j^T)^K(\alpha)) : T$ is $\Pi_2^1$-wellfounded, $\beta < (j^T)^K(\alpha)\}$, where $(j^T)^K$ denotes the direct limit of $(j^{T'})^K$ for $T'$ a finite subtree of $T$.

By elementarity and absoluteness, it suffices to show that $M_{2, \infty}$ is a model of this axiom. Fix $\alpha < \delta^1_3$.

Case 1: $\text{cf} \mathbb{L}_{\delta^1_3}[T_2](\alpha) = \omega$.

Then $M_{2, \infty} \models \"\text{cf}(\alpha) \text{ is not measurable}\"$. So when $\alpha < \xi < \delta^1_3$, $(M_{2, \infty})^{\text{Coll}(\omega, \xi)} \models \"\text{\pi}_{K|\xi, \infty} \text{ is continuous at } \alpha\"$.

Case 2: $\text{cf} \mathbb{L}_{\delta^1_3}[T_2](\alpha) = u_1$.

Let $F : u_1 \to \alpha$ be order preserving and cofinal, $F \in \mathbb{L}_{\delta^1_3}[T_2]$. Let $z \in \mathbb{R}$ so that $F$ is $\Delta_1^1$-definable over $L_{\kappa^3_2}[T_2, z]$ from $\{T_2, z\}$. Let $P' \in \mathcal{F}_{2, z}$ and $\bar{t}, F \in P$ so that $\pi_{P, \infty}(\bar{t}, F) = (\alpha, F)$. Let $Q = (L[S_3])^P$. So for any $\Pi_2^1$-wellfounded $T$, $Q^T = (L[S_3])^{P_T}$ and $j^T_Q = j^T_P \mid Q^T$. By absoluteness, it suffices to show that

$$\alpha = \sup\{\pi_{P_T, \infty}(\beta) : T \text{ is } \Pi_2^1\text{-wellfounded, } \beta < j^T_P(\alpha)\}.$$  

This would follow from

$$u_1 = \sup\{\pi_{P_T, \infty}(\beta) : T \text{ is } \Pi_2^1\text{-wellfounded, } \beta < (u_1)^{P_T}\}.$$  

The last equality is because $\{\pi_{P_T, \infty} \circ j^{T', T}_P((\text{seed}_{(1, 0)})^{P_T}) : T \text{ is } \Pi_2^1\text{-wellfounded, } T' \text{ is a finite subtree of } T, (0) \in T'\}$ is a subset of the right hand side and has order type $\omega_1$.

Case 3: $\text{cf} \mathbb{L}_{\delta^1_3}[T_2](\alpha) = u_2$.

Similar to Case 2. \qed

**Definition 5.39.** If $\mathcal{N}$ is a structure that satisfies Axioms \textbf{III} in Definition \textbf{5.14} and the universality of level $\leq 2$ ultrapowers axiom, then

$$\mathcal{G}_{\mathcal{N}}$$

is the direct system consisting of models $\mathcal{N}^{T}$ for which $T$ is a $\Pi_2^1$-wellfounded level $\leq 2$ tree and maps $\pi_{\mathcal{N}^{T}, \mathcal{N}^{T'}} : \mathcal{N}^{T} \to \mathcal{N}^{T'}$ for $\pi$ minimally factoring $T, T'$. Define

$$\mathcal{N}_{\infty} = \text{dirlim } \mathcal{G}_{\mathcal{N}},$$

$$\pi_{\mathcal{N}^{T}, \mathcal{N}_{\infty}} : \mathcal{N}^{T} \to \mathcal{N}_{\infty} \text{ is tail of the direct limit map.}$$
If in addition, \( \mathcal{N} \) is countable \( \Pi^1_3 \)-iterable mouse, then \( \mathcal{G}_\mathcal{N} \) is a subsystem of \( \mathcal{I}_\mathcal{N} \). By Lemma 5.38 \( \mathcal{G}_\mathcal{N} \) is dense in \( \mathcal{I}_\mathcal{N} \), so there is no ambiguity in the notation \( \mathcal{N}_\infty \).

**Lemma 5.40.** Suppose \( \mathcal{N} \) is a countable \( \Pi^1_3 \)-iterable mouse and satisfies Axioms 7.13 in Definition 5.14 and the universality of level \( \leq 2 \) ultrapowers axiom. If \( \pi : \mathcal{N} \to \mathcal{P} \) is an iteration map, then there exist a \( \Pi^1_2 \)-wellfounded \( T \) and \( \psi : \mathcal{P} \to \mathcal{N}^T \) such that \( \psi \circ \pi = j^T_N \) and \( \psi \) is essentially an iteration map.

The direct system \( \mathcal{G}_\mathcal{N} \) is useful even when \( \mathcal{N} \) is not \( \Pi^1_3 \)-iterable. In the proof of the level-4 Kechris-Martin theorem in Section 7, we will inevitably have to deal with partially iterable level-3 EM blueprints. The structure \( \mathcal{N} \) will be the EM model built from a partially iterable level-3 EM blueprint. The advantage of the (possibly illfounded) direct limit \( \mathcal{N}_\infty \) is that the order type of its ordinals is easily codable by a subset of \( u_\omega \). If \( X \) is a finite level \( \leq 2 \) tree, \( a \in \mathcal{N}, \vec{\beta} = (\beta_x)_{(x,\epsilon) \in \text{dom}(X)} \) is represented by both \( T \) and \( T' \), then 

\[
\pi_{\mathcal{N},T,\infty}(a) = \pi_{\mathcal{N},T',\infty}(a).
\]

We can define

\[
\pi_{\mathcal{N},X,\vec{\beta},\infty}(a) = \pi_{\mathcal{N},T,\infty} \circ j_{\mathcal{N},T}(a)
\]

for \( \vec{\beta} \) represented by \( T \). So

\[
\mathcal{N}_\infty = \{ \pi_{\mathcal{N},X,\vec{\beta},\infty}(a) : a \in \mathcal{N}, X \text{ finite level } \leq 2 \text{ tree}, \vec{\beta} \in [\omega_1]^{X^+} \}.
\]

Essentially, the inner model theoretic comparison between mice is replaced by the comparison between \( \Pi^1_2 \)-wellfounded level \( \leq 2 \) trees in Theorem 4.57.

A level \( \leq 3 \) code for an ordinal in \( \delta^1_3 \) is of the form

\[
(R, \vec{\gamma}, X, \vec{\beta}, \tau \vec{\sigma})
\]

such that \( R \) is a finite level-3 tree, \( \vec{\gamma} \) respects \( R \), \( X \) is a finite level \( \leq 2 \) tree, \( \vec{\beta} \) respects \( X \), and \( \sigma \) is an \( L^R \)-Skolem term for an ordinal. It codes the ordinal

\[
\left( R, \vec{\gamma}, X, \vec{\beta}, \tau \vec{\sigma} \right) = \pi_{\mathcal{M}^\#_{\infty}_R,X,\vec{\beta},\infty}(\sigma_{\mathcal{M}^\#_{\infty}_R,R}(\langle \xi_r \rangle_{r \in \text{dom}(R)})).
\]

By Lemmas 5.40, every ordinal in \( \delta^1_3 \) has a level \( \leq 3 \). The evaluation function on level \( \leq 3 \) codes is \( \Sigma^1_4(0^3\#) \), and hence definable over \( M^\#_{2,\infty}(0^3\#) \).

### 5.5 Level-3 indiscernibles

If \( \vec{\gamma} \) respects a level-3 tree \( R \), define

\[
c_{\vec{\gamma}} = (c_{R_{\text{tree}}(r), \vec{\gamma}}(r))_{r \in \text{dom}(R)}
\]
which strongly respects $R$. Combined with Lemma 4.61, this leads to the order of level-3 indiscernibles for $M_{2,\infty}$: $c^{(3)}_{r,\gamma} < c^{(3)}_{r',\gamma'}$ iff letting $(\gamma_i)_{1 \leq i \leq k}$ be the $Q$-approximation sequence of $\gamma$ and $(\gamma'_i)_{1 \leq i \leq k'}$ be the $Q'$-approximation sequence of $\gamma'$, then $(\gamma_i)_{1 \leq i \leq k} \prec \neg(1) <_{B_K} (\gamma'_i)_{1 \leq i \leq k'} \prec \neg(1)$. We prove the general remarkable property of $0^3#$ based on this order.

**Lemma 5.41** (General remarkability). Suppose $r$ and $r'$ both respect a finite level-3 tree $R$. Suppose $r \in \text{dom}(R)$ and for any $s \prec^R r$, $\gamma_s = \gamma'_s$. Then for any $L$-Skolem term $\tau$,

$$M_{2,\infty} \models \tau(c_{\gamma}) < c^{(3)}_{R_{\text{tree}}(r), \gamma_r} \rightarrow \tau(c_{\gamma'}) = \tau(c_{\gamma'}) .$$

**Proof.** Assume $\tau^{M_{2,\infty}}(c_{\gamma}) < c^{(3)}_{R_{\text{tree}}(r), \gamma_r}$. If $s \in \text{dom}(R)$ and $r \preceq^R s$ let $l(s)$ be the largest $l$ so that $\langle r \upharpoonright l \rangle^R = \langle s \upharpoonright l \rangle^R$. It is easy to find $\gamma^{l \downharpoonright l}_r \in [\delta^l_3]^{R_l}$ for $l \leq \text{lh}(r)$ so that $\gamma^{0}_r = \gamma_r$, $\gamma^{l \downharpoonright l}_r(r) = \gamma'_r$ and $\gamma^{l+1}_r = (r \preceq^R s \land l(s) = l)$. Thus, we may assume a fixed $l_0$ so that $\gamma_s \neq \gamma'_s$ implies $l(s) = l_0$. The case $l_0 = 0$ is just Lemma 5.24. Assume now $l_0 > 0$. Note that $l(s) = l_0$ also implies that $R_{\text{tree}}(s \upharpoonright l_0) = R_{\text{tree}}(r \upharpoonright l_0)$. A sliding argument similar to Lemma 5.24 reduces to the special case that $(\text{lh}(s) = \text{lh}(s') = l_0 + 1 \land l(s) = l(s') = l_0) \rightarrow \delta_s < \delta'_s$, then $\delta + \delta' = \varepsilon$, $\epsilon_{\rho(r)} = \delta_r$, $\epsilon_{\rho'(r)} = \delta'_r$. Put $\eta \in D$ iff $c^{(3)}_{\eta} = \eta$. By indiscernibility, we may assume that $\gamma, \gamma' \in [D]^{R_l}$, so that $c_{\gamma} = \gamma_r$, $c_{\gamma'} = \gamma'_r$. It is easy to construct $\delta^{l \downharpoonright l}_\xi \in [\delta^l_3]^{R_l}$ for $\xi < \gamma_r$ and $\eta < \xi < \gamma_r \rightarrow \delta^{l \downharpoonright l}_\xi \in [\delta^l_3]^{R_l}$, $\delta + \delta'^l = \varepsilon$, $\epsilon_{\rho(r)} = \delta_r$, $\epsilon_{\rho'(r)} = \delta'_r$. By indiscernibility, it suffices to show that $\varepsilon^{l \downharpoonright l}_\eta = \varepsilon^{l \downharpoonright l}_\eta$ for some (or equivalently, for any) $\eta < \xi < \gamma_r$. Suppose otherwise. By indiscernibility again, either $\eta < \xi < \gamma_r \rightarrow \epsilon^{l \downharpoonright l}_\eta > \varepsilon^{l \downharpoonright l}_\eta$ or $\eta < \xi < \gamma_r \rightarrow \epsilon^{l \downharpoonright l}_\eta < \varepsilon^{l \downharpoonright l}_\eta$. The former gives a descending chain of ordinals. The latter implies that $\gamma_r \leq \tau^{M_{2,\infty}}(c_{\gamma})$, contradicting to our assumption.

Recall that if $c < c'$ are consecutive $L[x]$-indiscernibles, then $L[x^\#] \models c < c^\#$. The level-3 version is similar.

**Lemma 5.42.** Assume $\Pi^4_1$-determinacy. For any $c^{(3)}_{\omega_1} < \xi \in I^{(3)}$, there is an $L$-Skolem term $\tau$ such that $M_{2,\infty}^{(3)}(0^3\#) \models \tau(\text{sup}(I^{(3)} \cap \xi), \cdot)$ is a surjection from $\text{sup}(I^{(3)} \cap \xi)$ onto $\xi$.

**Proof.** The evaluation function on level $\leq 3$ codes is $\Sigma^1_4(0^3\#)$, and hence is definable over $M^{(3)}_{2,\infty}(0^3\#)$. If $(R, \gamma, X, \beta, r')$ and $(R, \gamma', X, \beta', r')$ are both level $\leq 3$ codes an ordinal below $\xi$ and $\forall r (\gamma_r < \xi \rightarrow \gamma_r = \gamma'_r)$, then by Lemma 5.41 they must code the same ordinal. This easily defines a surjection from $\text{sup}(I^{(3)} \cap \xi)$ onto $\xi$ in $M^{(3)}_{2,\infty}(0^3\#)$.
By Lemma 4.61, for any finite level-3 tree $R$, if $A \prec A'$ then \( c_{A} < c_{A'} \) is true in \( 0^{3#}(R) \), if $A \sim A'$ then \( c_{A} = c_{A'} \) is true in \( 0^{3#}(R) \). For notational convenience, if $X$ is a finite level \( \leq 2 \) tree and $\gamma = [F]_{\mu X}$ is a limit ordinal, define $c^{(3)}_{X,\gamma} = [\alpha \mapsto c^{(3)}_{F(\alpha)}]_{\mu X}$; define $c^{(3)}_{\emptyset,\delta^1} = \delta^1_3$. Ordinals of the form $c^{(3)}_{X,\gamma}$ when $X \neq \emptyset$ are definable from elements in $I^{(3)}$ over $M^{\omega}_{2,\infty}$: If the $X$-approximation sequence of $\gamma$ is $(\gamma_i)_{1 \leq i \leq k}$, $(Q_i, (d_i, q_i, P_i)_{i < \lh(\gamma)})$ is the $X$-potential partial level \( \leq 2 \) tower induced by $\gamma_i$, $\pi : Q \to X$ is the induced level-2 factoring map, then

1. if $\gamma$ is of $X$-discontinuous type, then $c^{(3)}_{X,\gamma} = \pi^X(c^{(3)}_{Q,\gamma_k})$;

2. if $\gamma$ is of $X$-continuous type, $Q^-$ is the subtree of $Q$ obtained by removing $(d_k, q_k)$, then $c^{(3)}_{X,\gamma} = \pi^X \circ j^-_{Q^-(c^{(3)}_{Q,\gamma_k})}$.

Define $\bar{I}^{(3)}$ - the closure of $I^{(3)}$ under the order topology. Every ordinal in $\bar{I}^{(3)}$ is of the form $c^{(3)}_{X,\gamma}$ where $X$ is finite and $\gamma < \delta^1_3$ is a limit. Thus, if $A = (r, \pi, T) \in \text{desc}^{**}(R)$ and $\gamma$ strongly respects $R$, then $c^{(3)}_{T,\gamma_A} \in \bar{I}^{(3)}$ and is a limit point of $I^{(3)}$.

Given $\gamma_0, \ldots, \gamma_n, \gamma_0, \ldots, \gamma_n \in I^{(3)}$, $\gamma$ is a shift of $\gamma'$ if there exist a level-3 tree $R$, nodes $r_0, \ldots, r_n \in \text{dom}(R)$, $\delta, \delta'$ both respecting $R$ such that $\gamma_i = c^{(3)}_{R,\gamma_i}$, $\gamma_i' = c^{(3)}_{R,\gamma_i'}$ for any $i \leq n$. By indiscernability, if $\gamma$ is a shift of $\gamma'$, then for any $L$-formula $\varphi$,

$$M^{\omega}_{2,\infty} \models \varphi(\gamma) \leftrightarrow \varphi(\gamma').$$

**Lemma 5.43.** Suppose $R$ is a finite level-3 tree, $\tau$ is an $L$-Skolem term, $\gamma$ strongly respects $R$. Suppose $A = (r, \pi, T) \in \text{desc}^{**}(R)$. Then

$$\tau^{M^{\omega}_{2,\infty}}(c_\gamma) < c^{(3)}_{T,\gamma_A} \rightarrow \tau^{M^{\omega}_{2,\infty}}(c^{(3)}_{\gamma_A}) < \min(I^{(3)} \setminus \sup\{c^{(3)}_{T,\gamma_A'} : A' = (r', \pi', T') \prec^R R\}).$$

**Proof.** Suppose $\tau^{M^{\omega}_{2,\infty}}(c_\gamma) < c^{(3)}_{T,\gamma_A}$. Let $\delta = \min(I^{(3)} \setminus \sup\{c^{(3)}_{T,\gamma_A'} : A' = (r', \pi', T') \prec^R R\})$. We shall show that \( \{\delta' : c_\gamma(<\delta') \text{ is a shift of } c_\gamma(<\delta)\} \) is cofinal in $\gamma_A$. From this and indiscernability, $\tau^{M^{\omega}_{2,\infty}}(c_\gamma) < \delta$.

If $r = \emptyset$, then $\delta = c^{(3)}_{(\gamma_0)^a}$. Suppose now $r = (r, Q_1, (d_1, q, P_1)) \neq \emptyset$, $\lh(r) = k$, $\text{ucf}(R(r)) = (d^*, q^*)$, and if $d^* = 1$ put $q^* = q^*$, if $d^* = 2$ put $q^* = (q^*, P^*, b^*)$.

Case 1: $r$ is of discontinuous type, $\langle A \rangle$ ends with $-1$. 137
Let $s = \max_{<_{\mathcal{R}K}} R\{r, -\}$ and $s = (s, Q, (d, q, \vec{\beta}))$. Then $\delta = c_{T, \tau}(\gamma_{A}) + \omega$. It is easy to compute that $\{\delta' : c_{\gamma}(\delta') \text{ is a shift of } c_{\gamma}(\delta)\}$ is cofinal in $\gamma_{A}$.

Case 2: $r$ is of discontinuous type, $(A)$ ends with an ordinal.

If either $d = 1$ or $d = 2^\omega \land q \in \text{desc}(Q)$, let $\tau$ factor $(Q, T)$ so that $\tau$ agrees with $\pi$ on $\text{dom}(Q) \setminus \{(d', q')\}$, $\tau(d', q') = \text{pred}(\pi, T, (d', q'))$. Then $\delta = c_{T, \tau}(\gamma_{A}) + \omega$. Otherwise, let $U$ be the subtree of $T$ obtained by removing $\text{pred}(\pi, T, (d', q'))$ from its domain. Then $\delta = c_{U, \tau}(\gamma_{A}) + \omega$. In either case, $\{\delta' : c_{\gamma}(\delta') \text{ is a shift of } c_{\gamma}(\delta)\}$ is cofinal in $\gamma_{A}$.

Case 3: $r$ is of continuous type.

Similar to Cases 1 and 2.

\[\square\]

**Lemma 5.44.** Suppose $R$ is a finite level-3 tree and $A = (r, \pi, T) \in \text{desc}^{**}(R)$, $r \neq \emptyset$, $\gamma$ strongly respects $R$. Then $c_{T, \gamma}^{(3)}$ is a cardinal in $M^{2, \infty}_{\omega, \infty}$.

**Proof.** Otherwise, $\tau^{M_{2, \infty}}(\gamma_{A})$ is a wellordering on $\alpha =_{\text{DEF}} \text{card}^{M_{2, \infty}}(c_{T, \gamma_{A}}^{(3)})$ of order type $c_{T, \gamma_{A}}^{(3)}$ and $\alpha < c_{T, \gamma_{A}}^{(3)}$. Put $\beta = \min(I^{(3)} \setminus \sup\{c_{T, \gamma_{A}}^{(3)} : A' = (r', \pi', T') \rightarrow^{R} A\})$. By Lemma 5.43, $\alpha < \beta$. By Lemma 5.41, if $\delta$ respects $R$ and \(\forall\ s (s < \beta \rightarrow s = \gamma_{s})\), then $\tau^{M_{2, \infty}}(\gamma_{A}) = \tau^{M_{2, \infty}}(c_{\delta})$, and hence $\gamma_{A} = \delta_{A}$.

However, it is easy to find such $\delta$ satisfying $\delta_{A} > \gamma_{A}$.

\[\square\]

## 6 The boldface level-3 sharp

From now on, we assume $\Pi_{3}^{1}$-determinacy. Recall that $L[T_{3}] = \bigcup_{x \in \mathbb{R}} L[T_{3}, x]$. Every subset of $\delta_{3}$ in $L[T_{3}]$ is definable over $M_{2, \infty}(x)$ for some $x \in \mathbb{R}$. All the results in Section 3 relativize to any given real $x$. If $R$ is a $\Pi_{2}^{1}$-wellfounded level-3 tree, $M_{x}^{3} R$ is the EM model built from $x^{3}(R)$. $M_{x}^{3} R$, $M_{x}^{3} R$, $c_{x, Q, \gamma}^{(3)}$, $c_{x, \gamma}$, $c_{x, \gamma}$, $I_{x}$, $\vec{\gamma}_{x}$ have obvious meanings. Fixing $x$, the function $(Q, \gamma) \mapsto c_{x, Q, \gamma}^{(3)}$ is $\Sigma_{1}^{1}(x^{3})$ in the codes and hence is definable over $M_{2, \infty}(x^{3})$.

### 6.1 Homogeneity properties of $S_{3}$

A level $\leq 3$ tree is of the form $R = (0, 1, 0, 2, 3)$ so that $\leq_{2} R =_{\text{DEF}} (0, 1, 0, 2, 3) R$ is a level $\leq 2$ tree and $\leq_{3} R$ is a level-3 tree. If $T$ is a level $\leq 2$ tree and $Y$ is a level-3 tree then $T \oplus Y$ denotes the level $\leq 3$ tree $(0, T, 1, T, 2, Y)$. $R$ is $\Pi_{2}^{1}$-wellfounded iff $\leq_{2} R$ is $\Pi_{2}^{1}$-wellfounded and $\leq_{3} R$ is $\Pi_{2}^{1}$-wellfounded.

Suppose $R$ is a level $\leq 3$ tree. Define $\text{dom}(R) = \bigcup_{d} \{d\} \times \text{dom}(d^{4})$, $\text{desc}(R) = \bigcup_{d} \{d\} \times \text{desc}(d^{4})$. If $\beta$ respects $\leq_{2} R$ and $\gamma$ respects $\leq_{3} R$, define $\beta \oplus \gamma = \delta = (\delta_{i})_{(d, r) \in \text{dom}(R)}$ where $\delta_{r} = \beta_{r}$ for $(d, r) \in \text{dom}(\leq_{2} R)$ and
\[ \delta_r = \gamma_r \text{ for } r \in \text{dom}(\delta R). \] Define \( A \oplus B = \{ \vec{\beta} \oplus \vec{\gamma} : \vec{\beta} \in A, \vec{\gamma} \in B \}. \) If \( E \subseteq \omega_1 \) and \( C \subseteq \delta^{1}_3 \), define \([E, C]^{R \uparrow} = [E]^{\leq 2R \uparrow} \uplus [C]^{R \uparrow}. \) \( \vec{\delta} \) respects \( R \) iff \( \vec{\delta} \in [\omega_1, \delta^{1}_3]^{R \uparrow}. \) A finite level \( \leq 3 \) tree \( R \) induces a filter \( \mu^R \) on finite tuples in \( \delta^{1}_3 \), originated from the weak partition property of \( \delta^{1}_3 \) under AD. \( \mu^R \) is the higher level analog of the \( n \)-fold product of the club filter on \( \omega_1. \)

**Definition 6.1.** Assume \( \Pi^{1}_{3} \)-determinacy. Let \( R \) be a finite level \( \leq 3 \) tree.

We say

\[ A \in \mu^R \]

iff there are clubs \( E \subseteq \omega_1, C \subseteq \delta^{1}_3 \) such that \( E, C \in \mathbb{L}[T_3] \) and

\[ [E, C]^{R \uparrow} \subseteq A. \]

If \( Y \) is a finite level-3 tree, put \( A \in \mu^Y \) iff \([\omega_1]^{Q \uparrow} \uplus A \in \mu^{Q \uparrow Y} \).

\( \mu^R \) is an \( \mathbb{L}[T_3] \)-measure, the reason being as follows. Every \( A \in \mathbb{L}[T_3] \) is definable over \( M^\omega_{\infty}(x) \) from \( \{x\} \) for some real \( x \). By indiscernibility and remarkability, the section \( A^* =_{\text{DEF}} \{ \vec{\beta} : \vec{\beta} \oplus c^{(3)}_{x, \vec{\gamma}} \in A \} \) is invariant in \( \vec{\gamma} \in [\delta^{1}_3]^{R \uparrow}. \) So \( C = \{ c^{(3)}_{x, \vec{\xi}} : \vec{\xi} < \delta^{1}_3 \} \) and some \( E \) deciding the \( \mu^\leq 2R \)-measure of \( A^* \) works. \( \mu^R \) is the product measure on \( \mathbb{L}[T_3] \) of \( \mu^{\leq 2R} \) and \( \mu^{3R}. \)

Let \( J = j^R_{\mathbb{L}[T_3]} \) be the ultrapower map from \( \mathbb{L}[T_3] \) to \( \mathbb{L}[j^R(T_3)] \). For any \( x \in \mathbb{R}, j^R \) is elementary from \( L[T_3, x] \) to \( L[j^R(T_3), x] \). By indiscernibility and remarkability again, if \( \alpha < \delta^{1}_3 \) and \( F : [\omega_1, \delta^{1}_3]^{R \uparrow} \rightarrow \alpha, F \in \mathbb{L}[T_3] \), then there is \( x \in \mathbb{R} \) and \( G \in \mathbb{L}[T_2] \) such that for any \( \vec{\gamma} \in [\delta^{1}_3]^{R \uparrow}, \) for any \( \vec{\beta} \in [\omega_1]^{\leq 2R \uparrow}, F(\vec{\beta} \oplus \vec{\gamma}) = G(\vec{\beta}). \) Therefore,

\[ j^R \upharpoonright \delta^{1}_3 = j^{\leq 2R} \upharpoonright \delta^{1}_3. \]

For \( (d, r) \in \text{dom}(R) \cup \{(3, \emptyset)\}, \) let

\[ \text{seed}^{(d, r)} \]

be the element represented modulo \( \mu^R \) by the projection map \( \vec{\gamma} \mapsto d \gamma_r. \) If \( R \) is \( \Pi^{1}_{3} \)-wellfounded, the direct limit of \( j^{R, R' \uparrow} \) for \( R' \) a finite subtree of \( R'' \) and \( R'' \) a finite subtree of \( R \) is wellfounded, and we let \( j^R : \mathbb{L}[T_3] \rightarrow \mathbb{L}[j^R(T_3)] \) be the direct limit map; if \( R' \) is a finite subtree of \( R \) then \( j^{R', R} : \mathbb{L}[j^R(T_3)] \rightarrow \mathbb{L}[j^R(T_3)] \) is the tail of the direct limit map. If \( (d, r) \in \text{dom}(R'), R' \) is a finite subtree of \( R \), then \( \text{seed}^{(d, r)} = j^{R, R} \text{seed}^{(d, r)} \). Let

\[ \text{seed}^R = (\text{seed}^{(d, r)})_{(d, r) \in \text{dom}(R)}. \]

139
If $Y$ is a $\Pi^1_3$-wellfounded level-3 tree, then $j^Y = j^{Q_0 \oplus Y}$, seed$_Y = \text{seed}^{Q_0 \oplus Y}$, and seed$_Y = (\text{seed}^Y)_{j \in \text{dom}(Y)}$.

In particular, seed$_{R(3,0)} = j^R(\delta^3_1)$, seed$_{(d,r)} = \text{seed}^{\omega_1}_{(d,r)}$ when $d \leq 2$. If $A \in \text{desc}^*(R)$, $r \in \text{desc}^*(\tilde{R})$, $R'$ finite subtree of $R$, let $\text{seed}^{R,R}_{(3,A)} = j^{R,R}([3] \rightarrow \delta^3_A)[\omega^{\omega_1}]$. By Lemma 4.61, $\text{seed}^{R,(3,A)}_{(3,0)} < \text{seed}^{R,(3,A')}_r$ if and only if $A < A'$; $\text{seed}^{R,(3,A)}_{(3,0)} = \text{seed}^{R,(3,A')}_r$ if $A \sim A'$. seed$_{(3,A)}$ for finite $R$ is the higher level analog of uniform indiscernibles. If $Y$ is a $\Pi^1_3$-wellfounded level-3 tree and $A \in \text{desc}^*(Y)$, $y \in \text{desc}^*(Y)$, let seed$_A^Y = \text{seed}^{Q_0 \oplus Y}_A$, seed$_Y = \text{seed}^{Q_0 \oplus Y}_Y$.

We will show in Section 6.2 that seed$_{(3,A)} = \text{seed}^R_A$ for $A \in \text{desc}^*(R)$.

Under full AD, the set of seed$_{(3,0)}$ for finite $R$ is exactly $\{N_{\xi+1} : \omega \leq \xi < \omega^\omega\}$ by Martin [13, Theorem 4.17] and Jackson [12]. The set of seed$_{(3,0)}$ for finite $R$ and their limit points will be level-3 indiscernibles. The rest of this paper will contain a thorough analysis of the structure of level-3 uniform indiscernibles.

6.2 Level-3 uniform indiscernibles

**Definition 6.2.** If $R$ is a finite level $\leq 3$ tree, $\alpha$ is an $R$-uniform indiscernible if $\alpha \in \bigcap_{s \in \text{dom}(R)} j^R(\check{f}_s^x)$.

By Lemma 5.43, the set of $R$-uniform indiscernibles is the closure of $\{\text{seed}^{R,(3,A)}_{(3,0)} : A \in \text{desc}^*(R)\}$, which has order type $\omega_1 + 1$ if $[0]_R = \bar{\epsilon}$. By Lemmas 5.42 and 5.44, $\alpha$ is an $R$-uniform indiscernible if $\alpha \geq \delta^3_1$ is a cardinal in $\text{L}([j^R(T)])$. In particular, the least $R$-uniform indiscernible is $\delta^3_1$.

Recall that if $R$ is a level-3 tree, $s \in \text{dom}(R)$, then $R \upharpoonright s$ the subtree of $R$ whose domain consists of $r$ for which $\langle r \rangle <_{BK} \langle s \rangle$. If $R$ is a level-3 tree and $A \in \text{desc}^*(R)$, we let $R \upharpoonright A$ be the subtree of $R$ whose domain consists of $r$ for which $\langle r \rangle <_{BK} \langle A \rangle$. If $R$ is a level $\leq 3$ tree and $A \in \text{desc}^*(R)$, $s \in \text{dom}(R)$, let $R \upharpoonright (3,A) = \bigoplus \omega_1 \oplus (\delta^3 \upharpoonright A)$, $R \upharpoonright (3,s) = \bigoplus \omega_1 \oplus (\delta^3 \upharpoonright s)$.

**Lemma 6.3.** Assume $\Pi^1_1$-determinacy. Suppose $R$ is $\Pi^1_3$-wellfounded level $\leq 3$ tree and $A \in \text{desc}^*(R)$. Then $j^{R,(3,A)}_{(3,0)}$ is the identity on $\text{L}([j^{R,(3,A)}_r(T)])$. Furthermore, if $s \in \text{dom}(R)$ and $\text{lh}(s) = 1$, then $j^{R,(3,s)}_{(3,0)} = \text{seed}^{R,(3,s)}_{(3,0)}$.

**Proof.** Using a direct limit argument, it suffices to prove the case when $R$ is finite. We prove that $j^{R,(3,A)}_{(3,0)}(\delta^3_1)$ is contained in the range of $j^{R,(3,A)}_{(3,0)}$. Suppose $\alpha = [G]_{u \in R} < \text{seed}^{R,(3,A)}_{(3,0)}$, $x \in \mathbb{R}$, $\tau$ is an $\mathcal{L}$-Skolem term such that $G(\bar{\tau}) = \tau^{M_{\omega_1}((x,\bar{\tau}))}$ for any $\bar{\tau} \in [\omega_1,\delta^3_1 \upharpoonright R]$ and $G(\bar{\tau}) < \delta^3_1$ for $\mu^R$-a.e. $\bar{\tau}$. By Lemma 5.41, if $\bar{\beta}$ respects $\delta^3_1$, $\bar{\delta}$ both strongly respects $R$ and $\forall r (((3,r) \in \text{dom}(R \upharpoonright (3,A))) \rightarrow \delta^3_1 \upharpoonright \bar{\delta} = \delta^3_1 \upharpoonright \bar{\delta})$, then $\tau^{M_{\omega_1}((x,\bar{\beta} + \delta^3_1 \upharpoonright \bar{\delta}))} = \tau^{M_{\omega_1}((x,\bar{\beta} + \delta^3_1 \upharpoonright \bar{\delta}))}$.
$\tau_{M_2,\infty}(x, \vec{a} \oplus \iota_{(3)}_{x,\vec{a}})$. Using the fact that $(Q, \gamma) \mapsto \iota_{(3)}_{x,Q,\gamma}$ is definable over $M_2,\infty(x^{3#})$, we can find an $L$-Skolem term $\sigma$ such that for $\mu_R$-a.e. $\vec{y}$,

\[ G(\vec{y}) = \sigma_{M_2,\infty(x^{3#})}(x^{3#}, (d_{\vec{y}})(d, r) \in \text{dom}(R(3,A))). \]

Hence, $\alpha = j^{R(3,A)}(\beta)$ where $\beta = \sigma^{j^{R(3,A)}(M_2,\infty(x^{3#}))}(x^{3#}, \text{seed}^{R(3,A)})$. This also implies that $j^{R(3,A)}(\delta_3^1) \geq \text{seed}^{R(3,A)}$. The “furthermore” part is due to unboundedness of level-3 sharps. \[ \square \]

Suppose $Y$ is a level $\leq 3$ tree, $T$ is a level $\leq 2$ tree. A $(Y, T, *)$-description is of the form $B = (d, (y, \pi))$ so that $d = 3 \land (y, \pi) \in \text{desc}(Y, T, *)$ or $d \leq 2 \land (d, (y, \pi)) \in \text{desc}(\leq^{2}Y, T, *)$. As usual, $B = (d, (y, \pi))$ is abbreviated by $(d, y, \pi)$. If $Q$ is a finite level $\leq 2$ tree, a $(Y, T, Q)$-description is $(3, (y, \pi))$ so that $(y, \pi) \in \text{desc}(Y, T, Q)$. If $P$ is a finite level-1 tree, a $(Y, T, P)$-description is $(2, (y, \pi))$ so that $(2, (y, \pi)) \in \text{desc}(\leq^{2}Y, T, P)$. A $(Y, T, -1)$-description is $(1, (y, \emptyset))$ so that $y \in \text{dom}(Y)$. $\text{desc}(Y, T, *), \text{desc}(Y, T, Q)$, etc. denote the sets of relevant descriptions. If $Y, T$ are finite,

\[ \text{seed}^{Y,T}_{B} \in \mathbb{L}(j^{Y} \circ j^{T}(T_{3})) \]

is the element represented modulo $\mu^{Y}$ by $\text{id}^{Y,T}_{B}$.

Suppose that $R, Y$ are level $\leq 3$ trees and $T$ is a level $\leq 2$ tree. $\rho$ factors $(R, Y, T)$ if $\rho$ is a function on $\text{dom}(R)$, $\leq^{2}_{R} = \text{def} \rho \upharpoonright \text{dom}(\leq^{2}_{R})$ factors $(\leq^{2}_{R}, \leq^{2}_{Y}, T)$ and $\delta^{2}_{R} = \text{def} \rho \upharpoonright \text{dom}(\delta^{2}_{R})$ factors $(\delta^{2}_{R}, \delta^{2}_{Y}, T)$. $\rho$ factors $(R, Y)$ iff $\leq^{2}_{R} \rho \text{ factors } (\leq^{2}_{R}, \leq^{2}_{Y})$ and $\delta^{2}_{R} \rho \text{ factors } (\delta^{2}_{R}, \delta^{2}_{Y})$. Suppose that $\rho$ factors $(R, Y, T)$. If $F \in (\omega_1, \delta_3^1)^{V_{T}^{R}}$, then

\[ F_{\rho}^{T} : [\omega_1]^{T_{R}} \rightarrow [\omega_1, \delta_3^1]^{R_{T}} \]

is the function that sends $\vec{\xi}$ to $F^{T}_{\leq^{2}_{\rho}}(\vec{\xi}) \oplus F^{T}_{\delta^{2}_{\rho}}(\vec{\xi})$. If $T$ is finite,

\[ \text{id}^{Y,T}_{\rho} \]

is the function $[F]^{Yu} \mapsto [F_{\rho}]_{\mu^{R}}$. If $Y$ is also finite,

\[ \text{seed}^{Y,T}_{\rho} = [\text{id}^{Y,T}_{\rho}]_{\mu^{Y}} \in \mathbb{L}(j^{Y} \circ j^{T}(T_{3})). \]

By Łoś and Lemmas \[4.50 \, 4.51 \, 4.63 \, 4.64 \, 3.18 \, 4.46 \] for any $A \in \mu^{R}$, $\text{seed}^{Y,T}_{\rho} \in j^{Y} \circ j^{T}(A)$. We can unambiguously define

\[ \rho^{Y,T} : \mathbb{L}(j^{R}(T_{3})) \rightarrow \mathbb{L}(j^{Y} \circ j^{T}(T_{3})) \]

141
by sending \( j^R(F)(\mathrm{seed}^R) \) to \( j^Y \circ j^T(F)(\mathrm{seed}^T) \). In general, if \( Y, T \) are \( \Pi^1_3 \)-wellfounded and \( T \) is \( \Pi^1_2 \)-wellfounded, then \( \rho^{Y, T} \circ j^{R, R} = j^{Y, Y} \circ j^{(Y, T), T} \circ (\rho')^{Y, T} \) for \( R, Y, T \) finite subtrees of \( R, Y, T \) respectively and \( \rho' = \rho \restriction \text{dom}(R) \) factoring \( (R', Y', T') \), where \( j^{Y, (T', T)} = \cup_{x \in R} j^{Y} (j^{T', T} \restriction L[j^{T'}(T_3), x]) \). In particular, \( \rho^{Y, T} \circ j^{R}(\delta^3_3) = j^{Y}(\delta^3_3) \). If \( A \in \text{desc}^*(\bar{R}) \), then \( \rho^{Y, T}(\text{seed}^R_{(3, A)}) = \text{seed}^Y_{(3, \bar{R}(A))} \).

If \( Y \) is a level \( \leq 3 \) tree and \( T \) is a level \( \leq 2 \) tree, then

\[
Y \otimes T = (\leq^3 Y \otimes T) \oplus (Y \otimes T)
\]
is (modulo an isomorphism) a level \( \leq 3 \) tree. The domain of \( Y \otimes T \) consists of \( B = (d, (y, \pi)) \in \text{desc}(Y, T, *) \). So \( \rho \) factors \((R, Y, T)\) iff \( \rho \) factors \((R, Y \otimes T)\).

The identity map \( \text{id}_{Y \otimes T} : B \mapsto B \) factors \((Y \otimes T, Y, T)\).

**Lemma 6.4.** Suppose \( Y \) is a \( \Pi^1_3 \)-wellfounded level \( \leq 3 \) tree and \( T \) is a \( \Pi^1_2 \)-wellfounded level \( \leq 2 \) tree. Then \( L_{j^Y \otimes T(\delta^3_3)}[j^{Y, \otimes T}(T_3)] = L_{j^Y \otimes T(\delta^3_3)}[j^Y \circ j^T(T_3)] \) and \((\text{id}_{Y \otimes T})^{Y, T} \) is the identity map on \( L_{j^Y \otimes T(\delta^3_3)}[j^{Y, \otimes T}(T_3)] \).

**Proof.** Assume without loss of generality that \( Y, T \) are finite. Put \( R = Y \otimes T \) and \( \rho = \text{id}_{Y \otimes T} \). Then \( \rho^{Y, T}(u^R_{(3, A)}) = u^{Y, \otimes T}_{(3, \bar{R}(A))} \) for any \( A \in \text{desc}^*(\bar{R}) \) and \( \forall B \in \text{desc}^{(R)}(Y) \exists A \in \text{desc}^*(R) \bar{R}(A) \sim^*_Y B \). Recall that the set of \( L[j^R(T_3)]\)-cardinals in the interval \( [\delta^1_3, j^R(\delta^1_3)] \) is exactly \( \{ u^R_{(3, A)} : A \in \text{desc}^*(\bar{R}) \} \). For any \( x \in R \), \( \rho^{Y, T} \restriction L[j^R(T_3), x] \) is elementary from \( L[j^R(T_3), x] \) to \( L[j^Y \circ j^T(T_3), x] \). Hence, it suffices to show that \( \rho^{Y, T} \restriction \delta^1_3 \) is the identity and whenever \( \|A\|_{<^3} \) is a successor cardinal, then \( \rho^{Y, T} \) is continuous at \( u^R_{(3, A)} \).

By Lemma 4.53 and Corollary 4.15, \( j^R \restriction \delta^1_3 = j^{<^3 R} \restriction \delta^1_3 = j^{<^3 Y} \circ j^T \restriction \delta^1_3 = j^Y \circ j^T \restriction \delta^1_3 \). By indiscernibility and remar-kability, \( \rho^{Y, T} \restriction \delta^1_3 = j^{<^3 Y, \otimes T} \restriction \delta^1_3 \) is the identity map.

Suppose \( \|A\|_{<^3} \) is a successor cardinal and we prove that \( \rho^{Y, T} \) is continuous at \( \text{seed}^R_{(3, A)} \). Suppose \( \alpha < \text{seed}^Y_{(3, \bar{R}(A))} \). There is \( x \in \mathbb{R} \) such that

\[
\alpha < \min(j^Y(\bar{I}^3_x) \setminus \sup\{u^Y_{(3, B)} : B \prec^*_Y \bar{R}(A)\}).
\]

Let \( \beta = \min(j^R(I^3_x) \setminus \sup\{\text{seed}^R_{(3, B)} : B \prec^*_Y A\}) \). Then \( \beta < \text{seed}^R_{(3, A)} \) and by induction and ele-mentarity,

\[
\rho^{Y, T}(\beta) = \min(j^Y \circ j^T(\bar{I}^3_x) \setminus \sup\{u^Y_{(3, B)} : B \prec^*_Y \bar{R}(A)\}) \geq \alpha,
\]

the last inequality from \( j^T(\bar{I}^3_x) \subseteq \bar{I}^3_x \) and elementarity of \( j^Y \).

**Lemma 6.5.** Suppose \( Y, Y' \) are \( \Pi^1_3 \)-wellfounded level \( \leq 3 \) trees. Then \( [0]_{\mathcal{V}} = [0]_{\mathcal{V'}} \) iff \( j^Y(\delta^3_3) = j^{Y'}(\delta^3_3) \), \( [0]_{\mathcal{V}} < [0]_{\mathcal{V'}} \) iff \( j^Y(\delta^1_3) < j^{Y'}(\delta^1_3) \).

142
Lemma 6.3. \(j\) is an indiscernible.

Proof. If \([\emptyset]_{3Y} \leq [\emptyset]_{3Y'}\) then by Theorems 4.57 and 4.71 there exist a finite \(T\) and \(\rho\) that factors \((Y, Y', T)\). So \(\rho_{3Y'} \circ j^Y(\delta^4_3) = j^{Y'}(\delta^4_3)\), yielding \(j^Y(\delta^4_3) \leq j^{Y'}(\delta^4_3)\). So \([\emptyset]_{3Y'} = [\emptyset]_{3Y'}\) implies \(j^Y(\delta^4_3) = j^{Y'}(\delta^4_3)\). If \([\emptyset]_{3Y} < [\emptyset]_{3Y'}\), we further obtain \(B \in \text{desc}^{3}(Y', T, *)\) such that \(\text{lh}(B) = 1\) and \([B]_{3Y' \otimes T} = [\emptyset]_{3Y'}\).

Put \(Z = (Y' \otimes T) \mid (3, B)\). Then \(\rho\) factors \((Y, Z)\). Hence \(j^{Y'}(\delta^4_3) \leq j^{Z}(\delta^4_3)\).

By Lemma 6.3, the factor map \(j^{Z, Y' \otimes T}\) is the identity on \(L_{\xi}(\delta^4_3)\) and \(j^{Z}(\delta^4_3) = \text{seed}_{(3, B)}(\delta^4_3)\). By Lemma 6.4, \(\text{seed}_{(3, B)}(\delta^4_3) / \text{seed}_{(3, B)}(\delta^4_3) < j^{Y'} \circ j^{T}(\delta^4_3) = j^{Y'}(\delta^4_3)\).

\(\Box\)

Lemma 6.6. Suppose \(Y\) is a finite level \(\leq 3\) tree and \(A \in \text{desc}^{**}(3Y)\). Suppose \(\|A\|_{3Y} = \xi\) and \(R\) is a finite level \(\leq 3\) tree such that \([\emptyset]_{3R} = \hat{\xi}\). Then \(\text{seed}_{(3, A)} = j^{R}(\delta^4_3)\).

Proof. If \(A = (\emptyset, \emptyset, 0)\), this is exactly Lemma 6.5. Suppose \(A \neq (\emptyset, \emptyset, 0)\). Let \(T\) be a finite level \(\leq 3\) tree and let \(\rho\) minimally factor \((R, Y, T)\). Let \(B \in \text{desc}^{3}(Y, T, *)\) such that \(\text{lh}(B) = 1\) and \([B]_{3Y' \otimes T} = \hat{\xi}\). Put \(B = (y, \pi) \in \text{desc}^{3}(Y, T, Q)\). A routine computation gives \(A \sim_{\ast} (y, \pi, T \otimes Q)\).

So \(\text{seed}_{(3, B)}(\delta^4_3) = \text{seed}_{(3, A)}(\delta^4_3)\). Put \(Z = Y \otimes T \mid (3, B)\). Then \([\emptyset]_{3Z} = \emptyset_{3R}\). By Lemma 6.3, \(j^{Z}(\delta^4_3) = \text{seed}_{(3, B)}(\delta^4_3)\). By Lemma 6.5, \(j^{R}(\delta^4_3) = j^{Z}(\delta^4_3)\), and we are done.

\(\Box\)

Definition 6.7. In view of Lemma 6.6, we define the level-3 uniform indiscernibles:

1. \(u^{(3)}_{\xi+1} = j^{R}(\delta^4_3)\) when \(\xi < \omega^\omega\), \(R\) is a \(\Pi^1_3\)-wellfounded level \(\leq 3\) tree and \([\emptyset]_{3R} = \hat{\xi}\).

2. If \(0 < \xi \leq \omega^\omega\) is a limit, then \(u^{(3)}_{\xi} = \sup_{\eta < \xi} u^{(3)}_{\eta}\).

If \(R\) is a finite level \(\leq 3\) tree and \([\emptyset]_{3R} = \hat{\xi}\), then the set of \(R\)-uniform indiscernibles is \(\{u^{(3)}_{\eta} : 0 < \eta \leq \xi + 1\}\) and we have \(\text{seed}_{(3, A)}^{(3)} = u^{(3)}_{\eta+1}\) for \(\|A\|_{3R} = \eta\).

The next lemma is the higher level analog of \(\delta^4_2 = u_2\).

Lemma 6.8. Assume \(\Pi^1_3\)-determinacy. Then \(\delta^4_1 = u^{(3)}_2\).

Proof. If \(W\) is a \(\Sigma^1_1(x)\)-wellfounded relation on \(\mathbb{R}\), then \(W\) is \(\delta^4_1\)-Suslin via a tree in \(L[T, x]\), so by Kunen-Martin and Lemma 5.42, \(\text{rank}(W) < ((\delta^4_3)^+)^{L[T, x]} = \min(j^{R^0}(L^{(3)}_{\alpha^\#}) \setminus (\delta^4_3 + 1)) < u^{(3)}_2\) as \([\emptyset]_{R^0} = \omega = 1\). If \(\alpha < u^{(3)}_2\), pick \(x\) such that \(\alpha < \min(j^{R^0}(L^{(3)}_{\alpha^\#}) \setminus (\delta^4_3 + 1))\). Lemma 5.42 gives a surjection \(f : \delta^4_3 \to \alpha\) which is definable over \(j^{R^0}(M_{2, \infty}(x))\) from \(\delta^4_3, x\). From \(f\) we can define a \(\Lambda^1_4(x^\#)\)-wellordering of length \(\alpha\).

\(\Box\)
6.3 The level-4 Martin-Solovay tree

Let $R^\infty$ be the unique (up to an isomorphism) level-3 tree such that

1. for any finite level-3 tree $Y$, there exists $\rho$ which minimally factors $(Y, R)$;

2. if $r \in \text{dom}(R^\infty)$ then there exist a finite $Y$ and $\rho$ which minimally factors $(Y, R)$ such that $r \in \text{dom}(\rho)$.

In other words, $R^\infty$ is the minimum $\Pi_3^1$-wellfounded level-3 tree that is universal for finite level-3 trees in terms of minimal factorings. We fix the following representation of $R^\infty$, whose domain consists of finite tuples of ordinals in $\omega^{\omega^\omega}$:

1. $(\xi_1) \in \text{dom}(R^\infty)$ iff $0 < \xi_1 < \omega^{\omega^\omega}$. $R^\infty((\xi_1))$ is the $Q^0$-partial level $\leq 2$ tree induced by $\xi_1$.

2. If $r = (\xi_1, \ldots, \xi_{k-1}) \in \text{dom}(R^\infty)$, then $r^-((\xi_k)) \in \text{dom}(R^\infty)$ iff $\xi_k < \omega^{\omega^\omega}$ and there exists a completion $Q^+$ of $R^\infty(r)$ such that the $Q^+$-approximation sequence of $\xi_k$ is $(\xi_i)_{1 \leq i \leq k}$; if $r^-((\xi_k)) \in \text{dom}(R^\infty)$ and $Q^+$ is the unique such completion, then $R^\infty(r^-((\xi_k)))$ is the $Q^+$-partial level $\leq 2$ tree induced by $\xi_k$.

Therefore, $[\emptyset]_R = u_\omega$ and if $r = (\xi_1, \ldots, \xi_k) \in \text{dom}(R^\infty)$, then $[r]_R = \hat{\xi}_k$. If $Y$ is a finite level-3 tree, then the map $y \mapsto r_y$ minimally factors $(Y, R)$, where if $([y]_R)_{1 \leq i \leq \text{lh}(y)} = (\xi_1, \ldots, \xi_{\text{lh}(y)})$ then $r_y = (\xi_1, \ldots, \xi_{\text{lh}(y)})$. If $0 < \xi < \omega^{\omega^\omega}$, let $R^\infty_\xi = R^\infty | (\xi)$. By Lemma 6.3, if $0 < \xi \leq \omega^{\omega^\omega}$, then the factoring map $j_{R^\infty_\xi}^R_\xi : R^\infty_\xi \rightarrow R^\infty$ is the identity on $L_{j_{R^\infty_\xi}^R_\xi[\delta^1_{3}\upharpoonright T_3]}$ and $j_{R^\infty_\xi}^R_\xi(\delta^1_{3}) = \text{seed}(\xi_\xi) = u_\xi^{(3)}$.

In particular, $j_{R^\infty_\xi}^R_\xi(\delta^1_{3}) = u_\omega^{(3)}$.

$R^\infty$ will be the tree based on which level-3 sharp codes for ordinals below $u_\omega^{(3)}$ are defined. LO(3) is the set of $v \in \mathbb{R}$ such that $X_v$ is a linear ordering of $u_\omega$, where $v \mapsto X_v$ is the $\Delta^1_{3}$ surjection from $\mathbb{R}$ onto $P((V_\omega \cup u_\omega)^{<\omega})$, defined in Corollary 2.12 and renamed in the beginning of Section 4.9. WO(3) = WO(3)$_0$ is the set of $v \in \text{LO}(3)$ such that $X_v$ is a wellordering of $u_\omega$. WO(3)$_0$ is $\Pi^1_3$. For $v \in \text{WO}(3)$, put $\|v\| = \text{o.t.}(X_v)$. Every ordinal in $\delta^1_{3}$ is of the form $\|v\|$ for some $v \in \text{WO}(3)$.

A level-3 sharp code is a pair $(r^\tau, x^3\#)$ where $\tau$ is an $L^x_{\infty,R^\infty}$-Skolem term for an ordinal without free variables. For $0 < \xi \leq \omega^{\omega^\omega}$, $\text{WO}_{\xi}^{(3)}$ is the set of
level-3 sharp codes \( \langle \tau, x^3 \rangle \) such that \( \tau \) is an \( \mathcal{L}^{\omega_1^\omega} \)-Skolem term. \( \text{WO}^{(3)}_\xi \) is \( \Pi^1_1 \) for \( 0 < \xi \leq \omega^\omega \). The ordinal coded by \( \langle \tau, x^3 \rangle \) is

\[
\left| \langle \tau, x^3 \rangle \right| = \tau^{J_{\omega^\omega}(M_{\omega^\omega}^\omega); \text{seed}^{R^\omega}}.
\]

For each \( \xi \), \( \text{WO}^{(3)}_\xi \) is \( \Pi^1_1 \). By Lemma 6.9, if \( \langle \tau, x^3 \rangle \in \text{WO}^{(3)}_\xi \) and \( \tau = \sigma(x, c_{r_1}, \ldots) \), \( \sigma \) is an \( \mathcal{L} \)-Skolem term, then

\[
\left| \langle \tau, x^3 \rangle \right| = \sigma^{J_{\omega^\omega}(M_{\omega^\omega}^\omega)}(x, \text{seed}^{R^\omega}_{r_1}, \ldots).
\]

**Lemma 6.9.** The relations \( v, w \in \text{WO}^{(3)}_{\omega^\omega} \land |v| = |w| \) and \( v, w \in \text{WO}^{(3)}_{\omega^\omega} \land |v| < |w| \) are both \( \Delta^1_1 \).

**Proof.** \( \left| \langle \tau, x^3 \rangle \right| = \left| \langle \tau, x', x'' \rangle \right| \) if \( \tau = \sigma(x, c_{r_1}, \ldots) \) and \( \sigma' = \sigma(x, c_{r_1}, \ldots) \) are \( \mathcal{L} \)-Skolem terms, and for some finite level-3 tree \( Y \) and some \( \rho \) factoring \( (Y, R^\omega) \) such that \( \tau^{\omega^3 \omega^3} \subseteq \text{ran}(\rho) \), \( \sigma((x)^{\text{left}}, c_{p_1}^{\text{left}}, \ldots) = \sigma'((x)^{\text{right}}, c_{p_2}^{\text{right}}) \) is true in \( (x + x')^{3\omega} \).

Recall that \( \text{WO}^{(3)}_{\omega^\omega} \) is the set of (level-1) sharp codes for ordinals below \( u_{\omega^\omega} \). The connection between level-3 sharp codes and level-1 sharp codes or \( \text{WO} \) is also \( \Delta^1_1 \). For instance, the relation \( \langle v, x \rangle : v \in \text{WO}^{(3)}_{\omega^\omega} \land |v| = |x| \rangle \) is in \( \Delta^1_1 \).

If \( \Gamma \) is a pointclass, say that \( A \subseteq u_{\omega^\omega}^{(3)} \times R \) is in \( \Gamma \) iff \( \{ (v, x) : v \in \text{WO}^{(3)}_{\omega^\omega} \land |v|, x \in A \} \) is in \( \Gamma \). \( \Gamma \) acting on subsets of product spaces is defined in the obvious way.

**Lemma 6.10.**

1. Suppose that \( \xi \leq \eta < \omega^\omega \) and \( \rho \) factors \( (R^\omega_\xi, R^\omega_\eta) \). Then \( \rho^{R^\omega_\xi} \upharpoonright u^{(3)}_\xi \) is \( \Delta^1_1 \) uniformly in \( (\xi, \eta, \rho) \).

2. Suppose that \( \xi < \omega^\omega \) and \( Q, Q' \) are finite level \( \leq 2 \) trees, \( Q \) is a subtree of \( Q' \). Then \( j^{(Q, Q')}(R^\omega_\xi) \upharpoonright u^{(3)}_\xi \) is \( \Delta^1_1 \) uniformly in \( (\xi, Q, Q') \).

**Proof.**

1. \( \alpha < u^{(3)}_\xi \land \rho^{R^\omega_\xi} \upharpoonright (\alpha) = \beta \) iff there exist \( x \in R \), an \( \mathcal{L} \)-Skolem term \( \tau \) and \( r_1, \ldots, r_n \) such that \( r_i \in \text{dom}(R^\omega_\xi) \) for any \( i \) and \( \alpha = \langle \tau(x, c_{r_1}, \ldots, c_{r_n})^\tau, x^3 \rangle \), \( \beta = \langle \tau(x, c_{r_1}, \ldots, c_{r_n})^\tau, x^3 \rangle \).

2. \( \alpha < u^{(3)}_\xi \land j^{(Q, Q')}(\alpha) = \beta \) iff there exist \( x \in R \), an \( \mathcal{L} \)-Skolem term \( \tau \) and \( r_1, \ldots, r_n \) such that \( r_i \in \text{dom}(R^\omega_\xi) \) for any \( i \) and \( \alpha = \langle \tau(x, c_{r_1}, \ldots, c_{r_n})^\tau, x^3 \rangle \), \( \beta = \langle \tau(x, c_{r_1}, \ldots, c_{r_n})^\tau, x^3 \rangle \). \( \square \)
By Lemma 5.42, the set of uncountable $\mathbb{L}[j^{R^\infty}(T_3)]$-cardinals $\leq u^{(3)}_{\omega^\omega}$ is the closure of

$$\{u_n : 1 \leq n < \omega\} \cup \{\text{seed}^{R^\infty}_A : A \in \text{desc}^{**}(R^\infty)\}.$$  

By Lemma 6.4, if $A = (r, \pi, T) \in \text{desc}^{**}(R^\infty)$, $r = (r, Q, (d, q, P))$, $r = (\xi_1)_{1 \leq i \leq k}$ and $\text{seed}^{R^\infty}_A > \delta^1_3$ is a successor cardinal in $\mathbb{L}[j^{R^\infty}(T_3)]$, then $r$ is of discontinuous type, $\xi_k$ is a successor ordinal, and letting $r' = (\xi_1)_{1 \leq i < k} - (\xi_k - 1)$, $r' = (r', Q, (d, q, P))$, $A' = (r', \pi, T)$, then

$$\{\langle x^{3#}, \tau^{T}(\langle x, c_A \rangle) \rangle : x \in \mathbb{R}, \tau \text{ is an } \mathcal{L} \text{-Skolem term for an ordinal}\}$$

is a cofinal subset of $\text{seed}^{R^\infty}_A$.

A level-3 EM blueprint over a real $\Gamma$ is completely decided by $\Gamma(R^\infty)$. $\Gamma$ is coded into the real $z_T \in 2^\omega$ where $z(k) = 0 \leftrightarrow k \in \Gamma(R^\infty)$. We shall identify $\Gamma$ with $z_T$ when no confusion occurs. We define the level-4 Martin-Solovay tree $T_4$ which projects to $\{x^{3#} : x \in \mathbb{R}\}$. $T_4$ will be $\Delta^1_5$ as a subset of $(\omega \times u^{(3)}_{\omega^\omega})^c$, the complexity based on Lemma 6.10.

Let $T$ be a recursive tree so that $z \in [T]$ iff $z$ is a remarkable level-3 EM blueprint over a real. Let $(r_i)_{1 \leq i < \omega}$ be an effective enumeration of $\text{dom}(R^\infty)$ and let $(\tau_k)_{k < \omega}$ be an effective enumeration of all the $\mathcal{L}$-Skolem terms for an ordinal, where $\tau_k$ is $f(k) + 1$-ary. $T_4$ is the tree on $2 \times u^{(3)}_{\omega^\omega}$ where

$$(t, \vec{a}) \in T_4$$

iff $t \in T$ and

1. if $\xi \leq \eta < \omega^\omega$, $r_1, \ldots, r_{f(k)} \in \text{dom}(R^\infty_\xi)$, $r_1, \ldots, r_{f(l)} \in \text{dom}(R^\infty_\eta)$, $\rho$ factors $(R^\infty_\xi, R^\infty_\eta)$,

   (a) if $\tau^{R^\infty_\eta}(\vec{a}_1, c_{r_1}, \ldots, c_{r_{f(k)}})$ is true in $t$, then

   $$\rho^{R^\infty_\eta}(\alpha_k) = \alpha_l;$$

   (b) if $\tau^{R^\infty_\eta}(\vec{a}_1, c_{r_1}, \ldots, c_{r_{f(k)}}) < \tau^{R^\infty_\eta}(\vec{a}_1, c_{r_1}, \ldots, c_{r_{f(l)}})$ is true in $t$, then

   $$\rho^{R^\infty_\eta}(\alpha_k) < \alpha_l;$$

2. if $\xi < \omega^\omega$, $r_1, \ldots, r_{\max(f(k), f(l))} \in \text{dom}(R^\infty|\xi)$, $Q, Q'$ are finite level $\leq 2$ trees, $Q$ is a subtree of $Q'$, $\tau^{R^\infty_\eta}(\vec{a}_1, c_{r_1}, \ldots, c_{r_{f(k)}})$ is true in $t$, then $\tau^{R^\infty_\eta(Q, Q')}(\alpha_k) = \alpha_l$.

**Theorem 6.11.** $p[T_4] = \{x^{3#} : x \in \mathbb{R}\}$. Furthermore, for any $x \in \mathbb{R}$, $(\tau_k^{(j^{R^\infty_\eta}(M_{\infty}(x)))}(x, \text{seed}^{R^\infty_\eta}_1, \ldots, \text{seed}^{R^\infty_\eta}_{f(k)}))_{k<\omega}$ is the honest leftmost branch of $(T_4)_{x^{3#}}$.  

146
Proof. By definition, for any \( x, (x,^\#(\tau^j_{R^\omega}(\mathcal{M}^\omega_\infty(x))))(x, \text{seed}_{\tau_1}, \ldots, \text{seed}_{\tau(f(k))}) \ < \omega) \in [T_1] \). Suppose now \( (z, \vec{\beta}) \in p[T_1] \). Let \( x \) be a real so that \( z \) codes a remarkable level-3 EM blueprint \( \Gamma \) over \( x \). We need to show that \( z \) is iterable and for any \( k, \tau^j_{R^\omega}(\mathcal{M}^\omega_\infty(x))) \leq \beta_k \). For each \( k \), pick a finite subtree \( Y_k \) of \( R^\omega \) and \( F_k : [\delta^1_3]^* \rightarrow \delta^1_3 \) such that \( \{ r_1, \ldots, r_{f(k)} \} \subseteq \text{dom}(Y_k) \), \( F_k \in \mathbb{L}[T_3] \) and \( j^Y_k : R^\omega \to \delta^1_3 \) with \( \delta^1_3[Y_k] = \beta_k \). By \( \mathbb{L}[T_3] \)-countable completeness of the club filter on \( \delta^1_3 \), we can find a club \( C \) in \( \delta^1_3 \) such that \( C \in \mathbb{L}[T_3] \) and

1. if \( \xi \leq \eta < \omega^\omega, Y_k \) is a subtree of \( R^\omega \), \( Y \) is a subtree of \( R^\omega \), \( \rho \) factors \( (R^\omega, \eta) \), \( (\tau, \bar{\gamma}) \in [\delta^1_3]^* \), then \( j^Y_k : [\mathcal{M}^\omega_\infty(x)] \rightarrow \delta^1_3 \) such that \( \delta^1_3[Y_k] = \beta_k \). By \( \mathbb{L}[T_3] \)-countable completeness of the club filter on \( \delta^1_3 \), we can find a club \( C \) in \( \delta^1_3 \) such that \( C \in \mathbb{L}[T_3] \) and

2. if \( \xi < \omega^\omega, Y_k, Y \) are subtrees of \( R^\omega \), \( Q, Q' \) are finite level \( \leq 2 \), \( Q \) is a subtree of \( Q' \), \( \eta \) is true in \( \Gamma(\eta) \), \( \bar{\gamma} \in [\delta^1_3]^* \), then \( j^\omega_\omega(F_k(\bar{\gamma} | \text{dom}(Y_k))) = F_1(\bar{\gamma} | \text{dom}(Y_k)) \).

Suppose \( S \) is a \( \Pi^1_3 \)-wellfounded level-3 tree. We show that \( \mathcal{M}_{\Gamma, S} \) is a \( \Pi^1_3 \)-iterable \( \chi \)-mouse. Put \( N = \mathcal{M}_{\Gamma, S} \). Put \( \eta \in \mathcal{C}_0 \) iff \( C \cap \eta \) has order type \( \eta \), \( \eta \in D \) iff \( \mathcal{C}_0 \cap \eta \) has order type \( \eta \). Fix \( \bar{\gamma} \in [D]^\omega \). We define an embedding

\[ \theta : \text{Ord}^N \rightarrow \delta^1_3 \]

as follows. If \( \sigma \) is an \( \mathcal{L} \)-Skolem term, \( s_1, \ldots, s_n \in \text{dom}(R) \), \( R \) is a finite subtree of \( S \), \( a = (\sigma(c_{s_1}, \ldots, c_{s_n}))^N \), \( \rho \) factors \( (R, Y_k) \), \( \tau_k(c_{s_1}, \ldots, c_{r(f(k))}) \) is logically equivalent to \( \sigma(c_{\rho(s_1)}, \ldots, c_{\rho(s_n)}), \bar{\delta} \in [\mathcal{C}_0]^Y_k \), \( \delta_{\rho(s)} = \gamma_s \) for any \( s \in \text{dom}(R) \), we put

\[ \theta(a) = F_k(\bar{\delta}) \]

\( \theta \) is well defined: Suppose \( \sigma' \) is another \( \mathcal{L} \)-Skolem term, \( s'_1, \ldots, s'_n \in \text{dom}(R') \), \( R' \) is a finite subtree of \( S \), \( \sigma'(c_{s'_1}, \ldots, c_{s'_n}) \) is logically equivalent to \( \sigma'(c_{\rho(s'_1)}, \ldots, c_{\rho(s'_n)}) \) is logically equivalent to \( \sigma'(c_{s_1}, \ldots, c_{s_n}), \bar{\delta}' \in [\mathcal{C}_0]^Y_k \), \( \delta_{\rho(s)} = \gamma_s \) for any \( s \in \text{dom}(R') \). Pick \( \xi, \xi' < \omega^\omega \) such that \( Y_k \) is a subtree of \( R^\omega_k \), \( Y_{k'} \) is a subtree of \( R^\omega_{k'} \). Let \( (Y^*, \psi, \psi', \bar{\epsilon}) \) be the amalgamation of \( (Y_k, \bar{\delta}) \) and \( (Y_{k'}, \bar{\delta}') \), obtained by Lemma 4.73.
That is, $Y^*$ is a finite level-3 tree, $\psi$ factors $(Y_k, Y^*)$, $\psi'$ factors $(Y_k', Y^*)$, $\tilde{c} \in [C_0]^{Y^*+1}$, $\delta_y$ for $y \in \text{dom}(Y_k)$, $\epsilon_{\psi'(y)} = \delta_y$ for $y \in \text{dom}(Y_k')$. So $"\sigma(x, c_{\psi_0(p(s_1))}, \ldots) = \sigma'(x, c_{\psi'_0(p'(s'_1)), \ldots})"$ is true in $\Gamma(Y^*)$. Pick $\eta < \omega^{\omega^\omega}$ large enough so that there exist $\phi, \phi'$ factoring $(R_\xi^\infty, R_\eta^\infty)$, $(R_\xi^\infty, R_\eta^\infty)$ respectively and $\phi^*$ factoring $(Y^*, R_\eta^\infty)$ such that $\phi^* \circ \psi = \phi \upharpoonright \text{dom}(Y_k)$, $\phi^* \circ \psi' = \phi' \upharpoonright \text{dom}(Y_k')$. So $"\sigma(x, c_{\psi_0(p(s_1))}, \ldots) = \sigma'(x, c_{\psi'_0(p'(s'_1)), \ldots})"$ is true in $\Gamma(R^\infty)$. Let $\tau_l(x, c_{\tau_l(1)}, \ldots)$ be logically equivalent to $\sigma(x, c_{\psi_0(p(s_1))}, \ldots)$. The fact that every ordinal in $\text{Coll}(\omega, \xi)$ is $\Pi^3_3$-iterable and order preserving. In particular, for any $\gamma \in [D]^{R^\infty\downarrow}$, $\tau_k^{(M_{\bar{\omega}}(x)):\text{seed} R^\infty} \leq \beta_k$. We define an embedding $\theta : \{\tau_k^{(M_{\bar{\omega}}(x)):\gamma} : \gamma \in [D]^{R^\infty\downarrow}, k < \omega\} \to \delta_3^1$ by $\theta(\tau_k^{(M_{\bar{\omega}}(x)):\gamma}) = F_k(\gamma \upharpoonright \text{dom}(Y_k))$. A similar argument shows that $\theta$ is well defined and order preserving. In particular, for any $\gamma \in [D]^{R^\infty\downarrow}$, $\tau_k^{(M_{\bar{\omega}}(x)):\gamma} \leq F_k(\gamma \upharpoonright \text{dom}(Y_k))$. Hence, $\tau_k^{(M_{\bar{\omega}}(x)):\text{seed} R^\infty} \leq \beta_k$. 

\section{The level-4 sharp}

\subsection{The level-4 Kechris-Martin theorem}

For a countable structure $\mathcal{P}$ in the language of premice that satisfies Axioms [13] in Definition 5.14 and the universality of level $\leq 2$ ultrapowers axiom, the direct limit $\mathcal{P}_\infty$ is defined in Definition 7.39. The wellfounded part $\mathcal{P}_\infty$ is always transitive. Recall that every ordinal in $\mathcal{P}_\infty$ is of the form $\pi_{\mathcal{P}, X, \beta, \infty}(a)$ where $a \in \text{Ord}^\mathcal{P}$, $X$ is a finite level $\leq 2$ tree, $\beta \in [\omega_1]^{X\uparrow}$. The relation $"v \in \text{LO}^{(3)}(\beta)"$ codes the order type of $\text{Ord}^{\mathcal{P}_\infty}$ uniformly $\Delta^3_4$ in the code of $\mathcal{P}$.

For $x \in \mathbb{R}$, a putative $x$-3-sharp is a remarkable level-3 EM blueprint over $x$ that satisfies the universality of level $\leq 2$ ultrapowers axiom. Suppose $x^*$
is a putative $x$-3-sharp. For any limit ordinal $\alpha < \delta_3^1$, we can build an EM model

$$\mathcal{M}_{x^*, \alpha}$$

as follows. Let $R$ be a level-3 tree such that $[\emptyset]_R = \alpha$. Then $\mathcal{M}_{x^*, \alpha} = (\mathcal{M}_{x^*, R}^*)$. This definition is independent of the choice of $R$. Suppose $R'$ is another level-3 tree and $[\emptyset]_{R'} = \alpha$, and suppose without loss of generality that $\rho$ minimally factors $(R', R)$ by Theorem 4.71. Then $\rho_{x^*, R'} : \mathcal{M}_{x^*, R}^* \to \mathcal{M}_{x^*, R'}^*$ induces a canonical embedding $\phi : (\mathcal{M}_{x^*, R}^*)^\infty \to (\mathcal{M}_{x^*, R'}^*)^\infty$. Let $T$ be $\Pi^1_2$-wellfounded and let $\psi$ minimally factor $(R', R \otimes T)$. Then $\psi_{x^*, R, T} : \mathcal{M}_{x^*, R}^* \to \mathcal{M}_{x^*, R}^* \mathcal{M}_{x^*, R}^* \mathcal{M}_{x^*, R}^*$ induces a canonical embedding $\phi' : (\mathcal{M}_{x^*, R}^*)^\infty \to (\mathcal{M}_{x^*, R'}^*)^\infty$. By coherency, $\phi' \circ \phi = \text{id}$ and hence $\phi = \phi' = \text{id}$. We say that $x^*$ is $\alpha$-iterable iff $\alpha$ is in the wellfounded part of $\mathcal{M}_{x^*, \alpha}^*$.

A putative level-3 sharp code for an increasing function is $w = \langle \langle \tau^\omega, x^* \rangle \rangle$ such that $x^*$ is a putative $x$-3-sharp, $\tau$ is a unary $\mathcal{L}_\omega$-Skolem term and

$$\forall v, v'((v, v' \in \text{Ord} \land v < v') \to (\tau(v) \in \text{Ord} \land \tau(v) < \tau(v')))$$

is true in $x^*(\emptyset)$. The statement “$\langle \langle \tau^\omega, x^* \rangle \rangle$ is a putative level-3 sharp code for an increasing function, $x^*$ is $\alpha$-iterable, $r$ codes the order type of $\tau^\omega_{x^*, \alpha}(\alpha)$” about $\langle \langle \tau^\omega, x^* \rangle \rangle$ is $\Sigma_1^1$ in the code of $\alpha$. In addition, when $x^* = x^3\#$, $\langle \langle \tau^\omega, x^* \rangle \rangle$ is called a (true) level-3 sharp code for an increasing function.

**Lemma 7.1.** Assume $\Delta_1^1$-determinacy. Suppose $\kappa \leq u_{\omega_3}^{(3)}$ is an uncountable cardinal in $\text{L}[j^{R_\infty}(T_3)]$. If $A$ is a $\Sigma_1^1(x)$ subset of $\kappa$ and $\sup A < \kappa$, then $\exists w \in \Delta_3^1(x) \cap \text{WO}^{(3)}_{\omega_\omega} \ (\sup A < |w| < \kappa)$.

**Proof.** Let $x = 0$. The lemma is trivial if $\kappa$ is a limit cardinal in $\text{L}[j^{R_\infty}(T_3)]$. Suppose now $\kappa$ is a successor cardinal in $\text{L}[j^{R_\infty}(T_3)]$. Let $B$ be $\Pi^1_1$ such that $w \in \text{WO}^{(3)}_{\omega_\omega} \land |w| \in A \iff \exists y \ (w, y) \in B$.

**Case 1:** $\omega_1 \leq \kappa < u_{\omega_\omega}$.

The lower level proof in [23] carries over almost verbatim, except the game becomes $\Sigma_1^1$ for the winner and hence a $\Delta_1^1$ winning strategy can be found by Moschovakis third periodicity [36].

**Case 2:** $\kappa = \delta_3^1 = u_{\omega_1}^{(3)}$.

Suppose $A \subseteq \delta_3^1$ is $\Sigma_1^1$ and $\sup A < \delta_3^1$. Let $B$ be $\Pi^1_1$ such that $w \in \text{WO}^{(3)}_{\omega_\omega} \land |w| \in A \iff \exists y \ (w, y) \in B$. Consider the game in which I produces $v$, II produces $(w, y)$. II wins either $v \notin \text{WO}^{(3)}$ or $v, w \in \text{WO}^{(3)} \land |v| < |w| \land (w, y) \in B$. I has a winning strategy, and so has a $\Delta_1^1$ winning strategy $\tau$ by Moschovakis third periodicity. By boundedness, $\{||\tau \cdot x|| : x \in \mathbb{R}\}$ has a $\Delta_1^1(\tau)$ bound, hence has a $\Delta_1^1$ bound.
Case 3: \( \kappa = \text{seed}^R_A > \delta_3, A \in \text{desc}^*(R^\infty) \).

Put \( A = (r, \pi, T), r = (r, Q, (d, q, P)) \), \( r = (\xi_i)_{1 \leq i < k} \). Then \( r \) is of discontinuous type and \( \xi_k \) is a successor ordinal. Put \( r' = (\xi_i)_{1 \leq i < k - 1}, r' = (r', Q, (d, q, P)) \), \( A' = (r', \pi, T) \).

Consider the game in which I produces \( \langle r, \tau \rangle, a^* \), II produces \( (\langle \sigma, \tau \rangle, b^*), y \). II wins iff

1. If \( \langle \tau, \eta \rangle \) is a putative level-3 sharp code for an increasing function, then so is \( \langle \tau, \tau \rangle, b^* \). Moreover, for any \( \eta < \delta^1_3 \), if

\[
\begin{align*}
a^* & \text{ is } \eta \text{-iterable } \wedge \tau^{M^{a^* \eta}}(\eta) \in \text{wfp}(M^{a^* \eta}) \\
\end{align*}
\]

then

\[
\begin{align*}
b^* & \text{ is } \eta \text{-iterable } \wedge \sigma^{M^{b^* \eta}}(\eta) \in \text{wfp}(M^{b^* \eta}) \wedge \tau^{M^{a^* \eta}}(\eta) \in \text{wfp}(M^{a^* \eta}) < \sigma^{M^{a^* \eta}}(\eta).
\end{align*}
\]

2. If \( \langle \tau, \eta \rangle \) is a true level-3 sharp code for an increasing function, \( a^* = a^3^# \), then \( \langle \langle \tau \rangle \rangle^{\text{wfp}}(\langle \sigma, \tau \rangle, b^*), y \rangle \in B \).

This game is \( \Sigma^1_4 \) for Player I. Player I has a winning strategy, and so has a \( \Delta^1_5 \) winning strategy \( f \). Let \( \sigma \) be the \( L_\infty^\omega \)-Skolem term for \( c^{(3)}_{y, T, \pi, \tau}(c_\overline{z}^\omega) + \omega \) where \( \overline{z} = \gamma^3 \). Let

\[
w = \langle \tau, \gamma^3 \rangle^{\text{Skolem}}\]

So \( w \in \text{WO}^{(3)}_\omega \) is \( \Delta^1_5 \) and \( |w| < \text{seed}^R_A \). We show that \( \sup A < |w| \) using a boundedness argument. For each \( \eta < \delta^1_3 \), let \( Z_\eta \) be the set of \( r \in \text{LO}^{(3)} \) such that there are putative level-3 sharp codes for increasing function on ordinals \( \langle r, \tau \rangle, \), \( \langle \tau, \tau \rangle, b^* \) and an ordinal \( \beta \leq \eta \) such that

1. \( \langle r, \tau \rangle, a^* = f * \langle \sigma, \tau \rangle, b^* \rangle \);

2. for any \( \beta < \beta, b^* \) is \( \beta \)-iterable, \( \sigma^{M^{b^* \eta}}(\beta) \in \text{wfp}(M^{b^* \eta}), \sigma^{M^{a^* \eta}}(\beta) \leq \eta \);

3. \( a^* \) is \( \beta \)-iterable, \( \tau^{M^{a^* \eta}}(\beta) \) has order type coded in \( r \).

\( Z_\eta \) is a \( \Sigma^1_3 \) set in the code of \( \eta \). Since \( f \) is a winning strategy for I, \( Z_\eta \subseteq \text{WO}^{(3)} \).

By Corollary 5.3, \( \{ |r| : r \in Z_\eta \} \) is bounded by \( c^{(3)}_{f, \eta + \omega} \). Hence, if \( \langle \tau, \eta \rangle \) is a true level-3 sharp code for an increasing function \( g \) and \( \langle \tau, a^* \rangle = f * \langle \tau, b^* \rangle \), then \( \langle \tau, a^* \rangle \) is a true level-3 sharp code for an increasing function \( h \) and for any \( \eta < \delta^1_3 \) such that \( g^\eta \eta \subseteq \eta \), \( h(\eta) < c^{(3)}_{f, \eta + \omega} \). Let \( \eta \in C \) iff \( g^\eta \eta \subseteq \eta \). By Lemma 5.33, for any \( \gamma \in c^T(C), c^T(h)(\gamma) < c^{(3)}_{f, \gamma + \omega} \). Hence, \( \sup A < |w| \).
Based on Lemma 7.1, the proof of the following theorem is completely in parallel to the level-2 Kechris-Martin theorem in [23] or [39]. It is proved by induction on the $\mathbb{L}[j^{R^\omega}(T_3)]$-cardinality of $\text{sup}(A)$. A key step uses the following observation: by Lemma 5.42, if $|w| < \text{seed}_{R^\omega}^A$ and $A = \text{succ}_{R^\omega}(B)$ then there is a $\Delta^1_5(w)$ surjection from $\text{seed}_{R^\omega}^B$ onto $|w|$.

**Theorem 7.2.** Assume $\Delta^1_4$-determinacy. If $A$ is a nonempty $\Pi^1_5(x)$ subset of $u^{(3)}_{\omega^\omega}$, then $\exists w \in \Delta^1_3(x) (|w| \in A)$. So the pointclass $\Pi^1_5$ is closed under quantification over $u^{(3)}_{\omega^\omega}$.

**Definition 7.3.** $\kappa_5^x$ is the least $(T_4, x)$-admissible ordinal.

Using Theorem 7.2, we obtain the level-4 version of Theorem 2.1. The proof is parallel to [23] and [3], using Moschovakis set induction in one direction and the Becker-Kechris game in the other direction.

**Theorem 7.4.** Assume $\Delta^1_4$-determinacy. Then for each $A \subseteq u^{(3)}_{\omega^\omega} \times \mathbb{R}$, the following are equivalent.

1. $A$ is $\Pi^1_5$.
2. There is a $\Sigma^1_1$ formula $\varphi$ such that $(\alpha, x) \in A$ iff $L_{\kappa_5^x}[T_4, x] \models \varphi(T_4, \alpha, x)$.

The ordinal $\kappa_5^x$ is defined in a different way in [27]:

\[
\begin{align*}
\lambda_5^x &= \sup\{|W| : W \text{ is a } \Delta^1_5(x) \text{ prewellordering on } \mathbb{R}\}, \\
\kappa_5^x &= \sup\{\lambda_5^{x,y} : M^\#_3(x) \not\models \Delta^1_5(x, y)\}.
\end{align*}
\]

In parallel to [23], these two definitions are equivalent, and in fact,

\[
\begin{align*}
\lambda_5^x &= \sup\{\xi < \kappa_5^x : \xi \text{ is } \Delta_1\text{-definable over } L_{\kappa_5^x}[T_4, x] \text{ from } \{T_4, x\}\}, \\
\kappa_5^x &= \sup\{\text{o.t.}(W) : W \text{ is a } \Delta^1_5(x, <u^{(3)}_{\omega^\omega}) \text{ wellordering on } \mathbb{R}\}.
\end{align*}
\]

Moreover,

\[
\forall \alpha < u^{(3)}_{\omega^\omega} \exists w \in \text{WO}^{(3)}_{\omega^\omega} (|w| = \alpha \land \lambda_5^{x,w} < \kappa_5^x).
\]

### 7.2 The equivalence of $x^{4\#}$ and $M^\#_3(x)$

Suppose $x$ is a real and $\beta \leq u^{(3)}_{\omega^\omega}$. A subset $A \subseteq \mathbb{R}$ is $\beta$-$\Pi^1_5(x)$ iff there is a $\Pi^1_5(x)$ set $B \subseteq u^{(3)}_{\omega^\omega} \times \mathbb{R}$ such that $A = \text{Diff } B$. $\beta$-$\Pi^1_5(x)$ acting on product spaces of $\omega$ and $\mathbb{R}$ is defined in the obvious way. Lightface $\beta$-$\Pi^1_5$ and boldface $\beta$-$\Pi^1_5$ have the obvious meanings.

In parallel to the proof of Lemma 4.2, we have
Lemma 7.5. Assume $\Delta_1^1$-determinacy. Suppose $\xi < \omega^\omega$ and $m < \omega$. If $A$ is $(u^{(3)}_{\xi+1})^m-\Pi_3^1(x)$, then $A$ is $\mathcal{D}^2((\xi+1)-\Pi_3^1(x))$.

If $S$ is a finite regular level-3 tree, let $S^+$ be the level-3 tree extending $S$ where $\text{dom}(S^+) = \text{dom}(S) \cup \{(1)\}$ and $\text{cf}(S((1))) = 0$. Thus, $[\emptyset]_{S^+} = [\emptyset]_S + \omega$. If $\xi = (\xi_x)_{x \in \text{dom}(S^+)}$ respects $S^+$, put $\hat{\xi}^- = (\xi_x)_{x \in \text{dom}(S)}$.

If particular, $\mathcal{N}_{\xi_x,\alpha,\infty}(x) = \mathcal{M}_{2,\infty}(x)|_{\xi_x,\alpha}$.

Lemma 7.6. Assume $\Delta_1^1$-determinacy. Let $\xi < \omega^\omega$. If $A$ is $\mathcal{D}^2(\hat{\xi}-\Pi_3^1(x))$, then $A$ is $u^{(3)}_{\xi+2}\Pi_3^1(x)$.

Proof. Let $S$ be a regular level-3 tree such that $[\emptyset]_S = \hat{\xi}$. By Lemma 5.12 if $(y,r) \in \mathbb{R}^2$, $C \subseteq \mathbb{R}$ is $\hat{\xi}-\Pi_3^1(y,r)$, then we can effectively find a formula $\varphi$ such that Player I has a winning strategy in $G(C)$ iff

$$[\varphi(y,r)] \in (y,r)^3\#(S).$$

Suppose $A = \mathcal{D}B$, where $B \subseteq \mathbb{R}^2$ is $\mathcal{D}(\hat{\xi}-\Pi_3^1)$. Suppose $\varphi$ is an $\mathcal{L}$-formula such that

$$(y,r) \in B \leftrightarrow [\varphi(y,r, (c_x)_{x \in \text{dom}(S)})] \in (y,r)^3\#(S).$$

For ordinals $\xi$ respecting $S^+$, say that $M$ is a Kechris-Woodin non-determined set with respect to $(y,\xi)$ iff

1. $M$ is a countable subset of $\mathbb{R}$.
2. $M$ is closed under join and Turing reducibility.
3. $\forall \sigma \in M \exists v \in M \mathcal{N}_{\xi((1)),\infty}(y, \sigma \otimes v) \models \neg \varphi(y, \sigma \otimes v, \hat{\xi}^-)$.
4. $\forall \sigma \in M \exists v \in M \mathcal{N}_{\xi((1)),\infty}(y, \sigma \otimes v) \models \varphi(y, \sigma \otimes v, \hat{\xi}^-)$.

Say that $z$ is $(y,\hat{\xi})$-stable iff $z$ is not contained in any Kechris-Woodin non-determined set with respect to $(y,\hat{\xi})$. $z$ is $y$-stable iff $z$ is $(y,\hat{\xi})$-stable for all $\xi$ respecting $S^+$. The set of $(y,z)$ such that $z$ is $y$-stable is $\Pi_3^1$. By the proof of Kechris-Woodin [29], for all $y \in \mathbb{R}$, there is $z \in \mathbb{R}$ which is $y$-stable. Let $<_{\hat{\xi}}$ be the following wellfounded relation on the set of $z$ which is $(y,\hat{\xi})$-stable:

$$z' <_{\hat{\xi}} z \leftrightarrow z' \text{ is } (y,\hat{\xi})\text{-stable} \land z \leq_T z' \land \forall \sigma \leq_T z \exists v \leq_T z' \mathcal{N}_{\xi((1)),\infty}(y, \sigma \otimes v) \models \neg \varphi(y, \sigma \otimes v, \hat{\xi}^-) \land \forall \sigma \leq_T z \exists v \leq_T z' \mathcal{N}_{\xi((1)),\infty}(y, \sigma \otimes v) \models \varphi(y, \sigma \otimes v, \hat{\xi}^-).$$

152
If $z$ is $y$-stable, let $f^z_y$ be the function that sends $\vec{\xi}$ to the rank of $z$ in $<\vec{\xi}$. Then $f^z_y$ is a function into $\delta^1_3$. By $\Sigma^1_4$-absoluteness between $V$ and $\mathcal{N}_{Coll(\omega,\eta)}$, where $\mathcal{N} \in \mathcal{F}_{2,\infty}(y, z)$ and $\pi_{\mathcal{N}, \infty}(\eta) = \xi_{((1))}$, we can see $f^z_y \upharpoonright \{\vec{\xi} \in [\delta^1_3]^{S^+} : \xi_{((1))}\}$ is a cardinal cutpoint of $M_{2,\infty}(y, z)$ is definable over $M_{2,\infty}(y, z)$ in a uniform way, so there is a $L^S$-Skolem term $\tau$ such that for all $(y, z) \in \mathbb{R}^2$, if $z$ is $y$-stable, $\vec{\xi}$ respects $S$, $\xi_{((1))}$ is a cardinal cutpoint of $M_{2,\infty}(y, z)$, then

$$f^z_y(\vec{\xi}) = \tau_{M_{2,\infty}(y, z)}(y, z, \vec{\xi}).$$

Let

$$\beta^z_y = \tau_{S^+(M_{2,\infty}(y, z))}(y, z, \text{seed}^{R^+}).$$

The function $(y, z) \mapsto \beta^z_y$ is $\Delta^1_3$ in the level-3 sharp codes. The rest is in parallel to the proof of Lemma 4.3.

Lemma 7.5 and Lemma 7.6 are concluded in a simple equality between pointclasses.

**Theorem 7.7.** Assume $\Delta^1_4$-determinacy. Then for $x \in \mathbb{R}$,

$$\mathcal{O}^2(\omega^2) = u^{(3)}_{\omega^2}(x).$$

Hence by Theorem 4.5,

$$\mathcal{O}^1(\omega^2) = u^{(3)}_{\omega^2}(x).$$

The level-4 sharp is defined in parallel to the end of Section 4.1.

**Definition 7.8.**

$$\mathcal{O}^{T_4,x} = \{\langle \varphi^\gamma, \alpha \rangle : \varphi \text{ is a } \Sigma_1 \text{-formula, } \alpha < u^{(3)}_{\omega^}\text{-}\Pi^1_3(x) \}.$$ 

**Definition 7.9.**

$$\langle n, \varphi^\gamma, \psi^\gamma \rangle \in \{\langle n, \varphi^\gamma, \psi^\gamma \rangle : \exists \alpha < u^{(3)}_{\omega^}\text{-}\Pi^1_3(x) \land \forall \eta < \alpha(\varphi^\gamma, \psi^\gamma, \eta) \in \mathcal{O}^{T_4,x} \}.$$ 

Applying Theorem 4.7 to the space $\omega$, in combination with Theorem 4.4, we reach the equivalence between $x^4$ and $M^\#_3(x)$.

**Theorem 7.10.** Assume $\Delta^1_4$-determinacy. Then $x^4$ is many-one equivalent to $M^\#_3(x)$, the many-one reductions being independent of $x$.  

153
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