Sparse-grid polynomial interpolation approximation and integration for parametric and stochastic elliptic PDEs with lognormal inputs

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June 10, 2019

Abstract

By combining a certain approximation property in the spatial domain, and weighted $\ell_2$-summability of the Hermite polynomial expansion coefficients in the parametric domain obtained in [M. Bachmayr, A. Cohen, R. DeVore and G. Migliorati, ESAIM Math. Model. Numer. Anal. 51(2017), 341-363] and [M. Bachmayr, A. Cohen, D. Dũng and C. Schwab, SIAM J. Numer. Anal. 55(2017), 2151-2186], we investigate linear non-adaptive methods of fully discrete polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs. We explicitly construct such methods and prove corresponding convergence rates of the approximations by them. The linear non-adaptive methods of fully discrete polynomial interpolation approximation are sparse-grid collocation methods which are certain sums taken over finite nested Smolyak-type indices sets $G(\xi)$ parametrized by positive number $\xi$, of mixed tensor products of dyadic scale successive differences of spatial approximations of particular solvers, and of successive differences of their parametric Lagrange interpolating polynomials. The Smolyak sparse grids in the parametric domain are constructed from the roots of Hermite polynomials or their improved modifications. Moreover, they generate fully discrete weighted quadrature formulas in a natural way for integration of the solution to parametric and stochastic elliptic PDEs and its linear functionals, and the error of the corresponding integration can be estimated via the error in the Bochner space $L_1(\mathbb{R}^\infty, V, \gamma)$ norm of the generating methods where $\gamma$ is the Gaussian probability measure on $\mathbb{R}^\infty$ and $V$ is the energy space. Our analysis leads to auxiliary convergence rates in parameter $\xi$ of these approximations when $\xi$ going to $\infty$. For a given $n \in \mathbb{N}$, we choose $\xi_n$ so that the cardinality of $G(\xi_n)$ which in some sense characterizes computation complexity, does not exceed $n$, and hence obtain the convergence rates in increasing $n$, of the fully discrete polynomial approximation and integration.

Keywords and Phrases: High-dimensional approximation; Parametric and stochastic elliptic PDEs; Lognormal inputs; Collocation approximation; Fully discrete non-adaptive polynomial interpolation approximation; Fully discrete non-adaptive integration.

1 Introduction

One of basic problems in Uncertainty Quantification are approximation and numerical methods for parametric and stochastic PDEs. Since the number of parametric variables may be very large or even infinite, they are treated as high-dimensional or infinite-dimensional approximation problems. Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the diffusion elliptic equation

$$-\text{div}(a \nabla u) = f \quad \text{in} \quad D, \quad u|_{\partial D} = 0,$$

(1.1)

for a given fixed right-hand side $f$ and spatially variable scalar diffusion coefficient $a$. Denote $V := H^1_0(D)$ - the energy space and $V' := H^{-1}(D)$. If $a$ satisfies the ellipticity assumption

$$0 < a_{\min} \leq a \leq a_{\max} < \infty,$$

by the well-known Lax-Milgram lemma, for any $f \in V'$, there exists a unique solution $u \in V$ in weak form which satisfies the variational equation

$$\int_D a \nabla u \cdot \nabla v \, dx = \langle f, v \rangle, \quad \forall v \in V.$$

We consider diffusion coefficients having a parametrized form $a = a(y)$, where $y = (y_1, y_2, ...)$ is a sequence of real-valued parameters ranging in the set $\mathbb{U}^\infty$ which is either $\mathbb{R}^\infty$ or $\mathbb{I}^\infty := [-1, 1]^\infty$. The resulting solution to parametric and stochastic elliptic PDEs map $y \mapsto u(y)$ acts from $\mathbb{U}^\infty$ to the solution to parametric and stochastic elliptic PDEs space $V$. The objective is to achieve numerical approximation of this complex map by a small number of parameters with some guaranteed error in a given norm. Depending on the nature of the modeled object, the parameter $y$ may be either deterministic or random. In the present paper, we consider the so-called lognormal case when $\mathbb{U}^\infty = \mathbb{R}^\infty$ and the diffusion coefficient $a$ is of the form

$$a(y) = \exp(b(y)), \quad b(y) = \sum_{j=1}^{\infty} y_j \psi_j,$$

(1.2)

where the $y_j$ are i.i.d. standard Gaussian random variables and $\psi_j \in L_\infty(D)$. Note that the above countably infinite-dimensional setting comprises its finite-dimensional counterpart by setting $\psi_j = 0$ for $j$ large enough.

We also briefly consider the affine case when $\mathbb{U}^\infty = \mathbb{I}^\infty$ and the diffusion coefficient $a$ is of the form

$$a(y) = \bar{a} + \sum_{j=1}^{\infty} y_j \psi_j.$$

(1.3)

In order to study fully discrete approximations of the solution $u(y)$ to the parametrized elliptic PDEs (1.1), we assume that $f \in L^2(D)$ and $a(y) \in W^1_\infty(D)$, and hence we obtain that $u(y)$ has the second higher regularity, i. e., $u(y) \in W$ where $W$ is the space

$$W := \{v \in V : \Delta v \in L^2(D)\}$$

equipped with the norm

$$\|v\|_W := \|\Delta v\|_{L^2(D)},$$

which coincides with the Sobolev space $V \cap H^2(D)$ with equivalent norms if the domain $D$ has $C^{1,1}$ smoothness, see [19, Theorem 2.5.1.1]. Moreover, we assume that there holds the following approximation property for the spaces $V$ and $W$.
Assumption I There are a sequence $(V_n)_{n \in \mathbb{N}_0}$ of subspaces $V_n \subset V$ of dimension $\leq n$, and sequence $(P_n)_{n \in \mathbb{N}_0}$ of linear operators from $V$ into $V_n$, and a number $\alpha > 0$ such that

$$\|P_n(v)\|_V \leq C, \quad \|v - P_n(v)\|_V \leq C n^{-\alpha} \|v\|_W, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in W. \quad (1.4)$$

A basic role in the approximation and integration for parametric and stochastic PDEs are generalized polynomial chaos expansions for the dependence on the parametric variables. We refer the reader to [9, 13, 20, 29, 28] and references there for different aspects in approximation for parametric and stochastic PDEs. In [6]–[11], based on the conditions $\left(\|\psi_j\|_{W_p(D)}\right)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ on the affine expansion (1.3) for some $0 < p < 1$, the authors have proven the $\ell_p$-summability of the coefficients in a Taylor or Legendre polynomials expansion and hence proposed best adaptive $n$-term approximation methods in energy norm by choosing the set of the $n$ most useful terms in these expansions. The obtained $n$-term approximands are then approximated by finite element methods. Similar results have been received in [24] for the lognormal case based on the conditions $\left(\|\psi_j\|_{W_p(D)}\right)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$ some $0 < p < 1$. In these papers, they did not take into account support properties of the functions $\psi_j$.

A different approach to studying summability that takes into account the support properties has been recently proposed in [1] for the affine case and [2] for the lognormal case. This approach leads to significant improvements on the results on $\ell_p$-summability when the functions $\psi_j$ have limited overlap, such as splines, finite elements or wavelet bases. These results by themselves do not imply practical applications, because they do not cover the approximation of the expansion coefficients which are functions of the spatial variable.

In the recent paper [3], the rates of fully discrete adaptive best $n$-term Taylor and Galerkin approximations for elliptic PDEs with affine or lognormal parametrizations of the diffusion coefficients have been obtained based on combining a certain approximation property on the spatial domain, and extensions of the results on $\ell_p$-summability of [1, 2] to higher-order Sobolev norms of corresponding Taylor, Jacobi and Hermite expansion coefficients. These results providing a benchmark for convergence rates, are not constructive. In the case when $\ell_p$-summable sequences of Sobolev norms of expansion coefficients have an $\ell_p$-summable majorant sequence, these convergence rates can be achieved by non-adaptive linear methods of Galerkin and collocation approximations in the affine case [9, 14, 15, 16, 32, 33]. However, this non-adaptive approach is not applicable for the improvement of $\ell_p$-summability in [1, 2, 3] since the weakened $\ell_p$-assumption leads only to the $\ell_p$-summability of expansion coefficients, but not to an $\ell_p$-summable majorant sequence. Non-adaptive (not fully discrete) approach has been considered in [1, Remark 3.2] and [2, Remark 5.1] for Galerkin approximation, in [18] for polynomial collocation approximation, and in [4] for weighted integration.

In the present paper, by combining spatial and parametric approximability, namely, the approximation property in Assumptions I in the spatial domain and weighted $\ell_2$-summability of the $V$ and $W$ norms of Hermite polynomial expansion coefficients obtained in [2, 3], we investigate linear non-adaptive methods of fully discrete Galerkin approximation and polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs (1.2). We explicitly construct such methods and prove corresponding convergence rates of the approximations by them. We show that the rate of best adaptive fully discrete $n$-term Galerkin approximation obtained in [3], is achieved by linear non-adaptive methods of fully discrete Galerkin approximation. The linear non-adaptive methods of fully discrete polynomial interpolation approximation are sparse-grid collocation methods which are certain sums taken over finite nested Smolyak-type indices sets $G(\xi)$ parametrized by positive number $\xi$, of tensor products of dyadic scale successive differences of spatial approximations of particular solvers, and of successive differences of their parametric Lagrange interpolating polynomials. The Smolyak sparse grids in the parametric domain are constructed from the roots of Hermite polynomials or their improved modifications. Moreover, these methods generate fully discrete weighted quadrature formulas in a natural way for integration of the solution $u(y)$ and its linear functionals, and the error of the
corresponding integration can be estimated via the error in the space \( L_1(\mathbb{R}^\infty, V, \gamma) \) norm of the generating methods where \( \gamma \) is the Gaussian probability measure on \( \mathbb{R}^\infty \). Our analysis leads to auxiliary convergence rates in parameter \( \xi \) of these approximations when \( \xi \) going to \( \infty \). For a given \( n \in \mathbb{N} \), we choose \( \xi_n \) so that the cardinality of \( G(\xi_n) \) which in some sense measures computation complexity, does not exceed \( n \), and hence obtain the convergence rates in increasing \( n \), of the fully discrete polynomial approximation and integration. The convergence rate of fully discrete integration better than the convergence rate of the generating fully discrete polynomial interpolation approximation due to the simple but useful observation that the integral \( \int v(y) \, d\gamma(y) \) is zero if \( v(y) \) is an odd function and \( \gamma \) is the Gaussian probability measure on \( \mathbb{R} \). (This property has been used in [32, 33, 34] for improving convergence rate of integration in the affine case.) This is the main contribution of our paper. We also briefly consider similar problems for parametric and stochastic elliptic PDEs with affine inputs (1.3) by using counterparts-results in [1, 3], and by-product problems of non-fully discrete polynomial interpolation approximation and integration similar to those treated in [18, 4]. In particular, the convergence rate of integration obtained in this paper is better than that in [4].

The paper is organized as follows. In Sections 2–4, we explicitly construct general fully discrete linear methods of Galerkin and polynomial interpolation approximations in the Bochner space \( L_\rho(\mathbb{R}^\infty, X^1, \gamma) \) and quadrature of functions taking values in \( X^2 \) and having a weighted \( \ell_2 \)-summability of Hermite expansion coefficients for Hilbert spaces \( X^1 \) and \( X^2 \) satisfying a certain “spatial” approximation property (see (2.3)). In particular, in Section 2, we prove convergence rates of general fully discrete linear Galerkin methods of approximation; in Section 3, we prove convergence rates of general fully discrete linear polynomial interpolation methods of approximation; in Section 4, we prove convergence rates of general fully discrete linear quadrature for integration. In Section 5, we apply the results of Sections 2–4 to obtain the main results of this paper on convergence rates of linear non-adaptive methods of fully discrete Galerkin and polynomial interpolation approximation as well as fully discrete weighted quadrature methods of integration for parametric and stochastic elliptic PDEs with lognormal inputs. In Section 6, by extending the theory in Sections 2–4, we briefly consider similar problems for parametric and stochastic elliptic PDEs with affine inputs. In all Sections 2–6, we also briefly investigate related non-fully approximation problems.

## 2 Galerkin approximation

In this section, we construct general fully discrete linear Galerkin methods of approximation in the Bochner space \( L_\rho(\mathbb{R}^\infty, X^1, \gamma) \) with \( 0 < \rho \leq 2 \) and the infinite tensor product Gaussian probability measure \( \gamma \) of functions taking values in \( X^2 \) and having a weighted \( \ell_2 \)-summability of Hermite expansion coefficients for Hilbert spaces \( X^1 \) and \( X^2 \) satisfying a certain “spatial” approximation property. In particular, we prove convergence rates for these methods of approximation.

We first recall a concept of infinite tensor product of probability measures. (For details see, e.g., [23, pp. 429–435].) Let \( \mu(y) \) be a probability measure on \( U \), where \( U \) is either \( \mathbb{R} \) or \( \mathbb{I} := [-1, 1] \). We introduce the probability measure \( \mu(y) \) on \( U^\infty \) as the infinite tensor product of probability measures \( \mu(y_j) \):

\[
\mu(y) := \bigotimes_{j \in \mathbb{N}} \mu(y_j), \quad y = (y_j)_{j \in \mathbb{N}} \in U^\infty.
\]

The sigma algebra for \( \mu(y) \) is generated by the set of cylinders \( A := \prod_{j \in \mathbb{N}} A_j \), where \( A_j \subset U \) are univariate \( \mu \)-measurable sets and only a finite number of \( A_i \) are different from \( U \). For such a set \( A \), we have \( \mu(A) := \prod_{j \in \mathbb{N}} \mu(A_j) \). If \( \varrho(y) \) is the density of \( \mu(y) \), i.e., \( d\mu(y) = \varrho(y) dy \), then we write

\[
d\mu(y) := \bigotimes_{j \in \mathbb{N}} \varrho(y_j) dy_j, \quad y = (y_j)_{j \in \mathbb{N}} \in U^\infty.
\]
Let $X$ be a Hilbert space and $0 < p < \infty$. The probability measure $\mu(y)$ induces the Bochner space $L_p(U^\infty, X, \mu)$ of strongly $\mu$-measurable mappings $v$ from $U^\infty$ to $X$ which are $p$-summable. The (quasi-)norm in $L_p(U^\infty, X, \mu)$ is defined by
\[
\|v\|_{L_p(U^\infty, X, \mu)} := \left(\int_{U^\infty} \|v(\cdot, y)\|^p_X d\mu(y)\right)^{1/p}.
\]

In the present paper, we focus our attention mainly to the lognormal case with $U^\infty = \mathbb{R}^\infty$ and $\mu(y) = \gamma(y)$, the infinite tensor product Gaussian probability measure. Let $\gamma(y)$ be the probability measure on $\mathbb{R}$ with the standard Gaussian density:
\[
d\gamma(y) := g(y)\,dy, \quad g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.
\]
Then the infinite tensor product Gaussian probability measure $\gamma(y)$ on $\mathbb{R}^\infty$ can be defined by
\[
d\gamma(y) := \bigotimes_{j \in \mathbb{N}} g(y_j) d(y_j), \quad y = (y_j)_{j \in \mathbb{N}} \in \mathbb{R}^\infty.
\]

A powerful strategy for the approximation of functions $v$ in $L_2(U^\infty, X, \gamma)$ is based on the truncation of polynomial expansions by the Hermite series
\[
v(y) = \sum_{s \in \mathbb{F}} v_s H_s(y), \quad v_s \in X.
\]
Here $\mathbb{F}$ is the set of all sequences of non-negative integers $s = (s_j)_{j \in \mathbb{N}}$ such that their support $\text{supp}(s) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set, and
\[
H_s(y) = \bigotimes_{j \in \mathbb{N}} H_{s_j}(y_j), \quad v_s := \int_{\mathbb{R}^\infty} v(y) H_s(y) d\gamma(y), \quad s \in \mathbb{F},
\]
with $(H_k)_{k \geq 0}$ being the Hermite polynomials normalized according to $\int_{\mathbb{R}} |H_k(y)|^2 g(y)\,dy = 1$. It is well-known that $(H_s)_{s \in \mathbb{F}}$ is an orthonormal basis of $L_2(\mathbb{R}^\infty, \mathbb{R}, \gamma)$. Moreover, $L_2(U^\infty, X, \gamma) = L_2(U^\infty, \mathbb{R}, \gamma) \otimes X$, and for every $v \in L_2(U^\infty, X, \gamma)$ represented by the series (2.1) there holds Parseval’s identity
\[
\|v\|_{L_2(U^\infty, X, \gamma)}^2 = \sum_{s \in \mathbb{F}} \|v_s\|_X^2.
\]

We make use the abbreviations: $L_p(\mathbb{R}^\infty, \mu) := L_p(\mathbb{R}^\infty, \mathbb{R}, \mu)$; $V_p(X) := L_p(\mathbb{R}^\infty, X, \gamma)$ for $0 < p < \infty$. We use letter $C$ to denote a general positive constant which may take different values, and $C_{p, q, n, D, \ldots}$ a constant depending on $p, q, \alpha, D, \ldots$.

Let $X^1$ and $X^2$ be Hilbert spaces. To construct general fully discrete linear Galerkin methods of approximation in the Bochner space $L_p(\mathbb{R}^\infty, X^1, \gamma)$ of functions taking values in $X^2$, we need the following assumption on approximation property for $X^1$ and $X^2$.

**Assumption II** The Hilbert space $X^2$ a linear subspace of the Hilbert space $X^1$, and $\| \cdot \|_{X^1} \leq C \| \cdot \|_{X^2}$. There are a sequence $(V_n)_{n \in \mathbb{N}_0}$ of subspaces $V_n \subset X^1$ of dimension $\leq n$, and sequence $(P_n)_{n \in \mathbb{N}_0}$ of linear operators from $X^1$ into $V_n$, and a number $\alpha > 0$ such that
\[
\|P_n(v)\|_{X^1} \leq C, \quad \|v - P_n(v)\|_{X^1} \leq Cn^{-\alpha}\|v\|_{X^2}, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in X^2.
\]
For \( k \in \mathbb{N}_0 \), we define
\[
\delta_k(v) := P_{2^k}(v) - P_{2^{k-1}}(v), \quad k \in \mathbb{N}, \quad \delta_0(v) = P_0(v).
\]

We can represent every \( v \in X^2 \) by the series
\[
v = \sum_{k=0}^{\infty} \delta_k(v)
\]
converging in \( X^1 \) and satisfying the estimate
\[
\|\delta_k(v)\|_{X^1} \leq C 2^{-\alpha k} \|v\|_{X^2}, \quad k \in \mathbb{N}_0.
\]

(2.4)

For a subset \( G \) in \( \mathbb{N}_0 \times \mathbb{F} \), denote by \( V(G) \) the subspace in \( V_2(X^1) \) of all functions \( v \) of the form
\[
v = \sum_{(k,s) \in G} v_k H_s, \quad v_k \in V_{2^k}.
\]

Let Assumption II hold for Hilbert spaces \( X^1 \) and \( X^2 \). We define the linear operator \( S_G : V_2(X^2) \to V(G) \) by
\[
S_G v := \sum_{(k,s) \in G} \delta_k(v_s) H_s
\]
for \( v \in V_2(X^2) \) represented by the series
\[
v = \sum_{s \in \mathbb{F}} v_s H_s, \quad v_s \in X^2.
\]

(2.5)

**Lemma 2.1** Let Assumption II hold for Hilbert spaces \( X^1 \) and \( X^2 \). Then for every \( v \in V_2(X^2) \),
\[
\lim_{K \to \infty} \|v - S_{G_K} v\|_{V_2(X^1)} = 0,
\]
where \( G_K := \{(k,s) \in \mathbb{N}_0 \times \mathbb{F} : 0 \leq k \leq K\} \).

**Proof.** Obviously, by the definition,
\[
S_{G_K} v = \sum_{s \in \mathbb{F}} \sum_{k=0}^{K} \delta_k(v_s) H_s = \sum_{s \in \mathbb{F}} P_{2^k}(v_s) H_s.
\]

From Parseval’s identity and (2.3) it follows that
\[
\|S_{G_K} v\|_{V_2(X^1)}^2 = \sum_{s \in \mathbb{F}} \|P_{2^k}(v_s)\|_{X^1}^2 \leq \sum_{s \in \mathbb{F}} \|v_s\|_{X^1}^2 + 2 \sum_{s \in \mathbb{F}} \|v_s - P_{2^k}(v_s)\|_{X^1}^2
\]
\[
\leq \sum_{s \in \mathbb{F}} \|v_s\|_{X^1}^2 + 2 C^2 2^{-\alpha k} \sum_{s \in \mathbb{F}} \|v_s\|_{X^2}^2 < \infty.
\]

(2.7)

This means that \( S_{G_K} v \in V_2(X^1) \). Hence, by Parseval’s identity and (2.3) we deduce that
\[
\|v - S_{G_K} v\|_{V_2(X^1)}^2 = \sum_{s \in \mathbb{F}} \|v_s - P_{2^k}(v_s)\|_{X^1}^2 \leq C^2 2^{-2\alpha K} \sum_{s \in \mathbb{F}} \|v_s\|_{X^2}^2 = C^2 2^{-2\alpha K} \|v\|_{V(X^2)}^2
\]
which prove the lemma.

\[
\text{D}
\]
Theorem 2.1 Let $0 < p \leq 2$. Let Assumption II hold for Hilbert spaces $X^1$ and $X^2$. Let $v \in V_2(X^2)$ be represented by the series (2.5). Assume that for $r = 1, 2$ there exist sequences $(\sigma_{r,s})_{s \in F}$ of numbers strictly larger than 1 such that

$$\sum_{s \in F} (\sigma_{r,s}\|v_s\|_{X^r})^2 < \infty$$

and $(\sigma_{r,s}^{-1})_{s \in F} \in \ell_{q_r}(F)$ for some $0 < q_1 \leq q_2 < \infty$. Define for $\xi > 0$

$$G(\xi) := \begin{cases} \{ (k, s) \in \mathbb{N}_0 \times F : 2^k \sigma_{2,s}^2 \leq \xi \} & \text{if } \alpha \leq 1/q_2; \\ \{ (k, s) \in \mathbb{N}_0 \times F : \sigma_{1,s}^q \leq \xi, \ 2^{\alpha q_1 k} \sigma_{2,s}^q \leq \xi \} & \text{if } \alpha > 1/q_2. \end{cases} \quad (2.8)$$

Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $\dim(V(G(\xi_n))) \leq n$ and

$$\|v - S_{G(\xi_n)}v\|_{V_2(X^1)} \leq C n^{-\min(\alpha, \beta)}. \quad (2.9)$$

The rate $\alpha$ corresponds to the approximation of a single function in $X^2$ as given by (2.3), and the rate $\beta$ is given by

$$\beta := 1 - \frac{\alpha}{q_1 \alpha + \delta}, \quad \delta := 1 - \frac{1}{q_1 / q_2}. \quad (2.10)$$

The constant $C$ in (2.9) is independent of $v$ and $n$.

Proof. Due to the inequality $\|v\|_{V_2(X^1)} \leq \|v\|_{V_2(X^1)}$, it is sufficient to prove the theorem for $p = 2$.

We first consider the case $\alpha \leq 1/q_2$. Let $\xi > 0$ be given and take arbitrary positive number $\varepsilon$. Since $G(\xi)$ is finite, from the definition of $G_K$ and Lemma 2.1 it follows that there exists $K = K(\xi, \varepsilon)$ such that $G(\xi) \subset G_K$ and

$$\|v - S_{G_K}v\|_{V_2(X^1)} \leq \varepsilon. \quad (2.11)$$

By the triangle inequality,

$$\|v - S_{G(\xi)}v\|_{V_2(X^1)} \leq \|v - S_{G_K}v\|_{V_2(X^1)} + \|S_{G_K}v - S_{G(\xi)}v\|_{V_2(X^1)}. \quad (2.12)$$

We have by Parseval's identity and (2.4) that

$$\|S_{G_K}v - S_{G(\xi)}v\|_{V_2(X^1)} = \left\| \sum_{s \in F} \delta_{k}(v_s) H_s - \sum_{s \in F} \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} \delta_{k}(v_s) H_s \right\|_{V_2(X^1)}$$

$$= \left\| \sum_{s \in F} \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} \delta_{k}(v_s) H_s \right\|_{V_2(X^1)}^2 \leq \sum_{s \in F} \left( \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} \|\delta_{k}(v_s)\|_{X^1} \right)^2 \leq C \sum_{s \in F} \left( \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} \|v_s\|_{X^2}^2 \right)^2 \leq C \sum_{s \in F} \|v_s\|_{X^2}^2 \left( \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} 2^{-\alpha k} \|v_s\|_{X^2} \right)^2 \leq C \sum_{s \in F} \|v_s\|_{X^2}^2 \left( \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} (\xi \sigma_{2,s}^2)^{-\alpha} \right)^2 \leq C \sum_{s \in F} \|v_s\|_{X^2}^2 \left( \sum_{\xi \sigma_{2,s}^2 < 2^k \leq 2^K} (\sigma_{2,s}^{-q_2})^{-2\alpha} \right).$$

Hence, by the inequalities $q_2 \alpha \leq 1$ and $\sigma_{2,s} > 1$ we derive that

$$\|S_{G_K}v - S_{G(\xi)}v\|_{V_2(X^1)}^2 \leq C \sum_{s \in F} (\sigma_{2,s}\|v_s\|_{X^2})^2 = C \xi^{-2\alpha}. \quad (2.13)$$

Therefore, we have

$$\|v - S_{G(\xi)}v\|_{V_2(X^1)} \leq \varepsilon + C \xi^{-2\alpha}. \quad (2.14)$$

Since $\xi > 0$ is arbitrary, we have

$$\|v - S_{G(\xi)}v\|_{V_2(X^1)} \leq C \xi^{-2\alpha}. \quad (2.15)$$

We can replace $G(\xi)$ by $G_K$ in (2.11) and get

$$\|v - S_{G_K}v\|_{V_2(X^1)} \leq C \xi^{-2\alpha}.$$
Since $\varepsilon > 0$ is arbitrary, from the last estimates and (2.11) and (2.12) we derive that

$$\|v - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)} \leq C \xi^{-\alpha}. \quad (2.13)$$

For the dimension of the space $V(G(\xi))$ we have that

$$\dim V(G(\xi)) \leq \sum_{(k,s)\in G(\xi)} \dim V_{2^k} \leq \sum_{(k,s)\in G(\xi)} 2^k$$

$$\leq \sum_{a\in F} \sum_{2^k \leq \xi \sigma_{2^k}^{-q_2}} 2^k \leq 2 \sum_{a\in F} \xi \sigma_{2^k}^{-q_2} = M \xi,$$

where $M := 2\left(\sigma_{2^k}^{-1}\right)^{q_2}$. For any $n \in \mathbb{N}$, letting $\xi_n$ be a number satisfying the inequalities

$$M \xi_n \leq n < 2M \xi_n, \quad (2.14)$$

we derive that $\dim V(G(\xi_n)) \leq n$. On the other hand, from (2.14) it follows that $\xi_n^{-\alpha} \leq (2M)^\alpha n^{-\alpha}$. This together with (2.13) proves that

$$\|v - S_{G(\xi_n)}v\|_{V_2(\mathcal{X}^1)} \leq C n^{-\alpha}, \quad \alpha \leq 1/q_2. \quad (2.15)$$

We now consider the case $\alpha > 1/q_2$. Putting

$$v_\xi := \sum_{\sigma_{1,\xi} \leq \xi} v_\sigma H_\sigma,$$

we get

$$\|v - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)} \leq \|v - v_\xi\|_{V_2(\mathcal{X}^1)} + \|v_\xi - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)}.$$

The norms in the right hand side can be estimated using Parseval’s identity and the hypothesis of the theorem. Thus, or the norm $\|v - v_\xi\|_{V_2(\mathcal{X}^1)}$ we have that

$$\|v - v_\xi\|_{V_2(\mathcal{X}^1)}^2 = \sum_{\sigma_{1,\xi} > \xi^{1/q_1}} \|v_\sigma\|_{X_1}^2 = \sum_{\sigma_{1,\xi} > \xi^{1/q_1}} (\sigma_{1,\xi}\|v_\sigma\|_{X_1})^2 \sigma_{1,\xi}^{-2}$$

$$\leq \xi^{-2/q_1} \sum_{\sigma_{1,\xi} > \xi^{1/q_1}} (\sigma_{1,\xi}\|v_\sigma\|_{X_1})^2 \leq C \xi^{-2/q_1}. \quad (2.16)$$

Anh for the norm $\|v_\xi - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)}$, with $N = N(\xi, s) := 2^{\log_2(\xi^{1/q_1}\sigma_{2^k}^{-1/n})}$ we have that

$$\|v_\xi - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)}^2 = \sum_{\sigma_{1,\xi} > \xi^{1/q_1}} \|v_\sigma - \sum_{2^{q_1<s} \sigma_{2^k} \leq \xi} \delta_k(v_\sigma)\|_{X_1}^2 \leq \sum_{a\in F} \|v_\sigma - P_N(v_\sigma)\|_{X_1}^2$$

$$\leq \sum_{a\in F} C N^{-2\alpha}\|v_\sigma\|_{X_2} \leq C \xi^{-2/q_1} \sum_{a\in F} (\sigma_{2^k\xi}\|v_\sigma\|_{X_1})^2 \leq C \xi^{-2/q_1}.$$

These estimates yield that

$$\|v - S_{G(\xi)}v\|_{V_2(\mathcal{X}^1)} \leq C \xi^{-1/q_1}. \quad (2.17)$$
For the dimension of the space $V(G(\xi))$, with $q := q_2\alpha > 1$ and $1/q' + 1/q = 1$ we have that

$$
\dim V(G(\xi)) \leq \sum_{(k,s) \in G(\xi)} \dim V_{2k} \leq \sum_{2^k \leq \xi} \sum_{2^{n_1+s_2} \leq \xi} 2^k
$$

$$
\leq 2 \sum_{\sigma_1 \leq \xi} \xi^{1/(q_1\alpha)} \sigma_2^{-1/\alpha} \leq 2 \xi^{1/(q_1\alpha)} \left( \sum_{\sigma_1 \leq \xi} \sigma_2^{-q_2} \right)^{1/q} \left( \sum_{\sigma_1 \leq \xi} 1 \right)^{1/q'}
$$

$$
\leq 2 \xi^{1/(q_1\alpha)} \left( \sum_{s \in F} \sigma_2^{-q_2} \right)^{1/q} \left( \sum_{s \in F} \xi^{q_1/s} \right)^{1/q'} = M \xi^{1+\delta/\alpha},
$$

where $M := 2 \left\| (\sigma_2^{-1}) \right\|_{\ell^q(F)} \left\| (\sigma_1^{-1}) \right\|_{\ell^q(F)}$. For any $n \in \mathbb{N}$, letting $\xi_n$ be a number satisfying the inequalities

$$
M \xi_n^{1+\delta/\alpha} \leq n < 2M \xi_n^{1+\delta/\alpha},
$$

(2.18)

we derive that $\dim V(G(\xi_n)) \leq n$. On the other hand, by (2.18),

$$
\xi_n^{-1/q_1} \leq (2M)^{\frac{a}{q_1\alpha}} n^{\frac{1}{q_1}}
$$

This together with (2.17) proves that

$$
\| v - S_{G(\xi_n)} v \|_{V_2(X^1)} \leq Cn^{-\beta}, \quad \alpha > 1/q_2.
$$

(2.19)

By combining the last estimate and (2.15) we obtain (2.9).

3 Polynomial interpolation approximation

In this section, we construct general fully discrete linear polynomial interpolation methods of approximation in the Bochner space $V_p(X^1)$ of functions taking values in $X^2$ and having a weighted $\ell_2$-summability of Hermite expansion coefficients for Hilbert spaces $X^1$ and $X^2$ satisfying a certain “spatial” approximation property. In particular, we prove convergence rates for these methods of approximation. We also briefly consider non-fully discrete linear polynomial interpolation methods of approximation.

3.1 Auxiliary results

Let $w = \exp(-Q)$, where $Q$ is an even function on $\mathbb{R}$ and $yQ'(y)$ is positive and increasing in $(0, \infty)$, with limits 0 and $\infty$ at 0 and $\infty$. For $n \in \mathbb{N}$, the $n$th Mhaskar-Rakhmanov-Saff number $a_n = a_n(w)$ is defined as the positive root of the equation

$$
n = 2 \pi \int_0^1 a_n y Q'(a_n y) \sqrt{1 - y^2} \, dy.
$$

From [25, Page 11] we have for the weight $\sqrt{g}$,

$$
a_n(\sqrt{g}) = \sqrt{n}.
$$

(3.1)

For $0 < p, q \leq \infty$, we introduce the quantity

$$
\delta(p, q) := \frac{1}{2} \left| \frac{1}{p} - \frac{1}{q} \right|.
$$
Lemma 3.1 Let $0 < p, q \leq \infty$. Then exists a positive constant $C_{p,q}$ such that for every polynomial $\varphi$ of degree $\leq n$, there holds the Nikol’skii-type inequality
\[
\|\varphi \sqrt{g}\|_{L_p(\mathbb{R})} \leq C_{p,q} n^{\delta(p,q)} \|\varphi \sqrt{g}\|_{L_q(\mathbb{R})}.
\]

Proof. This lemma is an immediate consequence of (3.1) and the inequality
\[
\|\varphi \sqrt{g}\|_{L_p(\mathbb{R})} \leq C_{p,q} N_n(p,q) \|\varphi \sqrt{g}\|_{L_q(\mathbb{R})}
\]
which follows from [25, Theorem 9.1, p. 61], where
\[
N_n(p,q) := \begin{cases} 
 a_n^{1/p - 1/q}, & p < q, \\
 \left( \frac{n}{a_n} \right)^{1/q - 1/p}, & p > q.
\end{cases}
\]

Lemma 3.2 We have
\[
\|H_n \sqrt{g}\|_{L_\infty(\mathbb{R})} \leq 1, \quad n \in \mathbb{N}_0.
\] (3.2)

Proof. From Cramér’s bound (see, e.g., [17, Page 208, (19)]) we have for every $n \in \mathbb{N}_0$ and every $x \in \mathbb{R}$,
\[
|H_n(x) \sqrt{g(x)}| \leq K (2\pi)^{-1/4},
\]
where $K := 1.086435$. This implies (3.2).

For our application the estimate (3.2) is sufficient, see [12] for an improvement.

For $\theta, \lambda \geq 0$, we define the sequence
\[
p_s(\theta, \lambda) := \prod_{j \in \mathbb{N}} (1 + \lambda s_j)^\theta, \quad s \in \mathcal{F}.
\]
(3.3)

Lemma 3.3 Let $0 < p \leq 2$ and $X$ be a Hilbert space. Let $v \in L_2(\mathbb{R}^\infty, X, \gamma)$ be represented by the series (2.1). Assume that there exists a sequence $\sigma = (\sigma_s)_{s \in \mathcal{F}}$ of positive numbers such that
\[
\sum_{s \in \mathcal{F}} (\sigma_s \|v_s\|_X)^2 < \infty.
\]

We have the following.

(i) If $(p_s(\theta, \lambda)\sigma_s^{-1})_{s \in \mathcal{F}} \in \ell_\bar{q}(\mathbb{F})$ for some $0 < q \leq 2$ and $\theta, \lambda \geq 0$, then $(p_s(\theta, \lambda)\|v_s\|_X)_{s \in \mathcal{F}} \in \ell_{\bar{q}}(\mathbb{F})$ for $\bar{q}$ such that $1/\bar{q} = 1/2 + 1/q$.

(ii) If $(\sigma_s^{-1})_{s \in \mathcal{F}} \in \ell_q(\mathbb{F})$ for some $0 < q \leq 2$, then the series (2.1) converges absolutely in $V_p(X)$ to $v$.

Proof. Since $\tau := 2/\bar{q} \geq 1$, with $1/\tau + 1/\tau' = 1$ and $p_s = p_s(\theta, \lambda)$ by the Hölder inequality we have that
\[
\sum_{s \in \mathcal{F}} (p_s \|v_s\|_X)^{\bar{q}} \leq \left( \sum_{s \in \mathcal{F}} (\sigma_s^2 \|v_s\|_X^2)^{\frac{q}{\bar{q}}} \right)^{1/\tau} \left( \sum_{s \in \mathcal{F}} (p_s^\bar{q} \sigma_s^{-q})^{\tau'} \right)^{1/\tau'}
\]
\[
= \left( \sum_{s \in \mathcal{F}} (\sigma_s^2 \|v_s\|_X^2)^{\frac{q}{\bar{q}}} \right)^{\frac{\bar{q}}{2}} \left( \sum_{s \in \mathcal{F}} (p_s \sigma_s^{-1})^q \right)^{1-\bar{q}/2} < \infty.
\]

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This proves the assertion (i).

We have by the inequality \( \bar{q} \leq 1 \) and (i) for \( \theta = \lambda = 0 \),

\[
\sum_{s \in \mathcal{F}} \|v_s H_s\|_{V_2(X)} = \sum_{s \in \mathcal{F}} \|v_s\|_{H_s} \leq \left( \sum_{s \in \mathcal{F}} \|v_s\|_{X}^{\bar{q}} \right)^{1/\bar{q}} < \infty.
\]

This yields that the series (2.1) absolutely converges in \( V_2(X) \) to \( v \). The assertion (ii) is proven for the case \( p = 2 \). The case \( 0 < p < 2 \) is derived from the case \( p = 2 \) and the inequality \( \| \cdot \|_{V_p(X)} \leq \| \cdot \|_{V_2(X)} \).

**Lemma 3.4** Let \( 0 < p \leq 2 \). Let Assumption II hold for Hilbert spaces \( X^1 \) and \( X^2 \), and let the assumptions of Lemma 3.3(ii) hold for the space \( X^2 \). Then every \( v \in V_2(X^2) \) can be represented as the series

\[
v = \sum_{(k,s) \in \mathbb{N}_0 \times \mathcal{F}} \delta_k(v_s) H_s
\]

converging absolutely in \( V_p(X^1) \).

**Proof.** Similarly to the proof of Lemma 3.3, it is sufficient to prove the lemma for the case \( p = 2 \). With \( \bar{q} \) as in Lemma 3.3, we have by (2.3), Lemma 3.3(ii) and the inequality \( \bar{q} \leq 1 \),

\[
\sum_{(k,s) \in \mathbb{N}_0 \times \mathcal{F}} \|\delta_k(v_s) H_s\|_{V_2(X^1)} = \sum_{(k,s) \in \mathbb{N}_0 \times \mathcal{F}} \|\delta_k(v_s)\|_{X^1} \leq C \sum_{(k,s) \in \mathbb{N}_0 \times \mathcal{F}} 2^{-\alpha k} \|v_s\|_{X^2}
\]

\[
\leq C \sum_{s \in \mathcal{F}} \|v_s\|_{X^2} \sum_{k \in \mathbb{N}_0} 2^{-\alpha k} \leq C \sum_{s \in \mathcal{F}} \|v_s\|_{X^2}
\]

\[
\leq C \left( \sum_{s \in \mathcal{F}} \|v_s\|_{X^2}^{\bar{q}} \right)^{1/\bar{q}} < \infty.
\]

This means that the series in (3.4) converges absolutely to \( v \), since by Lemma 2.6 the sum \( S_{G_K} \) converges in \( V_2(X) \) to \( v \) when \( K \to \infty \).

We will need the following two lemmata for application in estimating the convergence rate of the fully discrete polynomial interpolation approximation in this section and of integration in Section 4.

**Lemma 3.5** Under the hypothesis of Theorem 2.1, assume in addition that \( q_1 < 2 \). Define for \( \xi > 0 \)

\[
G(\xi) := \begin{cases} 
{k \in \mathbb{N}_0 \times \mathcal{F} : 2^k \sigma_{2i_s}^{q_1} \leq \xi} & \text{if } \alpha \leq 1/q_2 - 1/2; \\
{k \in \mathbb{N}_0 \times \mathcal{F} : \sigma_{1i_s}^{q_1} \leq \xi, 2^{\alpha k} \sigma_{2i_s}^{q_1} \leq \xi} & \text{if } \alpha > 1/q_2 - 1/2.
\end{cases}
\]

Then for each \( \xi > 0 \),

\[
\|v - S_{G(\xi_0)} v\|_{V_p(X^1)} \leq C \times \begin{cases} 
\xi^{-\alpha} & \text{if } \alpha \leq 1/q_2 - 1/2; \\
\xi^{1/q_1-1/2} & \text{if } \alpha > 1/q_2 - 1/2.
\end{cases}
\]

The rate \( \alpha \) is given by (2.3). The constant \( C \) in (3.6) is independent of \( v \) and \( \xi \).
Proof. Similarly to the proof of Lemma 3.3, it is sufficient to prove the lemma for the case \( p = 2 \). Since in the case \( \alpha \leq 1/q_2 - 1/2 \), the formulas (2.8) and (3.5) define the same set \( G(\xi) \) for \( \xi > 0 \), from (2.13) follows the lemma for this case. Let us consider the case \( \alpha > 1/q_2 - 1/2 \). Putting
\[
v_\xi := \sum_{\sigma_1^{1,\alpha} \leq \xi} v_s H_s,
\]
we get
\[
\|v - S_G(\xi)v\|_{V_2(X^1)} \leq \|v - v_\xi\|_{V_2(X^1)} + \|v_\xi - S_G(\xi)v\|_{V_2(X^1)}.
\]
(3.7)
It is well known that the unconditional convergence in a Banach space follows from the absolute convergence. Thus, by Lemma 3.4 the series (3.4) converges unconditionally to \( v \). Hence the norm \( \|v - v_\xi\|_{V_2(X^1)} \) can be estimated as
\[
\|v - v_\xi\|_{V_2(X^1)} \leq \sum_{\sigma_1^{1,\alpha} > \xi^{1/q_1}} \|v_s\|_{X^1} \|H_s\|_{L_2(\mathbb{R}^\infty, \gamma)} = \sum_{\sigma_1^{1,\alpha} > \xi^{1/q_1}} \|v_s\|_{X^1}
\]
\[
\leq \left( \sum_{\sigma_1^{1,\alpha} > \xi^{1/q_1}} (\sigma_1^{1,\alpha} \|v_s\|_{X^1})^2 \right)^{1/2} \left( \sum_{\sigma_1^{1,\alpha} > \xi^{1/q_1}} \sigma_1^{-2} \right)^{1/2}
\]
\[
\leq C \left( \sum_{\sigma_1^{1,\alpha} > \xi^{1/q_1}} \sigma_1^{-q_1} \sigma_1^{-1/(2 - q_1)} \right)^{1/2}
\]
\[
\leq C \xi^{-(1/q_1 - 1/2)} \left( \sum_{s \in F} \sigma_1^{-q_1} \right)^{1/2} \leq C \xi^{-(1/q_1 - 1/2)}.
\]
(3.8)
For the norm \( \|v_\xi - S_G(\xi)v\|_{V_2(X^1)} \), with \( N = N(\xi, s) := 2^{(\log_2(\xi^{1/(q_1\alpha)} \sigma_2^{-1/\alpha}))} \) we have
\[
\|v_\xi - S_G(\xi)v\|_{V_2(X^1)} \leq \sum_{\sigma_1^{1,\alpha} \leq \xi} \|v_s - P_N(v_s)\|_{X^1} \leq \sum_{\sigma_1^{1,\alpha} \leq \xi} N^{-\alpha} \|v_s\|_{X^2}
\]
\[
\leq C \sum_{\sigma_1^{1,\alpha} \leq \xi} (\xi^{1/(q_1\alpha)} \sigma_2^{-1/\alpha})^{-\alpha} \|v_s\|_{X^2} \leq C \xi^{-1/q_1} \sum_{\sigma_1^{1,\alpha} \leq \xi} \sigma_2^{\alpha} \|v_s\|_{X^2}
\]
\[
\leq C \xi^{-1/q_1} \left( \sum_{\sigma_1^{1,\alpha} \leq \xi} (\sigma_2^{\alpha} \|v_s\|_{X^2})^2 \right)^{1/2} \left( \sum_{\sigma_1^{1,\alpha} \leq \xi} 1 \right)^{1/2}
\]
\[
\leq C \xi^{-1/q_1} \left( \sum_{\sigma_1^{1,\alpha} \leq \xi} \sigma_1^{-q_1} \xi \right)^{1/2} \leq C \xi^{-(1/q_1 - 1/2)} \left( \sum_{s \in F} \sigma_1^{-q_1} \right)^{1/2} \leq C \xi^{-(1/q_1 - 1/2)}.
\]
This, (3.7) and (3.8) prove the lemma for the case \( \alpha > 1/q_2 - 1/2 \).

We make use the notation: \( F_{ev} := \{ s \in F : s_j \text{ even, } j \in \mathbb{N} \} \). The following lemma can be proven in a similar way.
Lemma 3.6 Let $0 < p \leq 2$. Let Assumption II hold for Hilbert spaces $X^1$ and $X^2$. Let $v \in V_2(X^2)$ be represented by the series
\[ v = \sum_{s \in \mathbb{E}} v_s H_s, \quad v_s \in X^2. \] (3.9)
Assume that for $r = 1, 2$ there exist sequences $(\sigma_{r,s})_{s \in \mathbb{E}}$ of numbers strictly larger than 1 such that
\[ \sum_{s \in \mathbb{E}} (\sigma_{r,s} \|v_s\|_{X^r})^2 < \infty \]
and $(\sigma_{r,-1})_{s \in \mathbb{E}} \in \ell_q(\mathbb{E})$ for some $0 < q_1 \leq q_2 < \infty$ with $q_1 < 2$. Define for $\xi > 0$,
\[ G_{ev}(\xi) := \begin{cases} \{(k, s) \in \mathbb{N}_0 \times \mathbb{E} : 2^k \sigma_{2,s}^q \leq \xi\} & \text{if } \alpha \leq 1/q_2 - 1/2; \\ \{(k, s) \in \mathbb{N}_0 \times \mathbb{E} : \sigma_{1,s}^q \leq \xi, 2^{\alpha q_1 k} \sigma_{2,s}^q \leq \xi\} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \] (3.10)
Then for each $\xi > 0$,
\[ \|v - S_{G_{ev}(\xi_\alpha)} v\|_{V_\rho(X^1)} \leq C \times \begin{cases} \xi^{-\alpha} & \text{if } \alpha \leq 1/q_2 - 1/2; \\ \xi^{-(1/q_1 - 1/2)} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases} \] (3.11)
The rate $\alpha$ is given by (2.3). The constant $C$ in (3.11) is independent of $v$ and $\xi$.

3.2 Interpolation approximation

For every $n \in \mathbb{N}_0$, let $Y_n = (y_{n,k})_{k=0}^n$ be a sequence of points in $\mathbb{R}$ such that
\[ -\infty < y_{n;0} < \cdots < y_{n;n-1} < y_{n;n} < +\infty; \quad y_{0;0} = 0. \] (3.12)
If $v$ is a function defined on $\mathbb{R}$ taking values in a Hilbert space $X$, we define the interpolation formula $I_n(v)$ for $n \in \mathbb{N}_0$ by
\[ I_n(v) := \sum_{k=0}^n v(y_{n,k}) \ell_{n,k}, \quad \ell_{n,k}(y) := \prod_{j=0, j \neq k}^{n} \frac{y - y_{n,j}}{y_{n,k} - y_{n,j}}, \] (3.13)
as the unique Lagrange polynomial interpolating $v$ at $y_{n,k}$. Notice that for a function $v$ on $\mathbb{R}$, the function $I_n(v)$ is polynomial of degree $\leq n$ and $I_n(\varphi) = \varphi$ for every polynomial $\varphi$ of degree $\leq n$.

For univariate functions on $\mathbb{R}$, let
\[ \lambda_n(Y_n) := \sup_{\|v\|_{L_\infty(\mathbb{R})} \leq 1} \|I_n(v)\|_{L_\infty(\mathbb{R})} \]
be the Lebesgue constant. We want to choose sequences $Y_n$ so that for some positive numbers $\tau$ and $C$, there holds the inequality
\[ \lambda_n(Y_n) \leq (C n + 1)^\tau, \quad \forall n \in \mathbb{N}_0. \] (3.14)

We present two examples of $Y_n$ satisfying (3.14). The first example is the strictly increasing sequence $Y_n^* = (y_{n,k}^*)_{k=0}^n$ of the roots of $H_{n+1}$. Indeed, it was proven by Matjila and Szabados [26, 27, 30] that
\[ \lambda_n(Y_n^*) \leq C(n + 1)^{1/6}, \quad n \in \mathbb{N}, \]
for some positive constant $C$ independent of $n$ (with the obvious inequality $\lambda_0(Y_0^*) \leq 1$). Hence, for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ independent of $n$ such that

$$\lambda_n(Y_n^*) \leq (C_\varepsilon n + 1)^{1/6 + \varepsilon}, \quad \forall n \in \mathbb{N}_0. \quad (3.15)$$

The minimum distance between consecutive roots $d_{n+1}$ satisfies the inequalities $\frac{\pi \sqrt{2}}{2n+3} < d_{n+1} < \frac{\sqrt{21}}{2n+3}$, see [31, pp. 130–131]. The sequences $Y_n^*$ have been used in [4] for sparse quadrature for high-dimensional integration with the measure $\gamma$, and in [18] polynomial interpolation approximation with the measure $\gamma$.

The inequality (3.15) can be improved if $Y_n^* \geq 2$ is slightly modified by the “method of adding points” suggested by Szabados [30] (for details, see also [25, Section 11]). More precisely, for $n > 2$, he added to $Y_n^* \geq 2$ two points $\pm \zeta_{n-1}$, near $\pm a_{n-1}(g)$, which are defined by the condition $|H_{n-1}|(\zeta_{n-1}) = \|H_{n-1}|(\zeta_{n-1})\|_{L_\infty(\mathbb{R})}$. By this way, he obtained the strictly increasing sequence

$$\bar{Y}_n^* := \{-\zeta_{n-2}, y_{n-2,0}; \ldots, y_{n-2,n-2}, \zeta_{n-1}\}$$

satisfying the inequality

$$\lambda_n(\bar{Y}_n^*) \leq C \log(n-1) \quad (n > 2)$$

which yields that for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ independent of $n$ such that

$$\lambda_n(\bar{Y}_n^*) \leq (C_\varepsilon n + 1)^\varepsilon, \quad \forall n \in \mathbb{N}_0.$$

For a given sequence $(Y_n)_{n=0}^\infty$, we define the univariate operator $\Delta_n^1$ for $n \in \mathbb{N}_0$ by

$$\Delta_n^1 := I_n - I_{n-1},$$

with the convention $I_{-1} = 0$.

**Lemma 3.7** Assume that $(Y_n)_{n=0}^\infty$ is a sequence such that every $Y_n$ satisfies the condition (3.14) for some positive numbers $\tau$ and $C$. Then for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ independent of $n$ such that for every function $v$ on $\mathbb{R}$,

$$\|\Delta_n^1(v)\sqrt{g}\|_{L_\infty(\mathbb{R})} \leq (C_\varepsilon n + 1)^{\tau + \varepsilon}\|v\sqrt{g}\|_{L_\infty(\mathbb{R})}, \quad \forall n \in \mathbb{N}_0, \quad (3.16)$$

whenever the norm in the right-hand side is finite.

**Proof.** From the assumptions we have that

$$\|\Delta_n^1(v)\sqrt{g}\|_{L_\infty(\mathbb{R})} \leq 2(Cn + 1)^\tau\|v\sqrt{g}\|_{L_\infty(\mathbb{R})}, \quad \forall n \in \mathbb{N}_0,$$

which similarly to (3.15) gives (3.16). 

We introduce the operator $\Delta_s^1$ for $s \in F$ by

$$\Delta_s^1(v) := \bigotimes_{j \in \mathbb{N}} \Delta_{s_j}^1(v)$$

for functions $v$ defined on $\mathbb{R}^\infty$ taking values in a Hilbert space $X$, where the univariate operator $\Delta_{s_j}^1$ is applied to the univariate function $v$ by considering $v$ as a function of variable $y_i$ with the other variables held fixed. Next, we introduce the operator $I_\Lambda$ for a given finite set $\Lambda \subseteq F$ by

$$I_\Lambda := \sum_{s \in \Lambda} \Delta_s^1.$$
Let Assumption II hold for Hilbert spaces $X^1$ and $X^2$. We introduce the operator $\mathcal{I}_G$ for a given finite set $G \subset \mathbb{N}_0 \times \mathbb{F}$ by

$$\mathcal{I}_G v := \sum_{(k,s) \in G} \delta_k \Delta^1_s(v)$$

for functions $v$ defined on $\mathbb{R}^\infty$ taking values in a Hilbert space $X^2$.

Notice that $\mathcal{I}_G v$ is a linear non-adaptive method of fully discrete polynomial interpolation approximation which is the sum taken over the indices set $G$, of mixed tensor products of dyadic scale successive differences of “spatial” approximations to $v$, and of successive differences of their parametric Lagrange interpolating polynomials. It has been introduced in [14] (see also [16]). A similar construction for the multi-index stochastic collocation method for computing the expected value of a functional of the solution to elliptic PDEs with random data has been introduced in [21, 22] by using Clenshaw–Curtis points for quadrature.

A set $\Lambda \subset \mathbb{F}$ is called downward closed if the inclusion $s \in \Lambda$ yields the inclusion $s' \in \Lambda$ for every $s' \in \mathbb{F}$ such that $s' \leq s$. The inequality $s' \leq s$ means that $s'_j \leq s_j$, $j \in \mathbb{N}$. A sequence $(\sigma_s)_{s \in \mathbb{F}}$ is called increasing if $\sigma_{s'} \leq \sigma_s$ for $s' \leq s$. Put $R_s := \{s' \in \mathbb{F}: s' \leq s\}$.

**Theorem 3.1** Let $0 < p \leq 2$. Let Assumption II hold for Hilbert spaces $X^1$ and $X^2$. Assume that $(Y_n)_{n \in \mathbb{N}_0}$ is a sequence such that every $Y_n$ satisfies the condition (3.14) for some positive numbers $\tau$ and $C$. Let $v \in \mathcal{V}_2(X^2)$ be represented by the series (2.5). Assume that for $r = 1, 2$ there exist increasing sequences $(\sigma_r,s)_{s \in \mathbb{F}}$ of numbers strictly larger than 1 such that

$$\sum_{s \in \mathbb{F}} (\sigma_{r,s}^\theta s^s(X^r))^2 < \infty$$

and $(p_2(2\theta, \lambda) \sigma_{r,s}^{-1})_{s \in \mathbb{F}} \in \ell_q(\mathbb{F})$ for some $0 < q_1 \leq q_2 < \infty$ with $q_1 < 2$, where

$$\theta := \tau + \varepsilon + 5/4, \quad \lambda := \max(C_{\infty,2}, C_{2,\infty}, C_\varepsilon, 1), \quad (3.17)$$

$C_{\infty,2}, C_{2,\infty}$ are as in Lemma 3.1, $\varepsilon$ is an arbitrary positive number and $C_\varepsilon$ is as in Lemma 3.7. For $\xi > 0$ let $G(\xi)$ be the set defined as in (3.5).

Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|G(\xi_n)| \leq n$ and

$$\|v - \mathcal{I}_G(\xi_n)v\|_{\mathcal{V}_2(X^1)} \leq C n^{-\min(\alpha, \beta)}. \quad (3.18)$$

The rate $\alpha$ corresponds to the approximation of a single function in $X^2$ as given by (2.3). The rate $\beta$ is given by

$$\beta := \left(\frac{1}{q_1} - \frac{1}{2}\right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{q_1} - \frac{1}{q_2}. \quad (3.19)$$

The constant $C$ in (3.18) is independent of $v$ and $n$.

**Proof.** Clearly, by the inequality $\|v\|_{\mathcal{V}_2(X^1)} \leq \|v\|_{\mathcal{V}_2(X^1)}$ it is sufficient to prove the theorem for $p = 2$. By Lemmata 3.3 and 3.4 the series (2.5) and (3.4) converge absolutely, and therefore, unconditionally in the Hilbert space $\mathcal{V}_2(X^1)$ to $v$. We have that $\Delta^1_s H_s = 0$ for every $s \not\leq s'$. Moreover, if $\Lambda \subset \mathbb{F}$ is downward closed set, $I_\Lambda H_s = H_s$ for every $s \in \Lambda$, and hence we can write

$$I_\Lambda v = I_\Lambda \left(\sum_{s \in \mathbb{F}} v_s H_s\right) = \sum_{s \in \mathbb{F}} v_s I_\Lambda H_s = \sum_{s \in \Lambda} v_s H_s + \sum_{s \not\in \Lambda} v_s I_{\Lambda \cap R_s} H_s. \quad (3.20)$$

Let us first consider the case $\alpha \leq 1/2q_1 - 1/2$. Let $\xi > 0$ be given. For $k \in \mathbb{N}_0$, put

$$\Lambda_k := \{s \in \mathbb{F}: (k,s) \in G(\xi)\} = \{s \in \mathbb{F}: \sigma_{2,k,s}^q \leq 2^{-k}\xi\}.$$
Observe that $\Lambda_k = \emptyset$ for all $k > k^* := \lceil \log_2 \xi \rceil$, and consequently, we have that

$$I_{G(\xi)}v = \sum_{k=0}^{k^*} \sum_{s \in \Lambda_k} \delta_k(v_s) H_s + \sum_{k=0}^{k^*} \delta_k(v_{\emptyset}) I_{\Lambda_k \cap R_s} H_s$$

Since the sequence $(\sigma_{2s})_{s \in \mathcal{F}}$ is increasing, $\Lambda_k$ are downward closed sets, and consequently, the sequence $\{\Lambda_k\}_{k=0}^{k^*}$ is nested in the inverse order, i.e., $\Lambda_{k'} \subset \Lambda_k$ if $k' > k$, and $\Lambda_0$ is the largest and $\Lambda_{k^*} = \emptyset$. Therefore, from the unconditional convergence of the series (3.4) to $v$, (3.21) and (3.20) we derive that

$$I_{G(\xi)}v = \sum_{k=0}^{k^*} \sum_{s \in \Lambda_k} \delta_k(v_s) H_s + \sum_{k=0}^{k^*} \delta_k(v_{\emptyset}) I_{\Lambda_k \cap R_s} H_s$$

This implies that

$$v - I_{G(\xi)}v = v - S_{G(\xi)}v - \sum_{k=0}^{k^*} \sum_{s \in \Lambda_k} \delta_k(v_s) I_{\Lambda_k \cap R_s} H_s.$$

Hence,

$$\|v - I_{G(\xi)}v\|_{Y_2(X^1)} \leq \|v - S_{G(\xi)}v\|_{Y_2(X^1)} + \sum_{(k,s) \notin G(\xi)} \|\delta_k(v_s)\|_{X^1} \|I_{\Lambda_k \cap R_s} H_s\|_{L_2(R^\infty, \gamma)}.$$  (3.22)

Lemma 3.5 gives

$$\|v - S_{G(\xi)}v\|_{Y_2(X^1)} \leq C \xi^{-\alpha}.  (3.23)$$

Let us estimate the sum in the right-hand side of (3.22). We have that

$$\|I_{\Lambda_k \cap R_s} H_s\|_{L_2(R^\infty, \gamma)} \leq \sum_{s' \in \Lambda_k \cap R_s} \|\Delta^{1}_{s'}(H_s)\|_{L_2(R^\infty, \gamma)}.$$  (3.24)

We estimate the norm in the sum in the right-hand side. Assuming $s \in \mathcal{F}$ to be such that $\text{supp}(s) \subset \{1, \ldots, J\}$, we have $\Delta^{1}_{s'}(H_s) = \prod_{j=1}^{J} \Delta^{1}_{s'_j}(H_{s_j})$. Since $\Delta^{1}_{s'_j}(H_{s_j})$ is a polynomial of degree $\leq s'_j$ in variable $y_j$, from Lemma 3.1 we obtain that

$$\|\Delta^{1}_{s'}(H_s)\|_{L_2(R^\infty, \gamma)} = \prod_{j=1}^{J} \|\Delta^{1}_{s'_j}(H_{s_j})\|_{L_2(R, \gamma)} = \prod_{j=1}^{J} \|\Delta^{1}_{s'_j}(H_{s_j})\|_{L_2(R)} \|\sqrt{\gamma}\|_{L_2(R)}.$$

where $\theta = 1/4$, $\lambda := C_{2, \infty}$ and $C_{2, \infty}$ is the constant in Lemma 3.1. Due to the assumption (3.14), we have by Lemmata 3.7 and 3.2 that

$$\|\Delta^{1}_{s'_j}(H_{s_j})\|_{L_\infty(R)} \leq (1 + C_{2, \infty} s'_j)^\tau \|H_{s_j}\|_{L_\infty(R)} \leq (1 + C_{2, \infty} s'_j)^\tau (1 + C_{\infty, 2, s_j})^{1/4} \|H_{s_j}\|_{L_2(R)} = (1 + C_{2, \infty} s'_j)^\tau (1 + C_{\infty, 2, s_j})^{1/4}$$

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Hence, similarly to (2.14)–(2.15), for each \(n\), and the uniform boundedness of the operators \(P\),

\[
\|\Delta^1_{s'}(H_n)\|_{L_2(\mathbb{R}^\infty, \gamma)} \leq p_s(\theta, \lambda) \leq p_s(\theta, \lambda),
\]

where

\[
\theta := \tau + \varepsilon + 1/4, \quad \lambda := \max\{C_{\infty, 2}, C_{2, \infty}, C_\varepsilon\}.
\]

Substituting \(\|\Delta^1_{s'}(H_n)\|_{L_2(\mathbb{R}^\infty, \gamma)}\) in (3.24) with the right-hand side of (3.25) gives that

\[
\|I_{\Lambda_{s'} \cap R_n} H\|_{L_2(\mathbb{R}^\infty, \gamma)} \leq \sum_{s' \in \Lambda_{s'} \cap R_n} p_s(\theta, \lambda) \leq |R_n| p_s(\theta, \lambda) \leq p_s(1, 1) p_s(\theta, \lambda).
\]

where \(\theta\) and \(\lambda\) are as in (3.26). Hence,

\[
\|I_{\Lambda_{s'} \cap R_n} H\|_{L_2(\mathbb{R}^\infty, \gamma)} \leq p_s(\theta, \lambda),
\]

where \(\theta\) and \(\lambda\) are as in (3.17). Denote by \(\Sigma(\xi)\) the sum in the right-hand side of (3.22). By using (3.27) and (2.4) we estimate \(\Sigma(\xi)\) as

\[
\Sigma(\xi) \leq C \sum_{(k,s) \in G(\xi)} 2^{-\alpha k} p_s(\theta, \lambda) \|v_s\|_{X^2} = C \sum_{s \in F} p_s(\theta, \lambda) \|v_s\|_{X^2} \sum_{2^k > \xi \sigma_{2,s}^{-q_2}} 2^{-\alpha k} \leq C \sum_{s \in F} p_s(\theta, \lambda) \|v_s\|_{X^2} \left(\xi \sigma_{2,s}^{-q_2}\right)^{-\alpha} \leq C \xi^{-\alpha} \sum_{s \in F} p_s(\theta, \lambda) \sigma_{2,s}^{q_2} \|v_s\|_{X^2}.
\]

By the inequalities \(2(1 - q_2 \alpha) \geq q_2\) and \(\sigma_{2,s} > 1\) and the assumptions we have that

\[
\sum_{s \in F} p_s(\theta, \lambda) \|v_s\|_{X^2} \leq \left(\sum_{s \in F} (\sigma_{2,s} \|v_s\|_{X^2})^2\right)^{1/2} \left(\sum_{s \in F} p_s^2(\theta, \lambda) \sigma_{2,s}^{-2(1-q_2 \alpha)}\right)^{1/2} \leq \left(\sum_{s \in F} (\sigma_{2,s} \|v_s\|_{X^2})^2\right)^{1/2} \left(\sum_{s \in F} p_s^2(\theta, \lambda) \sigma_{2,s}^{-q_2}\right)^{1/2} = C < \infty.
\]

Thus, we obtain the estimate

\[
\Sigma(\xi) := \sum_{(k,s) \in G(\xi)} \|\delta_k(v_s)\|_{X^1} \|I_{\Lambda_{s'} \cap R_n} H\|_{L_2(\mathbb{R}^\infty, \gamma)} \leq C \xi^{-\alpha}.
\]

This together with (3.22) and (3.23) implies that

\[
\|v - \mathcal{I}_G(\xi) u\|_{V_p(X^1)} \leq C \xi^{-\alpha}.
\]

Hence, similarly to (2.14)–(2.15), for each \(n \in \mathbb{N}\) we can find a number \(\xi_n\) such that \(|G(\xi_n)| \leq n\) and

\[
\|v - \mathcal{I}_G(\xi_n) u\|_{V_p(X^1)} \leq C n^{-\alpha}, \quad \alpha \leq 1/q_2 - 1/2.
\]

We now consider the case \(\alpha > 1/q_2 - 1/2\). Observe that the unconditional convergence of the series (2.5) and the uniform boundedness of the operators \(P_s\) in \(X^1\) imply that

\[
\delta_k \Delta^1_{s'} u = \Delta^1_{s'} \delta_k v = \sum_{s' \in F} \delta_k(v_{s'}) \Delta^1_{s'}(H_{s'})
\]

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and
\[ P_n v = \sum_{s \in \mathbb{F}} P_n(v_s)H_s \]
with convergence of the series in \( V_2(X^1) \). Put \( \Lambda(\xi) := \{ s \in \mathbb{F} : \sigma_{1,s}^q \leq \xi \} \) and \( B(\xi, s) := \{ k \in \mathbb{N}_0 : 2^{\alpha_{q,k}} \leq \xi \sigma_{2,s}^{-q} \} \) for \( \xi > 0 \). By using of these equalities and the unconditional convergence of the series (2.5) and (3.4), with \( N(\xi, s) := 2 \log_2(\xi^{1/(q_1-\alpha)} \sigma_{2,s}^{-1/(\alpha)}) \) we derive the estimates

\[
I_{G(\xi)} v = \sum_{(k, s) \in G(\xi)} \delta_k \Delta_s^1 v = \sum_{s \in \Lambda(\xi)} \Delta_s^1 \left( \sum_{k \in B(\xi, s)} \delta_k v \right)
= \sum_{s \in \Lambda(\xi)} P_N(\xi, s)(v_s)H_s + \sum_{s \not\in \Lambda(\xi)} P_N(\xi, s)(v_s)I_{\Lambda(\xi) \cap R_s}(H_s)
= \sum_{s \in \Lambda(\xi)} \left( \sum_{k \in B(\xi, s)} \delta_k v \right) H_s + \sum_{s \not\in \Lambda(\xi)} P_N(\xi, s)(v_s)I_{\Lambda(\xi) \cap R_s}(H_s)
= S_{G(\xi)} v + \sum_{s \not\in \Lambda(\xi)} P_N(\xi, s)(v_s)I_{\Lambda(\xi) \cap R_s}(H_s). \tag{3.31}
\]

Hence,

\[
\| v - I_{G(\xi)} v \|_{V_2(X^1)} \leq \| v - S_{G(\xi)} v \|_{V_2(X^1)} + \sum_{s \not\in \Lambda(\xi)} \| P_N(\xi, s)(v_s) \|_{X^1} \| I_{\Lambda(\xi) \cap R_s} H_s \|_{L_2(\mathbb{F}, \gamma)} \tag{3.32}
\]

Lemma 3.5 gives

\[
\| v - S_{G(\xi)} v \|_{V_2(X^1)} \leq C \xi^{-(1/q_1 - 1/2)}. \tag{3.33}
\]

The sum in the right-hand side of (3.32) can be estimated by

\[
\sum_{s \not\in \Lambda(\xi)} \| P_N(\xi, s)(v_s) \|_{X^1} \| I_{\Lambda(\xi) \cap R_s} H_s \|_{L_2(\mathbb{F}, \gamma)} \leq C \sum_{s \not\in \Lambda(\xi)} \| v_s \|_{X^1} \| I_{\Lambda(\xi) \cap R_s} H_s \|_{L_2(\mathbb{F}, \gamma)}.
\]

Similarly to (3.27), we have

\[
\| I_{\Lambda(\xi) \cap R_s} H_s \|_{L_2(\mathbb{F}, \gamma)} \leq p_s(\theta, \lambda)
\]

with the same \( \theta \) and \( \lambda \) as in (3.17). This gives the estimate

\[
\sum_{s \not\in \Lambda(\xi)} \| P_N(\xi, s)(v_s) \|_{X^1} \| I_{\Lambda(\xi) \cap R_s} H_s \|_{L_2(\mathbb{F}, \gamma)} \leq C \sum_{s \not\in \Lambda(\xi)} \| v_s \|_{X^1} p_s(\theta, \lambda). \tag{3.34}
\]

We have by the Hölder inequality and the hypothesis of the theorem,

\[
\sum_{s \not\in \Lambda(\xi)} \| v_s \|_{X^1} p_s(\theta, \lambda) \leq \left( \sum_{\sigma_{1,s} > \xi^{1/q_1}} (\sigma_{1,s} \| v_s \|_{X^1})^2 \right)^{1/2} \left( \sum_{\sigma_{1,s} > \xi^{1/q_1}} p_s^2(\theta, \lambda) \sigma_{1,s}^{-2} \right)^{1/2}
\leq C \left( \sum_{\sigma_{1,s} > \xi^{1/q_1}} p_s^2(\theta, \lambda) \sigma_{1,s}^{-2} \sigma_{1,s}^{-2(2-q_1)} \right)^{1/2}
\leq C \xi^{-(1/q_1 - 1/2)} \left( \sum_{s \in \mathbb{F}} p_s(2, \lambda) \sigma_{1,s}^{-q_1} \right)^{1/2} \leq C \xi^{-(1/q_1 - 1/2)}.
\]

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Combining (3.32)–(3.35) leads to the estimate
\[
\|v - I_{G(\xi)} v\|_{V_2(X^1)} \leq C \xi^{-1/n - 1/2}.
\]
(3.36)

Hence, similarly to (2.18)–(2.19), for each \( n \in \mathbb{N} \) we can find a number \( \xi_n \) such that \( |G(\xi_n)| \leq n \) and
\[
\|v - I_{G(\xi_n)} v\|_{V_p(X^1)} \leq C n^{-\alpha}, \quad \alpha > 1/q_2 - 1/2.
\]
(3.37)

By combining the last estimate and (3.30) we derive (3.18).

Denote by \( \Gamma_s \) and \( \Gamma(\Lambda) \) the set of interpolation points in the operators \( \Delta_s^\theta \), respectively.

**Corollary 3.1** Let \( 0 < p \leq 2 \). Let \( v \in V_p(X) \) be represented by the series (2.1) for a Hilbert space \( X \).

Assume that \( (Y_n)_{n \in \mathbb{N}_0} \) is a sequence such that every \( Y_n \) satisfies the condition (3.14) for some positive numbers \( \tau \) and \( C \). Assume that there exists an increasing sequence \( (\sigma_s)_{s \in \mathbb{F}} \) of numbers strictly larger than 1 such that
\[
\sum_{s \in \mathbb{F}} (\sigma_s \|v_s\|_X)^2 < \infty
\]
and \( (p_s(2\theta, \max(2, \lambda)\sigma_s^{-1}))_{s \in \mathbb{F}} \in \ell_q(\mathbb{F}) \) for some \( 0 < q < 2 \), where \( \theta \) and \( \lambda \) are as in (3.17). For \( \xi > 0 \) define
\[
\Lambda(\xi) := \{ s \in \mathbb{F} : \sigma_s^\alpha \leq \xi \}.
\]
(3.38)

Then for each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |\Gamma(\Lambda(\xi_n))| \leq n \) and
\[
\|v - I_{\Lambda(\xi_n)} v\|_{V_p(X)} \leq C n^{-(1/q - 1/2)}.
\]
(3.39)

The constant \( C \) in (3.39) is independent of \( v \) and \( n \).

**Proof.** Similarly to the proof of Theorem 3.1 it is sufficient to prove (3.39) for \( p = 2 \). In the same way as in proving (3.32), we can show that
\[
\|v - I_{\Lambda(\xi)} v\|_{V_2(X^1)} \leq \|v - S_{\Lambda(\xi)} v\|_{V_2(X^1)} + \sum_{s \notin \Lambda(\xi)} \|v_s\|_{X^1} \|I_{\Lambda(\xi) \cap R_s} H_s\|_{L_2(\mathbb{R}^\infty, \gamma)},
\]
where
\[
S_{\Lambda(\xi)} v := \sum_{s \in \Lambda(\xi)} v_s H_s.
\]

By estimating \( \|v - S_{\Lambda(\xi)} v\|_{V_2(X^1)} \) and \( \sum_{s \notin \Lambda(\xi)} \|v_s\|_{X^1} \|I_{\Lambda(\xi) \cap R_s} H_s\|_{L_2(\mathbb{R}^\infty, \gamma)} \) similarly to (2.16) and (3.34)–(3.35), respectively, we derive
\[
\|v - I_{\Lambda(\xi)} v\|_{V_2(X^1)} \leq C \xi^{-(1/q - 1/2)}.
\]

Since \( |\Gamma_s| \leq \Pi_{\ell \in \mathbb{N}} (2s + 1) = p_s(1, 2) \), we have from the definition
\[
|\Gamma(\Lambda(\xi))| \leq \sum_{s \in \Lambda(\xi)} |\Gamma_s| \leq \sum_{\xi \sigma_s^{-q} \geq 1} p_s(1, 2) \leq M \xi,
\]
where \( M := \sum_{s \in \mathbb{F}} p_s(1, 2) \sigma_s^{-q} < \infty \) by the assumption. For any \( n \in \mathbb{N} \), by choosing a number \( \xi_n \) satisfying the inequalities \( M \xi_n \leq n < 2M \xi_n \), we derive (3.39).
The bound \( \| v - I_{\Lambda_n}v \|_{L^2(\mathbb{R}^+, \mathcal{H}, \gamma)} \leq C n^{-(1/q-1/2)} \) has been obtained in [18] for a Hilbert space \( \mathcal{H} \), where \( \Lambda_n \) is the set of \( s \) corresponding to the \( n \) largest elements of an \( \ell_q \)-summable majorant of the sequence \((\sigma^{-1}_s p_s(\theta, \lambda))_{s \in \mathbb{F}}\).

**Theorem 3.2** Let \( 0 < p \leq 2 \). Let Assumption II hold for Hilbert spaces \( X^1 \) and \( X^2 \). Let \( v \in \mathcal{V}_2(X^2) \) be represented by the series (3.9). Assume that \((Y_n)_{n \in \mathbb{N}}\) is a sequence such that every \( Y_n \) satisfies the condition (3.14) for some positive numbers \( \tau \) and \( C \). Assume that for \( r = 1, 2 \) there exist increasing sequences \((\sigma_{r,s})_{s \in \mathbb{F}_{ev}}\) of numbers strictly larger than 1 such that

\[
\sum_{s \in \mathbb{F}_{ev}} (\sigma_{r,s} \| v_s \|_{X^r})^2 < \infty
\]

and \( (p_s(2\theta, \lambda)\sigma^{-1}_{r,s})_{s \in \mathbb{F}_{ev}} \in \ell_q(\mathbb{F}_{ev}) \) for some \( 0 < q_1 \leq q_2 < \infty \) with \( q_1 < 2 \), where \( \theta \) and \( \lambda \) are as in (3.17). For \( \xi > 0 \) let \( G_{ev}(\xi) \) be the set defined as in (3.10).

Then for each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |G(\xi_n)| \leq n \) and

\[
\| v - I_{G_{ev}(\xi_n)}v \|_{\mathcal{V}_2(X^1)} \leq C n^{-\min(\alpha, \beta)}.
\]  

(3.40)

The rate \( \alpha \) corresponds to the approximation of a single function in \( X^2 \) as given by (2.3). The rate \( \beta \) is given by (3.19). The constant \( C \) in (3.40) is independent of \( v \) and \( n \).

**Proof.** The proof of this theorem is similar to the proof of Theorem 3.1 with some modification. For example, all the indices sets are taken from the sets \( \mathbb{F}_{ev} \) and \( \mathbb{N}_0 \times \mathbb{F}_{ev} \) instead \( \mathbb{F} \) and \( \mathbb{N}_0 \times \mathbb{F} \); estimates similar to (3.23) and (3.33) are given by Lemma 3.6 instead Lemma 3.5.

Similarly to Corollary 3.1 we have the following

**Corollary 3.2** Let \( v \in \mathcal{V}_2(X) \) be represented by the series (3.9) for a Hilbert space \( X \). Assume that \((Y_n)_{n \in \mathbb{N}_0}\) is a sequence such that every \( Y_n \) satisfies the condition (3.14) for some positive numbers \( \tau \) and \( C \). Assume that there exists an increasing sequence \((\sigma_s)_{s \in \mathbb{F}_{ev}}\) of numbers strictly larger than 1 such that

\[
\sum_{s \in \mathbb{F}_{ev}} (\sigma_s \| v_s \|_{X})^2 < \infty
\]

and \( (p_s(2\theta, \max(2, \lambda))\sigma^{-1}_s)_{s \in \mathbb{F}_{ev}} \in \ell_q(\mathbb{F}_{ev}) \) for some \( 0 < q < 2 \), where \( \theta \) and \( \lambda \) are as in (3.17). For \( \xi > 0 \), define

\[
\Lambda_{ev}(\xi) := \{ s \in \mathbb{F}_{ev} : \sigma^2_s \leq \xi \}.
\]  

(3.41)

Then for each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |\Gamma(\Lambda_{ev}(\xi_n))| \leq n \) and

\[
\| v - I_{\Lambda_{ev}(\xi_n)}v \|_{\mathcal{V}_p(X)} \leq C n^{-(1/q-1/2)}.
\]  

(3.42)

The constant \( C \) in (3.42) is independent of \( v \) and \( n \).

Theorem 3.2 and Corollary 3.2 will be applied in proving the convergence rates of fully and non-fully discrete integration in the next section.
4 Integration

In this section, we construct general fully discrete linear methods for integration of functions taking values in $X^2$ and having a weighted $\ell_2$-summability of Hermite expansion coefficients for Hilbert spaces $X^1$ and $X^2$ satisfying a certain "spatial" approximation property, and their bounded linear functionals. In particular, we give convergence rates for these methods of integration which are derived from results on convergence rate of polynomial interpolation approximation in $V_1(X^1)$ in Theorem 3.2. We also briefly consider non-fully discrete methods for integration.

If $v$ is a function defined on $\mathbb{R}$ taking values in a Hilbert space $X$, an interpolation formula $I_n(v)$ of the form (3.13) generates the quadrature formula defined as

$$Q_n(v) := \int_{\mathbb{R}} I_n(v)(y) \, d\gamma(y) = \sum_{k=0}^{n} \omega_{n;k} v(y_{n;k}),$$

where

$$\omega_{n;k} := \int_{\mathbb{R}} \ell_{n;k}(y) \, d\gamma(y).$$

Notice that

$$Q_n(\varphi) = \int_{\mathbb{R}} \varphi(y) \, d\gamma(y)$$

for every polynomial $\varphi$ of degree $\leq n$, due to the identity $I_n(\varphi) = \varphi$.

For integration purpose, we additionally assume that the sequence $Y_n$ as in (3.12) is symmetric for every $n \in \mathbb{N}_0$, i.e., $y_{n,n-k} = y_{n;k}$ for $k = 0, \ldots, n$. The sequences $Y_{n}^*$ of the of the roots of the Hermite polynomials $H_{n+1}$ and their modifications $\bar{Y}_{n}^*$ are symmetric. Also, for the sequence $Y_{n}^*$, it is well-known that

$$\omega_{n;k} = \frac{1}{(n+1)H_n^2(y_{n;k})}.$$ 

For a given sequence $(Y_n)_{n=0}^{\infty}$, we define the univariate operator $\Delta_n^Q$ for $n \in \mathbb{N}_0$ by

$$\Delta_n^Q := Q_n - Q_{n-1},$$

with the convention $Q_{-1} := 0$. We introduce the operator $\Delta_s^Q$ for $s \in F$ by

$$\Delta_s^Q(v) := \bigotimes_{j \in \mathbb{N}} \Delta_{s_j}^Q (v)$$

for $\gamma$-measurable functions $v$ on $\mathbb{R}^\infty$ taking values in a Hilbert space $X$, where the univariate operator $\Delta_{s_j}^Q$ is applied to the univariate function $v$ by considering $v$ as a function of variable $y_i$ with the other variables held fixed.

We introduce the operator $Q_\Lambda$ for a finite set $\Lambda \subset F$ by

$$Q_\Lambda := \sum_{s \in \Lambda} \Delta_s^Q.$$ 

Let Assumption II hold for Hilbert spaces $X^1$ and $X^2$. We introduce the operator $Q_G$ for a finite set $G \subset \mathbb{N}_0 \times F$ by

$$Q_G v := \sum_{(k,s) \in G} \delta_k \Delta_{s}^Q(v)$$
for \( \gamma \)-measurable functions \( v \) on \( \mathbb{R}^\infty \) taking values in a Hilbert space \( X^2 \). Notice that

\[
Q_{\Lambda} v = \int_{\mathbb{R}^\infty} I_{\Lambda} v(y) \, d\gamma(y),
\]

and

\[
Q_{G} v = \int_{\mathbb{R}^\infty} I_{G} v(y) \, d\gamma(y).
\] (4.1)

For a function \( v \) defined on \( \mathbb{R}^\infty \) taking values in a Hilbert space \( X \) such that \( v \in \mathcal{V}_2(X) \) and is represented by the series (2.1), consider the function \( v_{ev} \in \mathcal{V}_2(X) \) defined by

\[
v_{ev} := \sum_{s \in F_{ev}} v_s H_s.
\]

From the obvious equality \( \int_{\mathbb{R}} v(y) \, d\gamma(y) = 0 \) for every odd function \( v \), we have that

\[
\int_{\mathbb{R}^\infty} H_s(y) \, d\gamma(y) = 0, \ s \notin F_{ev},
\]

and hence,

\[
\int_{\mathbb{R}^\infty} v(y) \, d\gamma(y) = \int_{\mathbb{R}^\infty} v_{ev}(y) \, d\gamma(y).
\] (4.2)

Moreover, if \( Y_n \) is symmetric for every \( n \in \mathbb{N}_0 \),

\[
\Delta_{s'}^Q H_s(y) = 0, \ s \notin F_{ev}, \ s' \in F.
\] (4.3)

Let Assumption II hold for Hilbert spaces \( X^1 \) and \( X^2 \). If \( v \) is a function defined on \( \mathbb{R}^\infty \) taking values in \( X^2 \) and \( \phi \in (X^1)' \) a bounded linear functional on \( X^1 \). For a finite set \( G \subset \mathbb{N}_0 \times F \), the quadrature formula \( Q_{\Lambda} v \) generates the quadrature formula \( Q_{G} \langle \phi, v \rangle \) for integration of \( \langle \phi, v \rangle \) by

\[
Q_{G} \langle \phi, v \rangle := \sum_{(k, s) \in G} \langle \phi, \delta_k \Delta_s^Q v \rangle = \int_{\mathbb{R}^\infty} \langle \phi, Q_{G} v(y) \rangle \, d\gamma(y).
\]

**Theorem 4.1** Under the hypothesis of Theorem 3.1, assume additionally that the sequences \( Y_n, n \in \mathbb{N}_0 \), are symmetric. For \( \xi > 0 \) let \( G_{ev}(\xi) \) be the set defined as in (3.10). Then we have the following.

(i) For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |G_{ev}(\xi_n)| \leq n \) and

\[
\left\| \int_{\mathbb{R}^\infty} v(y) \, d\gamma(y) - Q_{G_{ev}(\xi_n)} v \right\|_{X^1} \leq C n^{-\min(\alpha, \beta)}.
\] (4.4)

(ii) Let \( \phi \in (X^1)' \) be a bounded linear functional on \( X^1 \). Then for each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |G_{ev}(\xi_n)| \leq n \) and

\[
\left| \int_{\mathbb{R}^\infty} \langle \phi, v(y) \rangle \, d\gamma(y) - Q_{G_{ev}(\xi_n)} \langle \phi, v \rangle \right| \leq C n^{-\min(\alpha, \beta)}.
\] (4.5)

The rate \( \alpha \) corresponds to the approximation of a single function in \( X^2 \) as given by (2.3). The rate \( \beta \) is given by (3.19). The constants \( C \) in (4.4) and (4.5) are independent of \( v \) and \( n \).
Proof. For a given \( n \in \mathbb{N} \), we approximate the integral in the right-hand side of (4.2) by \( \mathcal{Q}_{G_{\nu v}}(\xi_n) \) where \( \xi_n \) is as in Theorem 3.2. By Lemmata 3.3 and 3.4 the series (2.5) and (3.4) converge absolutely, and therefore, unconditionally in the Hilbert space \( V_1(X^1) \) to \( v \). Hence, by (4.3) we derive that \( \mathcal{Q}_{G_{\nu v}}(\xi_n) v = \mathcal{Q}_{G_{\nu v}}(\xi_n) v_{\nu v} \). Due to (4.1) and (4.2) there holds the equality
\[
\int_{\mathbb{R}^\infty} v(y) \, d\gamma(y) - \mathcal{Q}_{G_{\nu v}}(\xi_n) v = \int_{\mathbb{R}^\infty} (v_{\nu v}(y) - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}(y)) \, d\gamma(y).
\] (4.6)

Hence, applying (3.40) in Theorem 3.2 for \( p = 1 \), we obtain (i):
\[
\left\| \int_{\mathbb{R}^\infty} v(y) \, d\gamma(y) - \mathcal{Q}_{G_{\nu v}}(\xi_n) v \right\|_{X^1} \leq \left\| v_{\nu v} - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v} \right\|_{V_1(X^1)} \leq C n^{-\min(\alpha, \beta)}.
\]

For a given \( n \in \mathbb{N} \), we approximate the integral \( \int_{\mathbb{R}^\infty} \langle \phi, v(y) \rangle \, d\gamma(y) \) by \( \mathcal{Q}_{\Lambda_{\nu v}}(\xi_n) \langle \phi, v \rangle \) where \( \xi_n \) is as in Corollary 3.2. Similarly to (4.6), there holds the equality
\[
\int_{\mathbb{R}^\infty} \langle \phi, v_{\nu v}(y) \rangle \, d\gamma(y) - \mathcal{Q}_{G_{\nu v}}(\xi_n) \langle \phi, v_{\nu v}(y) \rangle = \int_{\mathbb{R}^\infty} \langle \phi, v_{\nu v}(y) - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}(y) \rangle \, d\gamma(y).
\]

Hence, applying (3.40) in Theorem 3.2 for \( p = 1 \), we prove (ii):
\[
\left| \int_{\mathbb{R}^\infty} \langle \phi, v_{\nu v}(y) \rangle \, d\gamma(y) - \mathcal{Q}_{G_{\nu v}}(\xi_n) \langle \phi, v \rangle \right| \leq \int_{\mathbb{R}^\infty} \left| \langle \phi, v_{\nu v}(y) - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}(y) \rangle \right| \, d\gamma(y)
\leq \int_{\mathbb{R}^\infty} \|\phi\|_{X^1} \|v_{\nu v}(y) - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}(y)\|_{X^1} \, d\gamma(y)
\leq C \int_{\mathbb{R}^\infty} \|v_{\nu v}(y) - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}(y)\|_{X^1} \, d\gamma(y)
\leq C \|v_{\nu v} - \mathcal{I}_{G_{\nu v}}(\xi_n) v_{\nu v}\|_{V_1(X^1)} \leq C n^{-\min(\alpha, \beta)}.
\]

Let \( v \) be a \( \gamma \)-measurable on \( \mathbb{R}^\infty \) taking values in a Hilbert space \( X \) and \( \phi \in X' \) a bounded linear functional on \( X \). Denote by \( \langle \phi, v \rangle \) the value of \( \phi \) in \( v \). For a finite set \( \Lambda \subset F \), the quadrature formula \( Q_{\Lambda} v \) generates the quadrature formula \( Q_{\Lambda}(\phi, v) \) for integration of \( \langle \phi, v \rangle \) by
\[
Q_{\Lambda}(\phi, v) := \sum_{\alpha \in \Lambda} (\phi, \Delta_\alpha^2 v) = \int_{\mathbb{R}^\infty} \langle \phi, I_{\Lambda} v(y) \rangle \, d\gamma(y).
\]

Similarly to the proof of Theorem 4.1, applying (3.42) in Corollary 3.2 for \( p = 1 \), we can derive the following

**Corollary 4.1** Under the hypothesis of Corollary 3.1, assume additionally that the sequences \( Y_n, n \in \mathbb{N}_0 \), are symmetric. For \( \xi > 0 \) let \( \Lambda_{\nu v}(\xi) \) be the set defined as in (3.41). Then we have the following.

(i) For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |\Gamma(\Lambda_{\nu v}(\xi_n))| \leq n \) and
\[
\left\| \int_{\mathbb{R}^\infty} v(y) \, d\gamma(y) - \mathcal{Q}_{\Lambda_{\nu v}}(\xi_n) v \right\|_{X} \leq C n^{-\frac{1}{q} - \frac{1}{2}}.
\] (4.7)
Lemma 5.2 Let $\phi \in X'$ be a bounded linear functional on $X$. Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|\Gamma(\Lambda_{ev}(\xi_n))| \leq n$ and

$$\left| \int_{\mathbb{R}^\infty} \langle \phi, v(y) \rangle \, d\gamma(y) - Q_{\Lambda_{ev}(\xi_n)}(\phi, v) \right| \leq Cn^{-(1/q-1/2)}. \quad (4.8)$$

The constants $C$ in (4.7) and (4.8) are independent of $v$ and $n$.

5 Elliptic PDEs with lognormal inputs

In this section, we apply the results in Sections 2–4 to fully discrete linear Galerkin and polynomial interpolation approximations as well as integration for parametric and stochastic elliptic PDEs with lognormal inputs (1.2).

We approximate the solution $u(y)$ to the parametrized elliptic PDEs (1.1) by truncation of the Hermite series

$$u(y) = \sum_{s \in \mathcal{F}} u_s H_s(y), \quad u_s \in V. \quad (5.1)$$

For convenience, we introduce the convention $W^1 := V$ and $W^2 := W$. Constructions of fully discrete approximations and integration are based on the approximation property (1.4) in Assumption I and the weighted $\ell$-summability of the series $(\|u_s\|_{W^r})_{s \in \mathcal{F}}$ in following lemma which has been proven in [2, 3].

Lemma 5.1 Let $r = 1, 2$. Assume that the right side $f$ in (1.1) belongs to $H^{r-2}(D)$, that the domain $D$ has $C^{r-2,1}$ smoothness, that all functions $\psi_j$ belong to $W^{r-1,\infty}(D)$. Assume that there exist a number $0 < q_r < \infty$ and a sequence $\rho_r = \{\rho_{r,j}\}_{j \in \mathbb{N}}$ of positive numbers such that $(\rho_{r,j}^{-1})_{j \in \mathbb{N}} \in \ell_{q_r}(\mathbb{N})$ and

$$\left\| \sum_{j \in \mathbb{N}} \rho_{r,j} |\psi_j| \right\|_{L^\infty(D)} + \sup_{\|\alpha\| \leq r-1} \left\| \sum_{j \in \mathbb{N}} \rho_{r,j} |D^\alpha \psi_j| \right\|_{L^\infty(D)} < \infty.$$

Then we have that for any $\eta \in \mathbb{N},$

$$\sum_{s \in \mathcal{F}} (\sigma_{r,s} \|u_s\|_{W^r})^2 < \infty \quad \text{with} \quad \sigma_{r,s}^2 := \sum_{\|s'\|_{\ell_{\infty}(\mathbb{N})} \leq \eta} \left( \frac{s}{s'} \right)^{q_r/2} \rho_{r,s'}^2.$$

We make use the following notation: for $\nu \in \mathbb{N},$

$$\mathcal{F}_\nu := \{s \in \mathcal{F} : s_j \in \mathbb{N}_{0,\nu}, \ j \in \mathbb{N} \}; \quad \mathcal{N}_{0,\nu} := \{n \in \mathbb{N} : n = 0, \nu, \nu + 1, \ldots \}.$$

The set $\mathcal{F}_\nu$ has been introduced in [34]. The set $\mathcal{F}_2$ plays an important role in establishing improved convergence rates for sparse-grid Smolyak quadrature in [33, 34].

The following lemma is a generalization of [2, Lemma 5.1].

Lemma 5.2 Let $0 < q < \infty$, $\eta \in \mathbb{N}$, $\rho = \{\rho_j\}_{j \in \mathbb{N}}$ of positive numbers such the sequence $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$. Let $\theta, \lambda$ be arbitrary positive numbers and $(\rho_n(\theta, \lambda))_{s \in \mathcal{F}}$ the sequence given in (3.3). Let for numbers $\eta \in \mathbb{N}$ the sequence $(\sigma_s)_{s \in \mathcal{F}}$ be defined by

$$\sigma_s^2 := \sum_{\|s'\|_{\ell_{\infty}(\mathbb{N})} \leq \eta} \left( \frac{s}{s'} \right)^{q_r/2} \rho_{s'}^2.$$
Then for any \( \eta > \frac{2\nu(\theta+1)}{q} \), we have
\[
\sum_{s \in \mathbb{F}_\nu} p_s(\theta, \lambda) \sigma_{sx}^{-q/\nu} < \infty.
\]

**Proof.** With \( \theta' := 2\theta \nu/q \), we have that
\[
\sum_{s \in \mathbb{F}_\nu} p_s(\theta, \lambda) \sigma_{sx}^{-q/\nu} = \prod_{j \in \mathbb{N}} \sum_{n \in \mathbb{N}_0} \left( \sum_{k=0}^\eta \binom{n}{k} (1 + \lambda \rho_j^{\theta'} \rho_j^{2k})^{-q/2\nu} \right) =: \prod_{j \in \mathbb{N}} B_j,
\]
and
\[
B_j \leq \sum_{n \in \mathbb{N}_0, n < \eta} \left( \binom{n}{\min(n, \eta)} (1 + \lambda n)^{\theta'} \rho_j^{2 \min(n, \eta)} \right)^{-q/2\nu} \leq \sum_{n \in \mathbb{N}_0, n < \eta} \left( \binom{n}{\min(n, \eta)} (1 + \lambda n)^{\theta'} \rho_j^{2n} \right)^{-q/2\nu} + \sum_{n \geq \eta} \left( (1 + \lambda n)^{\theta'} \rho_j^{2\eta} \right)^{-q/2\nu} \leq \sum_{n \in \mathbb{N}_0, n < \eta} (1 + \lambda n)^{\theta} \rho_j^{-nq/\nu} + \rho_j^{-\eta q/\nu} \sum_{n \geq \eta} \binom{n}{\eta}^{-q/2\nu} (1 + \lambda n)^{\theta} =: B_{j,1} + B_{j,2}.
\]

We estimate \( B_{j,1} \) and \( B_{j,2} \). We have
\[
B_{j,1} \leq 1 + \sum_{n=\nu}^{\eta-1} (1 + \lambda n)^\theta \rho_j^{-nq/\nu} \leq 1 + (1 + (\eta - 1)\lambda)^\theta \sum_{n=\nu}^{\eta-1} \rho_j^{-nq/\nu}.
\]

From the inequalities \( \left( \frac{2}{\eta} \right)^\eta \leq \binom{\eta}{\nu} \) and \( \eta q/2\nu - \theta > 1 \) we derive that
\[
B_{j,2} \leq \rho_j^{-\eta q/\nu} \sum_{n \geq \eta} \binom{n}{\eta}^{-q/2\nu} (1 + \lambda n)^\theta \leq C \rho_j^{-\eta q/\nu} \sum_{n \geq \eta} n^{-q(\eta q/2\nu - \theta)} \leq C \rho_j^{-\eta q/\nu}.
\]

Summing up we obtain that
\[
B_j \leq B_{j,1} + B_{j,2} \leq 1 + C \sum_{n=\nu}^{\eta} \rho_j^{-nq/\nu}.
\]

Since the sequence \( (\rho_j^{-1})_{j \in \mathbb{N}} \) belongs to \( \ell_q(\mathbb{N}) \), there exists \( j^* \) large enough such that \( \rho_j > 1 \) for all \( j \geq j^* \). Hence, there exists a constant \( C \) independent of \( j \) such that \( B_j \leq 1 + C \rho_j^{-q} \) for all \( j \in \mathbb{N} \), and consequently,
\[
\sum_{s \in \mathbb{F}_\nu} p_s(\theta, \lambda) \sigma_{sx}^{-q/2\nu} \leq \prod_{j \in \mathbb{N}} B_j \leq \prod_{j \in \mathbb{N}} (1 + C \rho_j^{-q}) \leq \exp \left( \sum_{j \in \mathbb{N}} C \rho_j^{-q} \right) < \infty.
\]

To treat fully discrete approximations we assume that \( f \in L_2(D) \) and there holds the approximation property (1.4) in Assumption I for the spaces \( V \) and \( W \). Notice here that classical error estimates [5]
yield the convergence rate $\alpha = 1/d$ by using Lagrange finite elements of order at least 1 on quasi-uniform partitions. Also, the spaces $W$ do not always coincide with $H^2(D)$. For example, for $d = 2$, we know that $W$ is strictly larger than $H^2(D)$ when $D$ is a polygon with re-entrant corner. In this case, it is well known that the optimal rate $\alpha = 1/2$ is yet attained, when using spaces $V_n$ associated to meshes $(T_n)_{n>0}$ with proper refinement near the re-entrant corners where the functions $v \in W$ might have singularities.

**Theorem 5.1** Let $0 < p \leq 2$. Let $f \in L_2(D)$ and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ and $W^2 = W$ with some $0 < q_1 \leq q_2 < \infty$. For $\xi > 0$ let $G(\xi)$ be the set defined as in (2.8).

Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $\dim(V(G(\xi_n))) \leq n$ and

$$\|u - S_{G(\xi_n)}u\|_{V_p(V)} \leq Cn^{-\min(\alpha,\beta)}.$$  

(5.2)

The rate $\alpha$ corresponds to the spatial approximation of a single function in $W$ as given by (1.4), and the rate $\beta$ is given by (2.10). The constant $C$ in (5.2) is independent of $u$ and $n$.

**Proof.** To prove the theorem it is sufficient to notice that the assumptions of Theorem 2.1 are satisfied for $X^1 = V$ and $X^2 = W$. This can be done by using Lemmata 5.1 and 5.2. (By multiplying the sequences $p_\tau$ in Lemma 5.1 with a positive constant we can get $\sigma_{r,s} > 1$ for $s \in \mathbb{F}$.)

The rate $\min(\alpha,\beta)$ in (5.2) is the rate of best adaptive $n$-term Galerkin approximation in $V_2(V)$ based on $\ell_{p_\tau}$-summability of $(\|u_s\|_V)_{s \in \mathbb{F}}$ and $\ell_{p_\tau}$-summability of $(\|u_s\|_W)_{s \in \mathbb{F}}$ proven in [3], where $1/p_\tau = 1/q_\tau + 1/2$ for $\tau = 1, 2$.

In the same way, from Theorems 2.1 and 3.1 and Corollary 3.1 we derive the following two theorems and corollary.

**Theorem 5.2** Let $0 < p \leq 2$. Let $f \in L_2(D)$ and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces $W^1 = V$ and $W^2 = W$ with some $0 < q_1 \leq q_2 < \infty$ with $q_1 < 2$. Assume that $(Y_n)_{n \in \mathbb{N}_0}$ is a sequence such that every $Y_n$ satisfies the condition (3.14) for some positive numbers $\tau$ and $C$. For $\xi > 0$ let $G(\xi)$ be the set defined as in (3.5).

Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|G(\xi_n)| \leq n$ and

$$\|u - I_{G(\xi_n)}u\|_{V_p(V)} \leq Cn^{-\min(\alpha,\beta)}.$$  

(5.3)

The rate $\alpha$ corresponds to the spatial approximation of a single function in $W$ as given by (1.4). The rate $\beta$ is given by (3.19). The constant $C$ in (5.3) is independent of $u$ and $n$.

Observe that the approximands $I_{G(\xi_n)}u$ in this theorem can be represented in the form

$$I_{G(\xi_n)}u = \sum_{k=0}^{k_n} \delta_k \left( \sum_{s \in \Lambda_k(\xi_n)} \Delta_{s}^1 \right) u = \sum_{k=0}^{k_n} \delta_k I_{\Lambda_k(\xi_n)}u,$$

where $k_n := [\log_2 \xi_n]$ and for $k \in \mathbb{N}_0$ and $\xi > 0$,

$$\Lambda_k(\xi) := \begin{cases} \{s \in \mathbb{F} : \sigma_{2,s}^\alpha \leq 2^{-k}\xi \} & \text{if } \alpha \leq 1/q_2; \\ \{s \in \mathbb{F} : \sigma_{1,s}^\alpha, \sigma_{2,s}^\alpha \leq \xi, \sigma_{2,s}^\alpha \leq 2^{-k}\xi \} & \text{if } \alpha > 1/q_2. \end{cases}$$

Moreover, $\Lambda_k(\xi_n)$ are downward closed sets, and consequently, the sequence $\{\Lambda_k(\xi_n)\}_{k=0}^{k_n}$ is nested in the inverse order, i.e., $\Lambda_{k'}(\xi_n) \subset \Lambda_k(\xi_n)$ if $k' > k$, and $\Lambda_0(\xi_n)$ is the largest and $\Lambda_{k_n}(\xi_n) = \{0_\mathbb{F}\}$. Further, the
fully discrete polynomial interpolation approximation by operators \( I_{G(\xi_n)} \) is a collocation approximation based on the finite number \(|\Gamma(\Lambda_0(\xi_n))| \leq \sum_{e \in \Lambda_0(\xi_n)} p_e(1,2) \) of the particular solvers \( u(y), y \in \Gamma(\Lambda_0(\xi_n)) \), where, we recall, \( \Gamma(\Lambda_0(\xi_n)) = \cup_{s \in \Lambda_0(\xi_n)} \Gamma_s \) and \( \Gamma_s = \{ y_{s,-c;m} : e \in E_s; m_j = 0, ..., s, j \in \mathbb{N} \} \) (\( E_s \) denotes the subset in \( F \) of all \( e \) such that \( e_j \) is 1 or 0 if \( s_j > 0 \), and \( e_j \) is 0 if \( s_j = 0 \), and \( y_{s,-c;m} \) := \((y_{sj,m_j})_{j \in \mathbb{N}}\)).

**Corollary 5.1** Let 0 < \( p \leq 2 \). Under the hypothesis of Lemma 5.1 for the spaces \( W^1 = V \) with some \( 0 < q < 2 \), for \( \xi > 0 \) let \( \Lambda(\xi) \) be the set defined as in (3.38). Then for each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |\Gamma(\Lambda(\xi_n))| \leq n \) and
\[
\|u - I_{\Lambda(\xi_n)}u\|_{V,\phi(V)} \leq C n^{-\frac{1}{q} - \frac{1}{2}}.
\] (5.4)

The constant \( C \) in (5.4) is independent of \( v \) and \( n \).

The rate in Corollary 5.1 has been obtained in [18] for a similar approximation in \( V_2(V) \).

**Theorem 5.3** Let \( f \in L_2(D) \) and Assumption I hold. Let the assumptions of Lemma 5.1 hold for the spaces \( W^1 = V \) and \( W^2 = W \) with some \( 0 < q_1 \leq q_2 < \infty \) with \( q_1 < 4 \). Assume that \( (Y_n)_{n \in \mathbb{N}^0} \) is a symmetric sequence such that every \( Y_n \) satisfies the condition (3.14) for some positive numbers \( \tau, C \). For \( \xi > 0 \) let \( G_{ev}(\xi) \) be the set defined as in (3.10). Then we have the following.

\( (i) \) For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |G_{ev}(\xi_n)| \leq n \) and
\[
\left\| \int_{\mathbb{R}^\infty} u(y) d\gamma(y) - Q_{G_{ev}(\xi_n)} u \right\|_{V,\phi(V)} \leq C n^{-\min(\alpha, \beta)}.
\] (5.5)

\( (ii) \) Let \( \phi \in V' \) be a bounded linear functional on \( V \). For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \( |G_{ev}(\xi_n)| \leq n \) and
\[
\left| \int_{\mathbb{R}^\infty} \langle \phi, u(y) \rangle d\gamma(y) - Q_{G_{ev}(\xi_n)} \langle \phi, u \rangle \right| \leq C n^{-\min(\alpha, \beta)}.
\] (5.6)

The rate \( \alpha \) corresponds to the spatial approximation of a single function in \( W \) as given by (1.4). The rate \( \beta \) is given by
\[
\beta := \left( \frac{2}{q_1} - 1, \frac{2}{q_2} \right), \quad \delta := \frac{2}{q_1} - \frac{2}{q_2}.
\] (5.7)

The constants \( C \) in (5.5) and (5.6) are independent of \( u \) and \( n \).

**Proof.** Observe that \( F_{ev} \subset F_2 \). From Lemma 5.1 and Lemma 5.2 we can see that the assumptions of Theorem 3.1 hold for \( X^1 = V \) and \( X^2 = W \) with some \( 0 < q_1, q_2 < 2 < \infty \) and \( q_1/2 < 2 \). Hence, by applying Theorem 4.1 we prove the theorem. \( \square \)

Observe that the rate in (5.5) and (5.6) can be improved as \( \min(\alpha, \frac{2}{q_1}, \frac{2}{q_2}) \) if the sequences \((\|u_{s}\|_{V})_{s \in \mathbb{F}}\) and \((\|u_{s}\|_{V'})_{s \in \mathbb{F}}\) have \( \ell_{p_1}\)- and \( \ell_{p_2}\)-summable majorant sequences, respectively, where \( 1/p_1 = 1/q_1 + 1/2 \) and \( 1/p_2 = 1/q_2 + 1/2 \).

In the same way, from Corollary 4.1 we derive the following

**Corollary 5.2** Let the assumptions of Lemma 5.1 hold for the spaces \( W^1 = V \) with some \( 0 < q < 4 \). Assume that \( (Y_n)_{n \in \mathbb{N}^0} \) is a symmetric sequence such that every \( Y_n \) satisfies the condition (3.14) for some positive numbers \( \tau, C \). For \( \xi > 0 \) let \( \Lambda_{ev}(\xi) \) be the set defined as in (3.41). Then we have the following.
(i) For each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|\Gamma(\Lambda_{ev}(\xi_n))| \leq n$ and
\[
\left\| \int_{\mathbb{R}^d} u(y) \, d\gamma(y) - Q_{\Lambda_{ev}(\xi_n)} u \right\|_V \leq C n^{-2/(q-1/2)}.
\] (5.8)

(ii) Let $\phi \in V'$ a bounded linear functional on $V$. For each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|\Gamma(\Lambda_{ev}(\xi_n))| \leq n$ and
\[
\left| \int_{\mathbb{R}^d} \langle \phi, u(y) \rangle \, d\gamma(y) - Q_{\Lambda_{ev}(\xi_n)} \langle \phi, u \rangle \right| \leq C n^{-2/(q-1/2)}.
\] (5.9)

The constants $C$ in (5.8) and (5.9) are independent of $u$ and $n$.

The rate $2/q - 1/2$ in Corollary 5.2 is an improvement of the rate $1/q - 1/2$ which has been recently obtained in [4]. Observe that this rate can be improved as $2/q$ if the sequence $(\|u_s\|_V)_{s \in F}$ has an $\ell_q$-summable majorant sequence, where $1/p = 1/q + 1/2$.

6 Elliptic PDEs with affine inputs

The theory of non-addaptive fully discrete approximations in Bochner spaces with infinite tensor product Gaussian measure in Sections 2–4 can be extended to other situations. In this section, we present some results on similar problems for parametric and stochastic PDEs with the affine inputs (1.3).

In the affine case, for given $a, b > -1$, we consider the orthogonal Jacobi expansion of the solution $u(y)$ of the form
\[
\sum_{s \in F} u_s J_s(y), \quad J_s(y) = \bigotimes_{j \in \mathbb{N}} J_{s_j}(y_j), \quad u_s := \int_1^y u(y) J_s(y) \, d\sigma_{a,b}(y),
\] where
\[
d\sigma_{a,b}(y) := \bigotimes_{j \in \mathbb{N}} \delta_{a,b}(y_j) \, dy_j,
\] (6.1)

\[
\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)}{2^{a+b+1} \Gamma(a+1) \Gamma(b+1)},
\]
and $(J_k)_{k \geq 0}$ is the sequence of Jacobi polynomials on $I := [-1,1]$ normalized with respect to the Jacobi probability measure $\int |J_k(y)|^2 \delta_{a,b}(y) \, dy = 1$. One has the Rodrigues’ formula
\[
J_k(y) = \frac{c_k^{a,b}}{k!^2} (1-y)^{-a}(1+y)^{-b} \frac{d^k}{dy^k} \left( (y^2 - 1)^k (1-y)^a(1+y)^b \right),
\]
where $c_k^{a,b} := 1$ and
\[
c_k^{a,b} := \sqrt{\frac{(2k + a + b + 1)k!\Gamma(k+a+b+1)\Gamma(a+1)\Gamma(b+1)}{\Gamma(k+a+1)\Gamma(k+b+1)\Gamma(a+b+2)}}, \quad k \in \mathbb{N}.
\]
Examples corresponding to the values $a = b = 0$ is the family of the Legendre polynomials, and to the values $a = b = -1/2$ the family of the Chebyshev polynomials.

We introduce the space $W^r := \{ v \in V : \Delta v \in H^{r-2}(D) \}$ for $r \geq 2$ with the convention $W^1 := V$. This space is equipped with the norm $\| v \|_{W^r} := \| \Delta v \|_{H^{r-2}(D)}$, and coincides with the Sobolev space $V \cap H^{r-2}(D)$ with equivalent norms if the domain $D$ has $C^{r-1,1}$ smoothness, see [19, Theorem 2.5.1.1]. The following lemma has been proven in [1, 3].
Lemma 6.1  For a given $\rho \in \mathbb{N}$, assume that $\bar{a} \in L^\infty(D)$ is such that $\text{ess inf} \bar{a} > 0$, and that there exists a sequence $\rho_r = (\rho_{r,j})_{j \in \mathbb{N}}$ of positive numbers such that

$$\left\| \sum_{j \in \mathbb{N}} \rho_{r,j} |\psi_j| \bar{a} \right\|_{L^\infty(D)} < 1.$$  

Assume that the right side $f$ in (1.1) belongs to $H^{r-2}(D)$, that the domain $D$ has $C^{r-2,1}$ smoothness, that $\bar{a}$ and all functions $\psi_j$ belong to $W^{r-1,\infty}(D)$ and that

$$\sup_{|\alpha| \leq r-1} \left\| \sum_{j \in \mathbb{N}} \rho_{r,j} |D^\alpha \psi_j| \right\|_{L^\infty(D)} < \infty.$$ 

Then

$$\sum_{\mathbf{s} \in \mathbf{F}_v} (\sigma_{r:s}) \|u_\mathbf{s}\|_{W^r}^2 < \infty, \quad \sigma_{r:s} := \rho_r \prod_{j \in \mathbb{N}} c_{s,j}.$$ 

Lemma 6.2  Let $0 < q < \infty$, $\rho = (\rho_j)_{j \in \mathbb{N}}$ of numbers larger than one such the sequence $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, $(\rho_s(\theta, \lambda))_{s \in \mathbf{F}}$ is a sequence of the form (3.3) with arbitrary nonnegative $\theta, \lambda$. Then for every $v \in \mathbb{N}$, we have

$$\sum_{\mathbf{s} \in \mathbf{F}_v} p_s(\theta, \lambda)(\rho^{-s})^{q/v} < \infty.$$ 

Proof. We have

$$\sum_{\mathbf{s} \in \mathbf{F}_v} p_s(\theta, \lambda)(\rho^{-s})^{q/v} = \prod_{j \in \mathbb{N}} \sum_{s_j \in \mathbb{N}_0, v} \rho_j^{-s_j q/v} (1 + \lambda s_j)\theta =: \prod_{j \in \mathbb{N}} A_j.$$ 

Since $\rho = (\rho_j)_{j \in \mathbb{N}}$ of numbers larger than one, and such the sequens $(\rho_j^{-1})_{j \in \mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$, we have $\min_{j \in \mathbb{N}} \rho_j > 1$. Hence, there exists a constant $C$ independent of $j$ such that

$$A_j = 1 + \sum_{k=v}^\infty \rho_j^{-k q/v} (1 + \lambda k)\theta \leq 1 + C \rho_j^{-q},$$ 

and consequently,

$$\sum_{\mathbf{s} \in \mathbf{F}_v} p_s(\theta, \lambda)(\rho^{-s})^{q/v} \leq \prod_{j \in \mathbb{N}} (1 + C \rho_j^{-q}) \leq \exp \left( \sum_{j \in \mathbb{N}} C \rho_j^{-q} \right) < \infty.$$  

Assumption III  There are a sequence $(V_n)_{n \in \mathbb{N}_0}$ of subspaces $V_n \subset V$ of dimension $\leq n$, and sequence $(P_n)_{n \in \mathbb{N}_0}$ of linear operators from $V$ into $V_n$, and a number $\alpha > 0$ such that

$$\|P_n(v)\|_V \leq C, \quad \|v - P_n(v)\|_V \leq C n^{-\alpha} \|v\|_{W^r}, \quad \forall n \in \mathbb{N}_0, \quad \forall v \in W^r. \quad (6.2)$$ 

In what follows, we make use the abbreviation: $V_{r,s}(V) := L_p(V^\infty, V, \sigma_{a,b})$ and assume that $r > 1$. From Lemmata 6.1 and 6.2 we can prove the following results on non-adaptive fully Galerkin and polynomial interpolation approximations and integration for the affine case.
Theorem 6.1 Let $0 < p \leq 2$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and $W^r$ with some $0 < q_1 \leq q_r < \infty$. For $\xi > 0$ let $G(\xi)$ be the set defined as in (2.8). Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $\dim(V(G(\xi_n))) \leq n$ and

$$
\|u - \mathcal{S}_{G(\xi_n)}u\|_{V_p(V)} \leq Cn^{-\min(\alpha,\beta)}. 
$$

(6.3)

The rate $\alpha$ corresponds to the spatial approximation of a single function in $W^r$ as given by (6.2), and the rate $\beta$ is given by (2.10). The constant $C$ in (6.3) is independent of $u$ and $n$.

The rate $\min(\alpha,\beta)$ in (6.3) is the same rate of fully discrete best adaptive $n$-term approximation in $\mathcal{V}_2(V)$ based on $\ell_p$-summability of $(\|u_s\|_V)_{s \in F}$ and $\ell_p$-summability of $(\|u_s\|_{W^r})_{s \in F}$ proven in [3], where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$. This rate can be achieved by fully discrete linear non-adaptive approximation when $(\|u_s\|_V)_{s \in F}$ and $(\|u_s\|_{W^r})_{s \in F}$ have $\ell_p$-summable and $\ell_p$-summable majorant sequences, respectively [33].

Theorem 6.2 Let $1 \leq p \leq \infty$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and $W^r$ with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 2$. For $\xi > 0$ let $G(\xi)$ be the set defined as in (3.5). Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $\dim(V(G(\xi_n))) \leq n$ and

$$
\|u - \mathcal{S}_{G(\xi_n)}u\|_{V_p(V)} \leq Cn^{-\min(\alpha,\beta)}. 
$$

(6.4)

The rate $\alpha$ corresponds to the spatial approximation of a single function in $W^r$ as given by (6.2). The rate $\beta$ is given by (3.19) The constant $C$ in (6.4) is independent of $u$ and $n$.

For polynomial interpolation approximation and integration, we keep all definitions and notations in Section 3 with a proper modification for the affine case. For example, for univariate interpolation and integration we take a sequence of points $Y_n = (y_{n,k})_{k=0}^{p}$ in $\mathbb{I}$ such that

$$
-\infty < y_{n,0} < \cdots < y_{n,n-1} < y_{n,n} < +\infty; \quad y_{n,0} = 0.
$$

Sequences of points $Y_n = (y_{n,k})_{k=0}^{p}$ satisfying the inequality (3.14), are the symmetric sequences of the Chebyshev points, the symmetric sequences of the Gauss-Lobatto (Clenshaw-Curtis) points and the nested sequence of the $\mathbb{R}$-Leja points, see [9] for details.

Theorem 6.3 Let $1 \leq p \leq \infty$. Let $f \in H^{r-2}(D)$ and Assumption III hold. Let the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and $W^r$ with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 2$. Assume that $(Y_n)_{n \in \mathbb{N}}$ is a sequence such that every $Y_n$ satisfies the condition (3.14) for some positive numbers $\tau$ and $C$. For $\xi > 0$ let $G(\xi)$ be the set defined as in (3.5). Then for each $n \in \mathbb{N}$ there exists a number $\xi_n$ such that $|G(\xi_n)| \leq n$ and

$$
\|u - \mathcal{I}_{G(\xi_n)}u\|_{V_p(V)} \leq Cn^{-\min(\alpha,\beta)}. 
$$

(6.5)

The rate $\alpha$ corresponds to the spatial approximation of a single function in $W^r$ as given by (6.2). The rate $\beta$ is given by (3.19) The constant $C$ in (6.5) is independent of $u$ and $n$.

The rates in (6.3)–(6.5) for some non-adaptive approximations have been proven in the case when $(\|u_s\|_V)_{s \in F}$ and $(\|u_s\|_{W^r})_{s \in F}$ have $\ell_p$-summable and $\ell_p$-summable majorant sequences, respectively, which are derived from the analyticity of the solution $u$, where $1/p_1 = 1/q_1 + 1/2$ and $1/p_r = 1/q_r + 1/2$, see [33].

Theorem 6.4 Let $f \in H^{r-2}(D)$ and Assumption III hold. Let $a = b$ in the definition of the Jacobi probability measure $\sigma_{a,b}(y)$ in (6.1), and the assumptions of Lemma 6.1 hold for the spaces $W^1 = V$ and $W^r$ with some $0 < q_1 \leq q_r < \infty$ with $q_1 < 4$. Assume that $(Y_n)_{n \in \mathbb{N}}$ is a sequence such that every $Y_n$ is symmetric and satisfies the condition (3.14) for some positive numbers $\tau$ and $C$. For $\xi > 0$ let $G_{ov}(\xi)$ be the set defined as in (3.10). Then we have the following.
(i) For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \(|G_{ev}(\xi_n)| \leq n\) and
\[
\left\| \int_{1}^{\infty} u(y) d\gamma(y) - Q_{G_{ev}(\xi_n)} u \right\|_V \leq Cn^{-\min(\alpha, \beta)}.
\] (6.6)

(ii) Let \( \phi \in V' \) be a bounded linear functional on \( V \). For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \(|G_{ev}(\xi_n)| \leq n\) and
\[
\left| \int_{1}^{\infty} \langle \phi, u(y) \rangle d\gamma(y) - Q_{G_{ev}(\xi_n)} \langle \phi, u \rangle \right| \leq Cn^{-\min(\alpha, \beta)}.
\] (6.7)

The rate \( \alpha \) corresponds to the spatial approximation of a single function in \( W^r \) as given by (6.2). The rate \( \beta \) is given by
\[
\beta := \left( \frac{2}{q_1} - \frac{1}{2} \right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{2}{q_1} - \frac{2}{q_r}.
\] (6.8)

The constants \( C \) in (6.6) and (6.7) are independent of \( u \) and \( n \).

The rate in (6.6)–(6.7) can be improved as \( \min(\alpha, \frac{2}{q_1} - \frac{\alpha}{\alpha + \delta}) \) if \( (\|u_s\|_V)_{s \in S} \) and \( (\|u_s\|_{W^r})_{s \in S} \) have \( \ell_{p_1}^- \) and \( \ell_{p_r}^- \)-summable majorant sequences, respectively, where \( 1/p_1 = 1/q_1 + 1/2 \) and \( 1/p_r = 1/q_r + 1/2 \), see [33].

**Corollary 6.1** Let \( a = b \) in the definition of the Jacobi probability measure \( \sigma_{a,b}(y) \) in (6.1), and the assumptions of Lemma 6.1 hold for the spaces \( W^1 = V \) with some \( 0 < q < 4 \). Assume that \( (Y_n)_{n \in \mathbb{N}_0} \) is a sequence such that every \( Y_n \) is symmetric and satisfies the condition (3.14) for some positive numbers \( \tau \) and \( C \). For \( \xi > 0 \) let \( \Lambda_{ev}(\xi) \) be the set defined as in (3.41). Then we have the following.

(i) For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \(|\Gamma(\Lambda_{ev}(\xi_n))| \leq n\) and
\[
\left\| \int_{1}^{\infty} u(y) d\gamma(y) - Q_{\Lambda_{ev}(\xi_n)} u \right\|_V \leq Cn^{-(2/q - 1/2)}.
\] (6.9)

(ii) Let \( \phi \in V' \) be a bounded linear functional on \( V \). For each \( n \in \mathbb{N} \) there exists a number \( \xi_n \) such that \(|\Gamma(\Lambda_{ev}(\xi_n))| \leq n\) and
\[
\left| \int_{1}^{\infty} \langle \phi, u(y) \rangle d\gamma(y) - Q_{\Lambda_{ev}(\xi_n)} \langle \phi, u \rangle \right| \leq Cn^{-(2/q - 1/2)}.
\] (6.10)

The constants \( C \) in (6.9) and (6.10) are independent of \( u \) and \( n \).

The rate in (6.9) and (6.10) can be improved as \( 2/q \) if the sequence \( (\|u_s\|_V)_{s \in S} \) has an \( \ell_{p}^- \)-summable majorant sequence, where \( 1/p = 1/q + 1/2 \), see [34].

**Acknowledgments.** This work is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 102.01-2017.05. It was partially supported by a grant from the Simion Foundation and EPSRC Grant Number EP/R014604/1. The author would like to thank the Isaac Newton Institute for Mathematical Sciences for partial support and hospitality during the programme Approximation, sampling and compression in data science when work on this paper was partially undertaken. He expresses special thanks to Dr. Nguyen Van Kien for valuable remarks and comments.
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