1 Introduction

Recently, we [9] and, independently, Du and Scott [11], defined an analogue of the $q$–Schur algebra [6, 7] for an Iwahori–Hecke algebra of type $B$. In this paper we study an analogue of the $q$–Schur algebra for an arbitrary Ariki–Koike algebra.

The Ariki–Koike algebra $\mathcal{H}$ is a cyclotomic algebra of type $G(r, 1, n)$ [2], and it becomes the Iwahori–Hecke algebra of type $A$ or $B$ when $r = 1$ or $2$ respectively. By working over a ring $R$ which contains a primitive $r$th root of unity, and by specializing the parameters appropriately, the Ariki–Koike algebra turns into the group algebra $R(C_r \wr \mathfrak{S}_n)$ of the wreath product of the cyclic group $C_r$ of order $r$ with the symmetric group $\mathfrak{S}_n$ of degree $n$.

For each multicomposition $\lambda$ of $n$, we construct a right ideal $M^\lambda$ of $\mathcal{H}$ (see Definition 3.8). The cyclotomic $q$–Schur algebra is then defined to be $\mathcal{S} = \text{End}_{\mathcal{H}} ( \bigoplus \lambda M^\lambda )$. (Under the specialization above where $\mathcal{H} \cong R(C_r \wr \mathfrak{S}_n)$, the module $M^\lambda$ becomes a module induced from a subgroup of the form $(C_r \times \cdots \times C_r) \rtimes \mathfrak{S}_\lambda$.)


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In this paper we construct a cellular basis for the cyclotomic \( q \)-Schur algebra. As a consequence we obtain a Weyl module \( W^\lambda \) for each multipartition \( \lambda \) of \( n \). We show that \( W^\lambda \) has simple head \( F^\lambda \) and that the set \( \{F^\lambda\} \), as \( \lambda \) ranges over the multipartitions of \( n \), is a complete set of non-isomorphic irreducible \( \mathcal{P} \)-modules. Using the cellular structure of \( \mathcal{P} \), it is now easy to see that the cyclotomic \( q \)-Schur algebra is quasi-hereditary.

In order to prove these results about the cyclotomic \( q \)-Schur algebra, we need to examine the ideals \( M^\lambda \) in some detail. Using the cellular structure of the Ariki–Koike algebra \( \mathcal{H} \) (cf. [10, 13]), we obtain a basis of \( M^\lambda \) and a special series of submodules of \( M^\lambda \), known as a Specht series. From the Specht series of \( M^\lambda \) we construct the cellular basis of \( \mathcal{P} \).

## 2 The Ariki–Koike algebra

Throughout this paper, \( r \) and \( n \) will be fixed positive integers with \( r \geq 1 \) and \( n \geq 1 \).

Let \( R \) be a commutative ring with 1 and let \( q, Q_1, \ldots, Q_r \) be elements of \( R \) with \( q \) invertible. The Ariki–Koike algebra \( \mathcal{H} \) is the associative unital \( R \)-algebra with generators \( T_0, T_1, \ldots, T_{n-1} \) subject to the following relations

\[
(T_0 - Q_1) \cdots (T_0 - Q_r) = 0 \\
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 \\
(T_i + 1)(T_i - q) = 0 \\
T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i \\
T_j T_j = T_j T_j
\]

for \( 0 \leq i < j \leq n - 1 \) and for \( 1 \leq i \leq n - 1 \) and for \( 1 \leq i \leq n - 2 \) and for \( 0 \leq i < j - 1 \leq n - 2 \).

Suppose that \( \pi \in R \) is a primitive \( r \)th root of 1 and that \( q = 1 \) and \( Q_k = \pi^k \) for \( k = 1, 2, \ldots, r \). Then it follows from the definition that \( \mathcal{H} \) is isomorphic to \( R(C_r \wr \mathfrak{S}_n) \).

Let \( \mathfrak{S}_n = \mathfrak{S}(\{1, 2, \ldots, n\}) \) be the symmetric group on \( \{1, 2, \ldots, n\} \) and let \( s_i = (i, i + 1) \) for \( 1 \leq i < n \). Then \( s_1, s_2, \ldots, s_{n-1} \) are the standard Coxeter generators of \( \mathfrak{S}_n \). If \( w \in \mathfrak{S}_n \) then a word \( w = s_{i_1} \cdots s_{i_k} \) for \( w \) is a reduced expression for \( w \) if \( k \) is minimal; in this case we say that \( w \) has length \( k \) and write \( \ell(w) = k \). Given a reduced expression \( s_{i_1} \cdots s_{i_k} \) for \( w \in \mathfrak{S}_n \), let \( T_w = T_{i_1} \cdots T_{i_k} \); the relations in \( \mathcal{H} \) ensure that \( T_w \) is independent of the choice of reduced expression. We denote by \( \mathcal{H}(\mathfrak{S}_n) \) the subalgebra of \( \mathcal{H} \) generated by \( T_1, T_2, \ldots, T_{n-1} \). Then \( \mathcal{H}(\mathfrak{S}_n) \) has basis \( \{T_w \mid w \in \mathfrak{S}_n\} \), and it is isomorphic to an Iwahori–Hecke algebra of type \( A \).

Define elements \( L_m = q^{1-m} T_{m-1} \cdots T_1 T_0 T_1 \cdots T_{m-1} \) for \( m = 1, 2, \ldots, n \); these are analogues of the \( q \)-Murphy operators of the Iwahori–Hecke algebras of type \( A \) [5, 15]. An easy calculation using the relations in \( \mathcal{H} \) (cf. [5, (2.1), (2.2)]) shows that we have the following results.

(2.1) Suppose that \( 1 \leq i \leq n - 1 \) and \( 1 \leq m \leq n \). Then

(i) \( L_i \) and \( L_m \) commute.

(ii) \( T_i \) and \( L_m \) commute if \( i \neq m - 1, m \).

(iii) \( T_i \) commutes with \( L_i L_{i+1} \) and \( L_i + L_{i+1} \).
(iv) If \( a \in R \) and \( i \neq m \) then \( T_i \) commutes with \( (L_1 - a)(L_2 - a) \ldots (L_m - a) \).

Using the elements \( T_w \) and \( L_m \) defined above, Ariki and Koike gave a basis for \( \mathcal{H} \) as follows.

\[ \text{(2.2) Theorem (Ariki–Koike [1, (3.10)]) The algebra } \mathcal{H} \text{ is a free } R\text{-module with basis} \]
\[ \{ L_1^{c_1} L_2^{c_2} \ldots L_n^{c_n} T_w \mid w \in \mathcal{S}_n \text{ and } 0 \leq c_m \leq r - 1 \text{ for } m = 1, 2, \ldots, n \}. \]

In particular, \( \mathcal{H} \) is free of rank \( r^n n! \).

\[ \text{(2.3) Let } * \text{ be the } R\text{-linear antiautomorphism of } \mathcal{H} \text{ determined by } T_i^* = T_i \text{ for all } i \text{ with } 0 \leq i \leq n - 1. \text{ Then } T_w^* = T_{w^{-1}} \text{ and } L_m^* = L_m \text{ for all } w \in \mathcal{S}_n \text{ and } m = 1, 2, \ldots, n. \]

We therefore have the following result.

\[ \text{(2.4) } \{ T_w L_1^{c_1} L_2^{c_2} \ldots L_n^{c_n} \mid w \in \mathcal{S}_n \text{ and } 0 \leq c_m \leq r - 1 \text{ for } m = 1, 2, \ldots, n \} \text{ is a basis of } \mathcal{H}. \]

3 A cellular basis of \( \mathcal{H} \)

In their paper [12], which introduced the concept of cellular algebras, Graham and Lehrer gave a cellular basis of \( \mathcal{H} \), using the Kazhdan–Lusztig basis of \( \mathcal{H}(\mathcal{S}_n) \). We require a different cellular basis, namely one similar to the basis of \( \mathcal{H}(\mathcal{S}_n) \) introduced by Murphy [16]. Although the construction of the cellular basis of \( \mathcal{H} \) in this section is similar to that in [10, 13], we are obliged to keep track of new information concerned with the cellular basis (see Corollary 3.24 below).

Consider the \( R\)-submodule of \( \mathcal{H} \) which is spanned by

\[ \{ L_1^{c_1} L_2^{c_2} \ldots L_n^{c_n} \mid 0 \leq c_i \leq r - 1 \text{ for } 1 \leq i \leq n \}. \]

When \( q = 1 \), this is a subalgebra of \( \mathcal{H} \), but one of the main difficulties of working with the Ariki–Koike algebra is that it is not a subalgebra in general. (To see this, consider \( L_2^2 \) when \( r = 2 \).) We shall need certain elements \( u_{a}^{+} \) of this \( R\)-submodule of \( \mathcal{H} \), and we introduce these now.

\[ \text{(3.1) Definition Suppose that } a = (a_1, \ldots, a_r) \text{ is an } r\text{-tuple of integers } a_i \text{ such that } 0 \leq a_i \leq n \text{ for all } i. \text{ Let } u_{a}^{+} = u_{a,1} u_{a,2} \ldots u_{a,r} \text{ where} \]
\[ u_{a,k} = \prod_{m=i}^{a_k} (L_m - Q_k) \text{ for } 1 \leq k \leq r. \]
(3.2) **Remarks**  
(i) Suppose that every $a_k$ is non-zero. Then $(L_1 - Q_k)$ is a factor of each $u_{a,b}$; so $u_a^+$ has a factor

$$
\prod_{k=1}^{r} (L_1 - Q_k) = \prod_{k=1}^{r} (T_0 - Q_k) = 0.
$$

Therefore, $u_a^+$ is zero in this case.

(ii) Rearranging the order of $a_1, a_2, \ldots, a_r$ amounts just to reordering the parameters $Q_1, Q_2, \ldots, Q_r$. For example, if we define $u_a^- = u_{(a_r,a_{r-1},\ldots,a_1)}^+$ then $u_a^-$ is obtained from $u_a^+$ by replacing $Q_k$ by $Q_{r-k+1}$ for $1 \leq k \leq r$.

(iii) In practice, we shall use $u_a^+$ only for $r$-tuples $a = (a_1, a_2, \ldots, a_r)$ such that $0 = a_1 \leq a_2 \leq \ldots \leq a_r \leq n$. Our last two remarks show that there is no loss in doing this, and that we could equally well work with the elements $u_a^-$ defined in Remark (ii).

(3.3) **Example** Suppose that $r = 4$, $n \geq 5$ and $a = (0, 2, 4, 5)$. Then

$$
u_a^+ = (L_1 - Q_2)(L_2 - Q_2) \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)$$

Our first lemma relates $u_a^+$ and $u_b^+$ when $b$ is obtained from $a$ by increasing a single part by one.

(3.4) **Lemma** Let $a = (a_1, a_2, \ldots, a_r)$ and assume that $1 \leq k \leq r$ and $a_k + 1 \leq n$. Let $b = (a_1, \ldots, a_{k-1}, a_k + 1, a_{k+1}, \ldots, a_r)$. Then, for some $h_1, h_2 \in \mathcal{H}(\mathcal{S}_n)$, we have $u_a^+ T_a T_{a-1} \ldots T_{1} T_0 = u_a^+ h_1 + u_b^+ h_2$.

**Proof:** The definition of $u_b^+$ gives $u_b^+ = u_a^+(L_{{a_k}+1} - Q_k)$. Hence,

$$u_a^+ T_a T_{a-1} \ldots T_{1} T_0 T_1 \ldots T_{a_k-1} T_{a_k} = q^{a_k} u_a^+ T_{a_k+1} = q^{a_k} Q_k u_a^+ + q^{a_k} u_b^+.$$  

The desired result follows by postmultiplying by $T_{a_k}^{-1} \ldots T_1^{-1}$.  

We next turn our attention to the subalgebra $\mathcal{H}(\mathcal{S}_n)$ of $\mathcal{H}$. Here we shall need the notation and combinatorics of multipartitions.

A composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a finite sequence of non-negative integers; we denote by $|\alpha|$ the sum of this sequence. A multicomposition of $n$ is an ordered $r$-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{k=1}^{r} |\lambda^{(k)}| = n$. We call $\lambda^{(k)}$ the $k$th component of $\lambda$. A partition is a composition whose parts are non-increasing; a multicomposition is a multipartition if all its components are partitions.

For each composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ with $|\alpha| = m$ we have a Young subgroup $\mathcal{S}_\alpha = \mathcal{S}_{\alpha_1} \times \mathcal{S}_{\alpha_2} \times \cdots$ of $\mathcal{S}_m$. Similarly, to each multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$ we associate the Young subgroup $\mathcal{S}_\lambda = \mathcal{S}_{\lambda^{(1)}} \times \mathcal{S}_{\lambda^{(2)}} \times \cdots \times \mathcal{S}_{\lambda^{(r)}}$ of $\mathcal{S}_n$.

We are now in a position to define certain key elements $m_\lambda$ of $\mathcal{H}$. The element $m_\lambda$ depends upon the Young subgroup $\mathcal{S}_\lambda$ and also involves one of the elements $u_a^+$ defined above.
(3.5) **Definition** Suppose that $\lambda$ is a multicomposition of $n$ and define $a = (a_1, a_2, \ldots, a_r)$ by $a_k = \sum_{i=1}^{k-1} |\lambda(i)|$. Let $x_\lambda = \sum_{w \in G_\lambda} T_w$ and set $m_\lambda = u_\lambda^- x_\lambda$.

(3.6) **Example** Suppose that $\lambda = ((2,1), (1^2), (2))$. Then $G_\lambda = G_2 \times G_1 \times G_1 \times G_1 \times G_2$, $a = (0, 3, 5)$, and

$$m_\lambda = (L_1 - Q_2)(L_2 - Q_3)(L_3 - Q_2)
\times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3)(L_5 - Q_3)
\times (1 + T_1)(1 + T_5).$$

(3.7) **Remark** If $\alpha = (|\lambda(1)|, |\lambda(2)|, \ldots, |\lambda(r)|)$ then all of the elements in $\mathcal{H}(G_\alpha)$ commute with $u_\alpha^+$ by (2.1)(iv). In particular, $m_\lambda = u_\alpha^- x_\lambda = x_\lambda u_\alpha^+$. Hence, $m_\lambda^* = m_\lambda$, where $*$ is the antiautomorphism of (2.3).

The $\mathcal{H}$–modules which will be our main concern in this paper are the right ideals generated by the $m_\lambda$, as $\lambda$ varies over the multicompositions of $n$. The cyclotomic $q$–Schur algebra will be built from endomorphisms between such right ideals.

(3.8) **Definition** Suppose that $\lambda$ is a multicomposition of $n$. Let $M^{\lambda} = m_\lambda \mathcal{H}$.

We leave the proof of the following remarks to the reader.

(3.9) **Remarks** (i) If the multicomposition $\mu$ is obtained from $\lambda$ by reordering the parts in each component then $M^\mu \cong M^\lambda$.

(ii) Suppose that $q = 1$ and that $Q_k = \pi^k$, for $k = 1, 2, \ldots, r$, where $\pi$ is a primitive $r$th root of unity in $R$. Then $\mathcal{H} \cong R(C_r \times G_n)$. Let $\alpha_k = |\lambda(k)|$ for $1 \leq k \leq r$. Then $M^\lambda$ is induced from a module $U$ for the subgroup $(C_r^{\alpha_1} \times C_r^{\alpha_2} \times \cdots \times C_r^{\alpha_r}) \times G_\lambda$. The restriction of $U$ to the subgroup $C_r^{\alpha_1} \times \cdots \times C_r^{\alpha_r}$ has the form $U_r^{\alpha_1} \otimes \cdots \otimes U_r^{\alpha_r}$ where $U_k$ has rank $k$ for $1 \leq k \leq r$. The restriction of $U$ to $G_\lambda$ is the trivial module.

We shall construct a basis of $M^\lambda$, and study $\mathcal{H}$–homomorphisms between the various modules $M^\lambda$. To this end, we introduce $\lambda$–tableaux.

The diagram of a composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ is $\{(i, j) \mid 1 \leq i \text{ and } 1 \leq j \leq \alpha_i\}$, which we regard as an array of nodes, or boxes, in the plane. The diagram of a multicomposition is the ordered $r$–tuple of the diagrams of its components.

Let $\lambda$ be a multicomposition of $n$. A $\lambda$–tableau $t = (t^{(1)}, \ldots, t^{(r)})$ is obtained from the diagram of $\lambda$ by replacing each node by one of the integers $1, 2, \ldots, n$, allowing no repeats. We call the tableaux $t^{(k)}$ the components of $t$.

(3.10) **Definition** (i) A $\lambda$–tableau is row standard if the entries in each row of each component increase from left to right.

(ii) A $\lambda$–tableau $t$ is standard if $\lambda$ is a multipartition of $n$, $t$ is row standard and the entries in each column of each component of $t$ increase from top to bottom.

(iii) If $\lambda$ is a multipartition of $n$, then let $\text{Std}(\lambda)$ be the set of standard $\lambda$–tableaux.
Note, particularly, that while row standard $\lambda$-tableaux are defined for all multicompositions $\lambda$, there exist standard $\lambda$-tableaux only if $\lambda$ is a multipartition of $n$.

We require partial orders on the set of multicompositions and on the set of row standard tableaux.

If $t$ is a row standard $\lambda$-tableau and $1 \leq m \leq n$, then the entries $1, 2, \ldots, m$ in $t$ occupy the diagram of a multicomposition; let $t \downarrow m$ denote this multicomposition. For example, $t \downarrow n = \lambda$. We use this notation in our next definition.

**Definition** Suppose that $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ are multicompositions of $n$.

(i) We say that $\lambda$ dominates $\mu$, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda^{(k)} \geq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{i=1}^{j} \mu^{(k)}$$

for all $k$ and $j$ with $1 \leq k \leq r$ and $j \geq 0$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ then we write $\lambda \triangleright \mu$.

(ii) Suppose that $s$ is a row standard $\lambda$-tableau and that $t$ is a row standard $\mu$-tableau. We say that $s$ dominates $t$, and write $s \triangleright t$ if $s \downarrow m \triangleright t \downarrow m$ for all $m$ with $1 \leq m \leq n$.

If $s \triangleright t$ and $s \neq t$ then we write $s \triangleright t$.

For example, if $n = r = 2$, then the multipartitions of 2 are ordered by $((2),(0)) \triangleright ((1^2),(0)) \triangleright ((1),(1)) \triangleright ((0),(2)) \triangleright ((0),(1^2))$.

Note that if $s$ is a row standard $\lambda$-tableau and $t$ is a row standard $\mu$ tableau such that $s \triangleright t$ then $\lambda \triangleright \mu$.

Our next definition gives another relation between tableaux.

**Definition** Suppose that $s$ is a tableau and that $1 < j < n$. We write $\text{comp}_s(j) = k$ if $j$ appears in the $k$th component $s^{(k)}$ of $s$.

Suppose that $t$ is another tableau. Then $\text{comp}_s = \text{comp}_t$ if $\text{comp}_s(j) = \text{comp}_t(j)$ for all $j$ with $1 \leq j \leq n$. We also write $\text{comp}_s \geq \text{comp}_t$ if $\text{comp}_s(j) \geq \text{comp}_t(j)$ for all $1 \leq j \leq n$; and $\text{comp}_s > \text{comp}_t$ if $\text{comp}_s > \text{comp}_t$ and $\text{comp}_s \neq \text{comp}_t$.

**Example** Let

$$s = \begin{pmatrix} 4 & 1 & 6 \\ 5 & 9 & 7 \end{pmatrix}, \quad t = \begin{pmatrix} 2 & 3 & 6 \\ 4 & 7 & 8 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \end{pmatrix} \begin{pmatrix} 5 & 9 \end{pmatrix}.$$ 

Then $s$ is row standard, and $t$ and $u$ are standard. We have $u \triangleright s$ and $u \triangleright t$ but $s$ and $t$ are incomparable in the dominance order. Also, $\text{comp}_t > \text{comp}_u$, but there are no other equations or inequalities between $\text{comp}_s$, $\text{comp}_t$ and $\text{comp}_u$.

Let $\lambda$ be a multicomposition of $n$. The symmetric group $S_n$ acts from the right on the set of $\lambda$–tableaux by permuting the entries in each tableau. Let $t^\rho$ be the $\lambda$–tableau where $1, 2, \ldots, n$ appear in order along the rows of the first component, and then along the rows
of the second component, and so on. The row stabilizer of \(t^\lambda\) is the Young subgroup \(\mathcal{S}_\lambda\) of \(\mathcal{S}_n\). For example, if \(\lambda = ((3, 2), (1^2), (3))\) then

\[
t^\lambda = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix}.
\]

For a row standard \(\lambda\)-tableau \(s\), let \(d(s)\) be the element of \(\mathcal{S}_n\) such that \(s = t^\lambda d(s)\). Then \(d(s)\) is a distinguished right coset representative of \(\mathcal{S}_\lambda\) in \(\mathcal{S}_n\) and we obtain, in this way, a correspondence between the set of row standard \(\lambda\)-tableaux and the set of right coset representatives of \(\mathcal{S}_\lambda\) in \(\mathcal{S}_n\).

Recall the antiautomorphism \(*\) of \(\mathcal{H}\) from (2.3).

**Definition** (3.14) Suppose that \(\lambda\) is a multicomposition of \(n\) and that \(s\) and \(t\) are row standard \(\lambda\)-tableaux. Let \(m_{st} = T^*_{d(s)} m_\lambda T^*_{d(t)}\).

Note that \(m_\lambda = m_{\lambda\lambda}\). Also, \(m_{st}^* = m_{ts}\) (cf. Remark 3.7).

One of the aims of this section is to show that the elements \(m_{st}\), as \((s, t)\) varies over the ordered pairs of standard tableaux of the same shape, give a cellular basis of \(\mathcal{H}\). We shall also establish useful properties of the right ideals \(M^\lambda\). Initially, though, we concentrate upon the two-sided ideal generated by \(m_\lambda\).

**Lemma** (3.15) Suppose that \(\lambda\) is a multicomposition of \(n\) and that \(s\) and \(t\) are row standard \(\lambda\)-tableaux. Let \(h \in \mathcal{H}(\mathcal{S}_n)\). Then \(m_{st} h\) is a linear combination of terms of the form \(m_{sv}\), where each \(v\) is a row standard \(\lambda\)-tableau.

**Proof:** Suppose that \(w \in \mathcal{S}_n\). Then there exist \(y \in \mathcal{S}_\lambda\) and a distinguished right coset representative \(d\) of \(\mathcal{S}_\lambda\) in \(\mathcal{S}_n\) such that \(w = yd\) and \(\ell(w) = \ell(y) + \ell(d)\). Hence, \(x_\lambda T_w = x_\lambda T_y T_d = q^{\ell(y)} x_\lambda T_d\). If \(m_\lambda = u^+_nx_\lambda\) then

\[
m_{st} T_w = T^*_{d(s)} u^+_n x_\lambda T_w = q^{\ell(y)} T^*_{d(s)} u^+_n x_\lambda T_d = q^{\ell(y)} m_{sv}
\]

where the tableau \(v = t^\lambda d\) is row standard.

Now, \(m_{st} h = m_{st} T_{d(t)} h\), and this is a linear combination of terms of the form \(m_{sv} T_w\) with \(w \in \mathcal{S}_n\). Therefore, the required result follows.

In order to apply a result of Murphy in Proposition 3.18 below, we require a combinatorial lemma which concerns the dominance order on row standard tableaux. A preliminary definition sets the scene.

**Definition** (3.16) We say that a tableau \(s\) is of the initial kind (for \(\lambda\)) if \(\text{comp}_s = \text{comp}_\lambda\).

Note that a \(\mu\)-tableau \(s\) can be of the initial kind for \(\lambda\) even though \(\mu \neq \lambda\).
Lemma 3.17 Suppose that $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is a multicomposition of $n$, and let $\alpha = (|\lambda^{(1)}|, \ldots, |\lambda^{(r)}|)$. Suppose that $w$ is a distinguished right coset representative of $S_\alpha$ in $S_n$ and that $s$ is a row standard $\lambda$-tableau of the initial kind. Then the following hold.

(i) The tableau $sw$ is row standard.
(ii) If $s$ is standard then $sw$ is standard.
(iii) If $t$ is a row standard $\lambda$-tableau of the initial kind with $s \triangleright t$ then $sw \triangleright tw$.

Proof: The fact that $w$ is a distinguished right coset representative of $S_\alpha$ in $S_n$ implies that whenever $x$ and $y$ are integers such that $x$ and $y$ are in the same component of $s$ and $x < y$, then $sw > yw$. Hence, (i) and (ii) are true.

Now assume that $t$ is a row standard $\lambda$-tableau of the initial kind, and that $s \triangleright t$. Then $d(s) \triangleright d(t)$ in the Bruhat–Chevalley order $\triangleright$ on $S_n$, by [4, (1.3)]. (Our notation for the order is such that $1 \triangleright v$ for all $v \in S_n$.) Since $w$ is a distinguished right coset representative of $S_\alpha$ in $S_n$, we also have $\ell(d(t)w) = \ell(d(t)) + \ell(w)$. The well-known “cancellation” property of the Bruhat–Chevalley order now lets us conclude that $d(s)w \triangleright d(t)w$. But $sw$ and $tw$ are row standard, so $d(s)w = d(sw)$ and $d(t)w = d(tw)$. By applying [4, (1.3)] again, we deduce that $sw \triangleright tw$. □

Proposition 3.18 Suppose that $\lambda$ is a multicomposition of $n$ and that $s$ and $t$ are row standard $\lambda$-tableaux. Then $m_{st}$ is a linear combination of terms of the form $m_{uw}$ where $u$ and $v$ are standard $\mu$-tableaux for some multipartition $\mu$ of $n$, and

(i) $u \triangleright s$ and $\operatorname{comp}_u = \operatorname{comp}_s$; and,
(ii) $v \triangleright t$ and $\operatorname{comp}_v = \operatorname{comp}_t$.

Proof: When $r = 1$, this is a Theorem of Murphy [16, (4.18)]. We deduce the general case from this.

Let $\alpha = (|\lambda^{(1)}|, \ldots, |\lambda^{(r)}|)$. We may write $s = s_1w_1$ and $t = t_2w_2$ where $s'$ and $t'$ are row standard $\lambda$-tableaux of the initial kind, and $w_1$ and $w_2$ are distinguished right coset representatives for $S_\alpha$ in $S_n$.

Let $a = (a_1, \ldots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|$, as in the definition of $m_\lambda$. We have $m_{st} = T_{T_{w_1}^*T_{w_2}}$ and $m_{s't'} = T_{T_{w_2}^*}u_\alpha^+x_\lambda T_{d(v')} = u_\alpha^+T_{T_{d(v')}^*}x_\lambda T_{d(v')}T_{d(v')}T_{d(v')}T_{d(v')}^*$ since $d(s') \in S_\alpha$.

We may write $T_{d(v')}^*x_\lambda T_{d(v')}^*$ as a product of $r$ commuting terms, one for each component of $\lambda$; say, $T_{d(v')}^*x_\lambda T_{d(v')}^* = x_1x_2 \ldots x_r$, where the $k$th term $x_k$ involves only elements $T_w$ with $a_k + 1, a_k + 2, \ldots, a_{k+1}$. For example, $x_1 = T_{d(1)}^*x_\lambda T_{d(1)}^*$ where $s_1'$ is the first component of $s'$ and $t_1'$ is the first component of $t'$. By applying Murphy’s result [16, (4.18)] to the Hecke algebra $H(S_{\alpha_1})$ we may write $x_1$ as a linear combination of terms $T_{d(v_1')}^*x_\mu T_{d(v_1')}^*$ where $u_1'$ and $v_1'$ are standard $\mu^{(1)}$-tableaux for some partition $\mu^{(1)}$ of $a_1$, and $u_1' \triangleright s_1'$ and $v_1' \triangleright t_1'$.

We can apply the same technique for the other factors $x_2, \ldots, x_r$, to conclude that $T_{d(v')}^*x_\lambda T_{d(v')}^*$ is a linear combination of terms of the form $T_{d(v')}^*x_\mu T_{d(v')}^*$ where $u'$ and $v'$ are standard $\mu$-tableaux for some multipartition $\mu$ of $n$, and $u' \triangleright s'$ and $v' \triangleright t'$. Also, $u'$ and $v'$
are of the initial kind, and \(|\mu^{(k)}| = |\lambda^{(k)}|\) for \(1 \leq k \leq r\). Therefore, \(m_{uv'}v'\) is a linear combination of terms \(m_{uv'}\) where the sum runs over the same set of pairs \((u', v')\). Consequently, \(m_{st}\) is a linear combination of terms \(T_{w_1}^* m_{uv'} v'\).

Now, \(T_{d(v')v}w_2 = T_{d(v)w} = T_{d(v')w_2}\), so \(T_{w_1}^* m_{uv'}v'w_2 = T_{w_1}^* m_{uv}v\) where \(v = v'w_2\). By Lemma 3.17, since \(v'\) is of the initial kind, \(v = v'w_2\) is standard, and \(v'w_2 \geq t'w_2\); that is, \(u \geq t\). Moreover, \(\text{comp}_{v'w_2} = \text{comp}_{v'w_2}\) since \(v'\) and \(t'\) are of the initial kind.

Similar remarks applied to \(T_{w_1}^* T_{d(v')}^*\) now complete the proof of the Proposition. □

(3.19) Corollary Suppose that \(\lambda\) is a multicomposition of \(n\) and that \(s\) and \(t\) are row standard \(\lambda\)-tableaux. If \(h \in \mathcal{H}(\mathfrak{S}_n)\) then \(m_{st}h\) is a linear combination of terms of the form \(m_{uv}\) where \(u\) and \(v\) are standard \(\mu\)-tableaux for some multipartition \(\mu\) of \(n\), and \(u \geq \lambda\), \(u \geq s\) and \(\text{comp}_u = \text{comp}_s\).

PROOF: Combine Lemma 3.15 and Proposition 3.18, and recall that \(u \geq s\) implies that \(\mu \geq \lambda\). □

Corollary 3.19 provides the kind of information we need when we multiply \(m_{st}\) by elements of \(\mathcal{H}(\mathfrak{S}_n)\). More complicated is our next proposition, which shows what happens when we multiply \(m_{st}\) by \(T_0\). It is vital to the proposition that \(\lambda\) is a multipartition, not merely a multicomposition.

(3.20) Proposition Suppose that \(\lambda\) is a multipartition of \(n\), and that \(s\) and \(t\) are standard \(\lambda\)-tableaux. Then \(m_{st}T_0 = x_1 + x_2\) where

(i) \(x_1\) is a linear combination of terms of the form \(m_{uv}\) where \(u\) and \(v\) are standard \(\lambda\)-tableaux, with \(u \geq s\) and \(\text{comp}_u = \text{comp}_s\), and

(ii) \(x_2\) is a linear combination of terms of the form \(m_{uv}\) where \(u\) and \(v\) are standard \(\mu\)-tableaux for some multipartition \(\mu\) of \(n\), with \(\mu \geq \lambda\) and \(\text{comp}_u \geq \text{comp}_s\).

PROOF: Let \(\alpha = (|\lambda^{(1)}|, \ldots, |\lambda^{(r)}|)\) and let \(a = (a_1, a_2, \ldots, a_r)\) where \(a_k = \sum_{i=1}^{k-1} |\lambda^{(i)}|\) for \(1 \leq k \leq r\). Write \(d(t) = yc\) with \(y \in \mathfrak{S}_a\) and \(c\) a distinguished right coset representative for \(\mathfrak{S}_\alpha\) in \(\mathfrak{S}_n\). Then the \(\alpha\)-tableau \(t^\alpha c\) is row standard. Assume that 1 is in row \(k\) of \(t^\alpha c\); thus, \(1 \leq k \leq r\).

Now, \(c = (a_k, a_k + 1)(a_k - 1, a_k) \ldots (1, 2)c', \) where \(\ell(c) = a_k + \ell(c')\) and \(c'\) fixes 1. Thus, \(T_c = T_{a_k}T_{a_k-1} \ldots T_1T_c\) and \(T_{t^t}T_0 = T_0T_{t^t}\). Let \(b = (a_1, \ldots, a_k + 1, \ldots, a_r)\). Then

\[ u_1^+ T_c T_0 = u_1^+ T_{a_k}T_{a_k-1} \ldots T_1T_0 T_{t^t} = u_1^+ a_1^+ + u_2^+ h_2 \]

for some \(h_1, h_2 \in \mathfrak{S}_a\) by Lemma 3.4. Since \(y \in \mathfrak{S}_\alpha\) and \(t^ty\) is standard, \(y\) fixes \(a_k + 1\); therefore, \(T_y\) commutes with \(u_1^+\). Hence, using Remark 3.7, we have

\[ m_{st}T_0 = T_{d(s)}^* u_1^+ a_1^+ \lambda T_{d(t)} T_0 = T_{d(s)}^* a_1^+ \lambda T_y T_c T_0 = T_{d(s)}^* \lambda T_y u_1^+ h_1 + u_2^+ h_2 \]

\[ = T_{d(s)}^* \lambda T_y (u_1^+ h_1 + u_2^+ h_2) = T_{d(s)}^* \lambda (u_1^+ T_y h_1 + u_2^+ T_y h_2). \]
The first term, $T_{d(s)}^* x_{\lambda} u_b^+ T_y h_1$, is a linear combination of terms of the required form by Corollary 3.19. If $k - 1$ then $u_i^+ = 0$ by Remark 3.2(i), and the proof is complete in this case. Assume therefore that $k \geq 2$. We turn out attention to the term $T_{d(s)}^* x_{\lambda} u_b^+ T_y h_2$. We now need to digress in order to prove that $T_{d(s)}^* x_{\lambda} u_b^+ T_y h_2$ has the required form.

Define the multicomposition $\nu = (\nu^{(1)}, \ldots, \nu(r))$ of $n$ by $\nu^{(i)} = \lambda^{(i)}$ for $i \neq k - 1, k$, $\nu^{(k-1)} = (\lambda_1^{(k-1)}, \lambda_2^{(k-1)}, \ldots, 1)$ and $\nu^{(k)} = (\lambda_1^{(k)} - 1, \lambda_2^{(k)}, \ldots)$. (Thus we reduce the first part of the $k$th component of $\lambda$ by 1, and adjoin a new part of size 1 to the end of the $(k - 1)$th component of $\lambda$.) Note that $\nu \triangleright \lambda$.

Let $l = \lambda_1^{(k)}$ be the first part of $\lambda^{(k)}$. The entries in the first row of the $k$th component of $t^\nu$ are then $a_k + 1, a_k + 2, \ldots, a_k + l$. Let $G_1$ be the symmetric group on these numbers, and let $G_2$ be the symmetric group on $a_k + 2, \ldots, a_k + l$. Let $c_1, c_2, \ldots, c_l$ be the distinguished right coset representatives for $G_2$ in $G_1$, ordered in terms of increasing length. Then the tableaux $t^\nu c_l$ are row standard, and they agree with $t^\nu$ except on the last row of the $(k - 1)$th component and the first row of the $k$th component. The last row of the $(k - 1)$th component of $t^\nu c_l$ contains the single entry $a_k + i$.

We are given that $s = t^\nu d(s)$ is a row standard $\lambda$-tableau. Let $u_i = t^\nu c_l d(s)$ for $1 \leq i \leq l$. Each $u_i$ agrees with $s$ except on the last row of the $(k - 1)$th component of $u_i$ and on the first row of the $k$th component of $u_i$. Furthermore, since $(a_k + 1)d(s) < \cdots < (a_k + l)d(s)$, it follows that each $u_i$ is row standard.

We also have $\text{comp}_u(j) = \text{comp}_u(j)$ if $j \notin \{(a_k + 1)d(s), \ldots, (a_k + l)d(s)\}$ and, for $j \in \{(a_k + 1)d(s), \ldots, (a_k + l)d(s)\}$, we have $\text{comp}_u(j) = k - 1$ or $k$ and $\text{comp}_u(j) = k$. Therefore, $\text{comp}_u \triangleright \text{comp}_u$.

Now $x_\lambda$ has a factor

$$\sum_{w \in G_1} T_w = \sum_{i=1}^l T_{c_l}^* \sum_{w \in G_2} T_w.$$ 

Hence, $x_\lambda = \sum_{i=1}^l T_{c_l}^* x_{\nu}$. Note that $m_{\nu} = x_{\nu} u_b^+$. Thus,

$$T_{d(s)}^* x_\lambda u_b^+ = T_{d(s)}^* \sum_{i=1}^l T_{c_l}^* x_{\nu} u_b^+ = \sum_{i=1}^l T_{c_l d(s)}^* x_{\nu} u_b^+ = \sum_{i=1}^l m_{u_i} u_{\nu}.$$ 

Hence, by Corollary 3.19, $T_{d(s)}^* x_\lambda u_b^+ T_y h_2$ is a linear combination of the required form.

This completes the proof of the Proposition. \hfill $\Box$

3.21 Definition Suppose that $\lambda$ is a multicomposition of $n$.

(i) Let $N^\lambda$ be the $R$-module spanned by

$$\left\{ m_{\mu \lambda} \mid s \text{ and } t \text{ are standard } \mu \text{-tableaux for some multipartition } \mu \text{ of } n \text{ with } \mu \triangleright \lambda \right\}.$$
(ii) Let $N^\lambda$ be the $R$-module spanned by

$$\left\{ m_{st} \mid s \text{ and } t \text{ are standard } \mu \text{-tableaux for some multipartition } \mu \text{ of } n \text{ with } \mu \trianglerighteq \lambda \right\}.$$ 

We now apply Propositions 3.18 and 3.20 to obtain a sequence of useful results.

**Proposition 3.22** Suppose that $\lambda$ is a multicomposition of $n$. Then $N^\lambda$ and $\overline{N^\lambda}$ are two-sided ideals of $\mathcal{H}$.

**Proof:** Corollary 3.19 shows that $N^\lambda$ is closed under postmultiplication by $T_1, T_2, \ldots, T_{n-1}$ and Proposition 3.20 shows that $N^\lambda$ is also closed under postmultiplication by $T_0$. Because $\mathcal{H}$ is generated by $T_0, T_1, \ldots, T_{n-1}$, we deduce that $N^\lambda$ is a right ideal of $\mathcal{H}$. Since $m_{st}^* = m_{ts}$, by applying the anti-automorphism $\ast$ we see that $N^\lambda$ is also a left ideal of $\mathcal{H}$.

Finally, $\overline{N^\lambda}$ is a two-sided ideal of $\mathcal{H}$ since $\overline{N^\lambda} = \sum_{\mu \trianglerighteq \lambda} N^\mu$. □

A proof very similar to that of Proposition 3.22 gives our next result.

**Proposition 3.23** Suppose that $\lambda$ is a multicomposition of $n$ and that $s$ is a row standard $\lambda$-tableau. Let $I_s$ be the $R$-module spanned by

$$\left\{ m_{uv} \mid u \text{ and } v \text{ are standard } \mu \text{-tableaux for some multipartition } \mu \text{ of } n \text{ with } \mu \trianglerighteq \lambda \text{ and } \text{comp}_s \geq \text{comp}_u \right\}.$$ 

Then $I_s$ is a right ideal of $\mathcal{H}$.

Recall that $M^\lambda = m_\lambda \mathcal{H}$.

**Corollary 3.24** Suppose that $\lambda$ is a multicomposition of $n$. Then every element of $M^\lambda$ is a linear combination of terms of the form $m_{uv}$ where $u$ and $v$ are standard $\mu$-tableaux for some multipartition $\mu$ of $n$ with $\mu \trianglerighteq \lambda$ and $\text{comp}_v \geq \text{comp}_u$.

**Proof:** Since $m_\lambda = m_{\lambda \lambda}$, Proposition 3.18 shows that $m_\lambda \in I_\lambda$ (note that $\lambda$ may not be a multipartition). Therefore, $M^\lambda \subseteq I_\lambda$ and the desired result follows by Proposition 3.23. □

**Proposition 3.25** Suppose that $\lambda$ is a multipartition of $n$ and that $t$ is a standard $\lambda$-tableau. Let $h \in \mathcal{H}$. Then for every standard $\lambda$-tableau $v$ there exists $r_v \in R$ such that, for all standard $\lambda$-tableau $s$, we have

$$m_{st}h \equiv \sum_{v \in \text{Std}(\lambda)} r_v m_{sv} \text{ mod } N^\lambda.$$
PROOF: Let $U$ be the $R$–module spanned by $\{ m_{\nu^t} \mid \nu^t \text{ is a standard } \lambda \text{–tableau} \}$. Note that $\nu^\lambda$ is the unique $\lambda$–tableau $u$ such that $u \succeq \nu^\lambda$. Therefore, $U + \overline{N^\lambda}$ is closed under postmultiplication by $T_1, T_2, \ldots, T_{n-1}$ by Proposition 3.18, and it is closed under postmultiplication by $T_0$ by Proposition 3.20. Hence $U + \overline{N^\lambda}$ is a right ideal of $\mathcal{H}$. Thus, for each $v \in \text{Std}(\lambda)$ there exists $r_v \in R$ such that

$$m_{\nu^\lambda} h \equiv \sum_{v \in \text{Std}(\lambda)} r_v m_{\nu^v} \mod \overline{N^\lambda}.$$ 

Multiply this congruence on the left by $T_d^*(\nu^\lambda)$, and use the fact that $\overline{N^\lambda}$ is an ideal, to obtain the congruence of the proposition.

(3.26) Theorem ([13, (1.7)]) The algebra $\mathcal{H}$ is a free $R$–module with basis

$$\mathcal{M} = \left\{ m_{st} \mid s \text{ and } t \text{ are standard } \lambda \text{–tableaux for some multipartition } \lambda \text{ of } n \right\}.$$ 

Moreover, $\mathcal{M}$ is a cellular basis of $\mathcal{H}$.

PROOF: Since $m_\lambda = 1$ when $\lambda = ((0), \ldots, (0), (1^n))$, Proposition 3.25 shows that $\mathcal{M}$ spans $\mathcal{H}$. By Theorem 2.2, $\mathcal{H}$ is free of rank $r^n n!$ Since this is also the cardinality of $\mathcal{M}$, we deduce that $\mathcal{M}$ is a basis of $\mathcal{H}$. Finally, the properties that $\mathcal{M}$ must satisfy in order to be a cellular basis of $\mathcal{H}$ (as given in [12, (1.1)]) are covered by Proposition 3.25 and the fact that $m_{st}^* = m_{ts}$.

(3.27) Definition We call $\mathcal{M}$ the standard basis of $\mathcal{H}$.

We can now apply Graham and Lehrer’s theory of cellular algebras [12] to describe the representation theory of $\mathcal{H}$.

(3.28) Definition Suppose that $\lambda$ is a multipartition of $n$. Let $z_\lambda = (\overline{N^\lambda} + m_\lambda) / \overline{N^\lambda}$. The Specht module $S^\lambda$ is the submodule of $\mathcal{H} / \overline{N^\lambda}$ given by $S^\lambda = z_\lambda \mathcal{H}$.

As in [12], or directly from Proposition 3.25 and Theorem 3.26, $S^\lambda$ is a free $R$–module with basis $\{ z_\lambda T_d(\nu^\lambda) \mid \nu^\lambda \text{ a standard } \lambda \text{–tableau} \}$.

Define a bilinear form $\langle \, , \, \rangle$ on the Specht module $S^\lambda$ by

$$m_\lambda T_d(s) T_d^*(t) m_\lambda \equiv (z_\lambda T_d(s), z_\lambda T_d(t)) m_\lambda \mod \overline{N^\lambda}.$$ 

By Proposition 3.25 (and the version of Proposition 3.25 obtained by applying the antiautomorphism $*$), the bilinear form is well defined (cf. [12]). Moreover, the bilinear form is symmetric and $\langle uh, v \rangle = \langle u, vh^* \rangle$ for all $u, v \in S^\lambda$ and all $h \in \mathcal{H}$ [12, (2.4)]. Consequently, $\text{rad } S^\lambda$, the radical of the bilinear form, is an $\mathcal{H}$–module.
(3.29) Definition Suppose that $\lambda$ is a multipartition of $n$. Let $D^\lambda = S^\lambda / \text{rad } S^\lambda$.

(3.30) Theorem Suppose that $R$ is a field. Then the non-zero $\mathcal{H}$ modules in

$$\{ D^\lambda \mid \lambda \text{ a multipartition of } n \}$$

form a complete set of non-isomorphic irreducible $\mathcal{H}$-modules. Moreover, each irreducible module $D^\lambda$ is absolutely irreducible.

PROOF: Since $\mathcal{H}$ is a cellular basis of $\mathcal{H}$ the Theorem follows from the general theory of cellular algebras [12, (3.4)].

The theory in [12] also shows that if $D^\mu \neq 0$ and $D^\mu$ is a composition factor of $S^\lambda$ then $\lambda \succeq \mu$.

When $r = 1$ the partitions $\mu$ for which $D^\mu \neq 0$ have been classified in [4]. When $r > 1$, if $q = 1$, or if the parameters $Q_k$ are powers of $q$, the multipartitions $\mu$ for which $D^\mu \neq 0$ are classified by the results of [12,13,14]; in type B, see also [8,10].

4 A Specht series for $M^\mu$

In the last section, we used various elements $m_\mu$ of $\mathcal{H}$ to construct a cellular basis of $\mathcal{H}$. We saw that the right ideal $M^\mu$ generated by $m_\mu$ is an analogue of an induced module; indeed, if $r = 1$ and $q = 1$ then $M^\mu$ is the permutation module of $\mathfrak{S}_n$ on the Young subgroup $\mathfrak{S}_\mu$. We shall use the right ideals $M^\mu$ of $\mathcal{H}$ to construct the cyclotomic $q$-Schur algebra, but, before that, we study the individual modules $M^\mu$ in more detail. In particular, we shall give a semistandard basis of $M^\mu$ and construct a Specht series for $M^\mu$.

We are going to define a new kind of $\lambda$-tableau. This will have entries which are ordered pairs $(i, k)$ where $i$ is a positive integer and $1 \leq k \leq r$. We denote such tableaux by capital letters.

(4.1) Definition Suppose that $\lambda$ is a multipartition of $n$ and that $\mu$ is a multicomposition of $n$. A $\lambda$-tableau $\mathfrak{s}$ has type $\mu$ if its entries are ordered pairs $(i, k)$, as above, and for all $m$ and $k$ the number of times $(i, k)$ is an entry in $\mathfrak{s}$ is $\mu_i^{(k)}$.

Next, we introduce a function which converts standard $\lambda$-tableaux (or, indeed any tableau whose entries are $1, 2, \ldots, n$) into a $\lambda$-tableau of type $\mu$.

(4.2) Definition Suppose that $\lambda$ is a multipartition of $n$ and that $\mu$ is a multicomposition of $n$. Let $\mathfrak{s}$ be a standard $\lambda$-tableau. Define $\mu(\mathfrak{s})$ to be the $\lambda$-tableau obtained from $\mathfrak{s}$ by replacing each entry $m$ in $\mathfrak{s}$ by $(i, k)$ if $m$ is in row $i$ of the $k$th component of $v^\mu$.

In our examples, we shall always write $i_k$ for the ordered pair $(i, k)$.
(4.3) Example Suppose that $\mu = ((3, 1), (2), (2, 1^2))$. Then

$$\nu^\mu = \begin{pmatrix} 1 & 2 & 3 \\ 4 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 9 \\ 10 \end{pmatrix} \quad \text{and} \quad \mu(\nu^\mu) = \begin{pmatrix} 1_1 & 1_1 & 1_1 \\ 2_1 \\ 1_2 & 1_2 \end{pmatrix}, \begin{pmatrix} 1_3 & 1_3 \\ 2_3 \end{pmatrix}.$$

Suppose that

$$s_1 = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 4 \\ 6 & 10 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 9 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 4 \\ 5 & 10 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 9 \end{pmatrix}$$

and

$$s_3 = \begin{pmatrix} 1 & 2 & 5 & 9 \\ 7 \\ 3 & 10 \end{pmatrix}, \begin{pmatrix} 4 & 6 \\ 8 \end{pmatrix}.$$ 

Then

$$\mu(s_1) = \mu(s_2) = \begin{pmatrix} 1_1 & 1_1 & 1_1 & 1_2 \\ 2_1 \end{pmatrix}, \begin{pmatrix} 1_2 & 3_3 \\ 2_3 \end{pmatrix}, \begin{pmatrix} 1_3 & 1_3 \\ 2_3 \end{pmatrix}$$

and

$$\mu(s_3) = \begin{pmatrix} 1_1 & 1_1 & 1_2 & 2_3 \\ 1_3 \end{pmatrix}, \begin{pmatrix} 1_1 & 3_3 \\ 2_3 \end{pmatrix}, \begin{pmatrix} 2_1 & 1_2 \\ 1_3 \end{pmatrix}.$$

Given two pairs $(i_1, k_1)$ and $(i_2, k_2)$, we write $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or $k_1 - k_2$ and $i_1 < i_2$. Note that if $s$ is a standard $\lambda$-tableau, then $\mu(s)$ is a $\lambda$-tableau of type $\mu$ whose entries are weakly increasing along rows and down columns. We next single out some of these tableaux $\mu(s)$.

(4.4) Definition Suppose that $\lambda$ is a multipartition of $n$ and that $\mu$ is a multicomposition of $n$. Let $S = (S^{(1)}, \ldots, S^{(r)})$ be a $\lambda$-tableau of type $\mu$. Then $S$ is semistandard if

(i) the entries in each row of each component $S^{(k)}$ of $S$ are non-decreasing; and

(ii) the entries in each column of each component $S^{(k)}$ of $S$ are strictly increasing; and

(iii) for each $k$ with $1 \leq k \leq r$, no entry in $S^{(k)}$ has the form $(i, l)$ with $l < k$.

Let $\mathcal{T}_0(\lambda, \mu)$ denote the set of semistandard $\lambda$-tableaux of type $\mu$.

For example, $\mu(s_1)$ in Example 4.3 is semistandard, but $\mu(s_3)$ is not, since the third part of Definition 4.4 is not satisfied.

The point of the third condition in Definition 4.4 is the following result (cf. Corollary 3.24).

(4.5) Suppose that $s$ is a standard $\lambda$-tableau. Then $\mu(s)$ satisfies Definition 4.4(iii) if and only if $\text{comp}_\mu \geq \text{comp}_s$.

If $\lambda$ is a multipartition and $S \in \mathcal{T}_0(\lambda, \mu)$, then it is clear that there exists a standard $\lambda$-tableau $s$ with $S = \mu(s)$. In order to say more about this, we introduce another definition.
(4.6) **Definition** If $s$ is a standard $\lambda$-tableau and $1 \leq m \leq n$ then let $\text{row}_s(m) = (i, k)$ if $m$ belongs to row $i$ of the $k$th component of $s$.

The definition of $\text{row}_s$ allows us to recover $s$ from the function $\text{row}_s$.

Suppose that $S \in T_0(\lambda, \mu)$ and let $s$ be a standard $\lambda$-tableau such that $\mu(s) = S$. If there exists an integer $i$ such that $\text{row}_s(i) \neq \text{row}_s(i+1)$ and $i$ and $i+1$ are in the same row of $t^t$ then the tableau $s_1 = s(i, i+1)$ is standard and $\mu(s_1) = S$. Also, $s \triangleright s_1$ if $\text{row}_s(i) < \text{row}_s(i+1)$, and $s_1 \triangleright s$ if $\text{row}_s(i) > \text{row}_s(i+1)$. Hence we have shown the following.

(4.7) Suppose that $S \in T_0(\lambda, \mu)$. Then there exist standard $\lambda$-tableaux $\text{first}(S)$ and $\text{last}(S)$ such that

(i) $\mu(\text{first}(S)) = \mu(\text{last}(S)) = S$; and,

(ii) if $s$ is any standard $\lambda$-tableau such that $\mu(s) = S$ then $\text{first}(S) \triangleright s \triangleright \text{last}(S)$.

(4.8) **Definition** Suppose that $\lambda$ is a multipartition of $n$ and that $\mu$ is a multicomposition of $n$. Let $S \in T_0(\lambda, \mu)$ and $t \in \text{Std}(\lambda)$. Then $m_{st}$ is the element of $H$ given by

$$m_{st} = \sum_{s \in \text{Std}(\lambda), \mu(s) = S} m_{st}.$$ 

Note that $m_{st}$ is a sum of standard basis elements $m_{st}$ where $\text{first}(S) \triangleright s \triangleright \text{last}(S)$.

(4.9) **Example** Suppose that $\lambda = ((4,3), (2,1), (2,1))$, $\mu = ((3^2,1), (1^2), (2,1^2))$ and

$$S = \left( \begin{array}{ccc} 1_1 & 1_1 & 2_1 \\ 2_1 & 2_1 & 3_1 \\ 1_2 & 3_3 \\ 2_2 \\ 3_3 \end{array} \right).$$

Then $S \in T_0(\lambda, \mu)$ and for any $t \in \text{Std}(\lambda)$ we have $m_{st} = m_{s_1 t} + m_{s_2 t} + m_{s_3 t}$ where

$$s_1 = \left( \begin{array}{ccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 13 \\ 9 & 12 \end{array} \right), \quad s_2 = \left( \begin{array}{ccc} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 \\ 8 & 13 \\ 9 & 12 \end{array} \right), \quad s_3 = \left( \begin{array}{ccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 \\ 8 & 13 \\ 9 & 12 \end{array} \right).$$

Here $s_1 = \text{first}(S)$ and $s_3 = \text{last}(S)$.

(4.10) **Lemma** Suppose that $\lambda$ is a multipartition of $n$ and that $\mu$ is a multicomposition of $n$. Let $S \in T_0(\lambda, \mu)$ and $t \in \text{Std}(\lambda)$. Then $m_{st} \in M^\mu$.

**Proof:** Let $a = (a_1, \ldots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu(i)|$, and $b = (b_1, \ldots, b_r)$ where $b_k = \sum_{i=1}^{k-1} |\lambda(i)|$. 

Let $s_1 = \text{first}(S)$ and let $d = d(s_1)$. Then, as in [4],

$$
\sum_{\mu(s) = 3} x_\lambda T_{d(s)} = \sum_{w \in S_\lambda \Delta S_\mu} T_w = h^* T_d x_\mu,
$$

where $h = \sum T_v$, the sum running over certain elements $v \in S_\lambda$. Since $h \in \mathcal{H}(S_\lambda)$, we have $h^* u_b^+ = u_b^+ h^*$; so we obtain

$$
m_{st} = \sum_{\mu(s) = 3} m_{st} = \sum_{\mu(s) = 3} T^*_{d(s)} x_\lambda u_b^+ T_{d(t)} = x_\mu T^*_d u_b^+ h^* T_{d(t)}.
$$

Therefore, since $m_{st} = x_\mu u_a^+$, it is sufficient to prove that $T^*_d u_b^+ \in u_a^+ \mathcal{H}$.

Assume that $1 \leq k \leq r$. Recall that $s_1 = \text{first}(S)$, $a_k = \sum_{i=1}^{k-1} |\mu(i)|$ and $b_k = \sum_{i=1}^{k-1} |\lambda(i)|$ and note that $a_k \leq b_k$ since $S$ is semistandard. Define $t_k$ to be the standard $\lambda$-tableau such that $1, 2, \ldots, a_k$ occupy the same positions in $s_1$ and $t_k$; $b_k + 1, \ldots, n$ occupy the same positions in $t^3$ and $t_k$; and that for $a_k + 1 \leq i < j \leq b_k$ we have $\text{row}_{u_b^+}(i) \leq \text{row}_{u_b^+}(j)$ — thus the numbers $a_k + 1, \ldots, b_k$ are inserted into $t_k$ in “row order”. The fact that $S$ is semistandard ensures that $t_k$ is well defined. Note that $t_1 = t^3$ and $t_r = s_1$.

Now define $w_k \in S_n$ inductively by requiring that $t_k = t^k w_1 w_2 \ldots w_k$. Then $w_1 = 1$ and $d = w_1 w_2 \ldots w_r$; moreover, $\ell(d) = \ell(w_1) + \ldots + \ell(w_r)$. By construction, the element $w_k$ fixes each of the numbers $1, 2, \ldots, a_k-1$ and it also fixes each of $b_k + 1, b_k + 2, \ldots, n$.

For $1 \leq k \leq r$, let $u_{a,k}$ and $u_{b,k}$ be as defined in Definition 3.1, and let $v_k = \prod_{m=a_k+1}^{b_k} (L_m - Q_k)$. Then $u_{b,k} = u_{a,k} v_k$.

Now, $T_{w_k}$ commutes with $u_{a,l}$ for $l < k$ (since $w_k$ fixes $1, 2, \ldots, a_k-1$), and $T_{w_k}$ commutes with $u_{b,l}$ for $l > k$ (since $w_k$ fixes $b_k + 1, b_k + 2, \ldots, n$); see (2.1(iv)). Therefore,

$$
T^*_d u_b^+ = T^*_{w_1} \ldots T^*_{w_r} u_{b,1} u_{b,2} \ldots u_{b,r} = T^*_{w_1} \ldots T^*_{w_r} u_{b,1} u_{b,2} \ldots u_{b,r} T^*_{w_1} = u_{a,1} u_{a,2} \ldots T^*_{w_1} u_{b,1} \ldots u_{b,r} T^*_{w_1} = \ldots = u_{a,1} \ldots u_{a,r} T^*_{w_1} v_{1} T^*_{w_1} = \ldots T^*_{w_1} v_{1} T^*_{w_1}.
$$

Thus, $T^*_d u_b^+ \in u_a^+ \mathcal{H}$, as required.

Having shown that $m_{st} \in M^\mu$, our next aim is to prove that the set of elements of the form $m_{st}$ give a basis of $M^\mu$.

Let $a = (a_1, \ldots, a_r)$ where $a_k = \sum_{i=1}^{k-1} |\mu(i)|$ as in the previous proof; then $m_\mu = u_a^+ x_\mu$. Since $u_a^+$ and $x_\mu$ commute, we have $M^\mu = m_\mu \mathcal{H} \subseteq u_a^+ \mathcal{H} \cap x_{\mu} \mathcal{H}$. We shall show that we have equality here.

We have seen that $\mathcal{H}$ has a standard basis which consists of elements $m_{st}$ with $s$ and $t$ standard $\lambda$-tableaux for some multipartition $\lambda$ of $n$. Suppose that $h \in \mathcal{H}$ and let $h =$
\[ \sum_{s \in T} r_{st} m_{st}, \text{ with } r_{st} \in R, \text{ be the unique expression for } h \text{ in terms of the standard basis. We say that } m_{st} \text{ is involved in } h \text{ if } r_{st} \neq 0. \]

(4.11) Lemma Suppose that \( \mu \) is a multipartition of \( n \) and that \( m_\mu = u_\mu^+ x_\mu \). Let \( h \in x_\mu \mathcal{H} \cap x_\mu^+ \mathcal{H} \) and suppose that \( (s, t) \) is a pair of standard tableaux of the same shape such that

(i) \( m_{st} \) is involved in \( h \); and,

(ii) if \( (u, v) \) is a pair of standard tableaux of the same shape such that \( s \triangleright u, t \triangleright v \) and \( (s, t) \neq (u, v) \) then \( m_{uv} \) is not involved in \( h \).

Let \( S = \mu(s) \). Then \( S \) is semistandard and \( s = \text{last}(S) \).

Proof: Since \( s \) is standard, the entries in \( S \) are non-decreasing down rows and columns.

Suppose that \( i \) and \( i + 1 \) belong to the same row of \( v^\mu \). Then \( T_i x_\mu = q x_\mu \), so \( T_i h = qh \). Therefore, by [16, (4.19)], \( i \) and \( i + 1 \) do not belong to the same column of \( s \). Hence the entries in \( S \) are strictly increasing down columns.

Since \( h \in x_\mu^+ \mathcal{H} \) we have \( \text{comp}_\mu \geq \text{comp}_s \) by Corollary 3.24 (applied to the multi-composition \( [(1^{\alpha_1}), (1^{\alpha_2}), \ldots, (1^{\alpha_r})] \), where \( \alpha = (|\mu(1)|, \ldots, |\mu(r)|) \)). Hence, by (4.5), \( S \) is semistandard.

Finally, suppose that \( s \neq \text{last}(S) \). Then there exist integers \( i \) and \( i + 1 \) in the same row of \( v^\mu \) such that the tableau \( s' = s(i, i + 1) \) is standard and \( s \triangleright s' \). As in [16, (4.19)], \( m_{st} \) is involved in \( T_i h = qh \), contradicting part (ii) of our hypothesis. Therefore, \( s = \text{last}(S) \). \( \square \)

(4.12) Corollary Suppose that \( \mu \) is a multi-composition of \( n \) and \( m_\mu = u_\mu^+ x_\mu \). Then \( u_\mu^+ \mathcal{H} \cap x_\mu \mathcal{H} \) is spanned by

\[ \left\{ m_{st} \mid S \in T_0(\lambda, \mu) \text{ and } t \in \text{Std}(\lambda) \text{ for some multipartition } \lambda \text{ of } n \right\}. \]

Proof: Note that \( m_{st} \in M^\mu \) by Lemma 4.10.

For each multipartition \( \lambda \) of \( n \) let

\[ T_\ell(\lambda, \mu) = \{ s \in \text{Std}(\lambda) \mid \mu(s) = \text{last}(S) \text{ for some } S \in T_\ell(\lambda, \mu) \}. \]

By Lemma 4.11 every non-zero element of \( u_\mu^+ \mathcal{H} \cap x_\mu \mathcal{H} \) involves a standard basis element \( m_{st} \) where \( s \in T_\ell(\lambda, \mu) \) and \( t \in \text{Std}(\lambda) \) for some multipartition \( \lambda \).

Now suppose that \( h \in u_\mu^+ \mathcal{H} \cap x_\mu \mathcal{H} \) and write \( h \) in terms of the standard basis; say, \( h = \sum r_{uv} m_{uv} \) with \( r_{uv} \in R \). Let

\[ h' = h - \sum_{\lambda} \sum_{s \in T_\ell(\lambda, \mu)} \sum_{t \in \text{Std}(\lambda)} r_{st} m_{\mu(s)t}. \]

Then \( h' \in u_\mu^+ \mathcal{H} \cap x_\mu \mathcal{H} \), but \( h' \) does not involve any term \( m_{st} \) for any \( s \in T_\ell(\lambda, \mu) \). Therefore, \( h' = 0 \), and we have obtained an expression for \( h \) as a linear combination of the required form. \( \square \)
(4.13) Corollary Suppose that $\mu$ is a multicomposition of $n$ and $m_\mu = u^+_{a,\mu} x_\mu$. Then $M^\mu = u^+_{a,\mu} \mathcal{H} \cap x_\mu \mathcal{H}$.

Proof: We have that $M^\mu \subseteq u^+_{a,\mu} \mathcal{H} \cap x_\mu \mathcal{H}$. The inclusion $u^+_{a,\mu} \mathcal{H} \cap x_\mu \mathcal{H} \subseteq M^\mu$ follows from Corollary 4.12 and Lemma 4.10.

The next theorem for Iwahori–Hecke algebras of type A (that is, the case $r = 1$), is due to Murphy [16]; for Iwahori–Hecke algebras of type B (that is, the case $r = 2$), the result is due to Du and Scott [11].

(4.14) Theorem Suppose that $\mu$ is a multicomposition of $n$. Then $M^\mu$ is free as an $R$-module with basis

$$\left\{ m_{\lambda t} \mid S \in T_0(\lambda, \mu) \text{ and } t \in \text{Std}(\lambda) \text{ for some multipartition } \lambda \text{ of } n \right\}.$$ 

Proof: By Lemma 4.10 each element $m_{\lambda t}$ belongs to $M^\mu$. Since distinct elements $m_{\lambda t}$ involve distinct standard basis elements $m_{\lambda t}$, the elements $m_{\lambda t}$ are linearly independent. Finally, Corollaries 4.12 and 4.13 show that the elements $m_{\lambda t}$ span $M^\mu$.

(4.15) Corollary Suppose that $\mu$ is a multicomposition of $n$. Then there exists a filtration of $M^\mu$,

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$$

such that for each $i$ with $1 \leq i \leq k$ there exists a multipartition $\lambda_i$ of $n$ with $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, if $\lambda$ is a multipartition of $n$, then the number of factors $S^{\lambda_i}$ isomorphic to $S^{\lambda}$ is equal to the number of semistandard $\lambda$-tableaux of type $\mu$.

Proof: Choose an ordering $S_1 > S_2 > \cdots > S_k$ on the set of semistandard tableaux of type $\mu$ such that $j > i$ if $\lambda_i > \lambda_j$ where $S_i \in T_0(\lambda_i, \mu)$ and $S_j \in T_0(\lambda_j, \mu)$. Let $M_i$ be the $R$-submodule of $M^\mu$ with basis $\{ m_{\lambda t} \mid j \geq i \text{ and } t \in \text{Std}(\lambda_j) \}$. Then

$$M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0,$$

and, by Proposition 3.25 and Theorem 4.14, each $M_i$ is a right ideal of $\mathcal{H}$.

Suppose that $1 \leq i \leq k$. Then $M_i \cap N^{\lambda_i} \subseteq M_{i-1}$. Hence, there is a well defined $\mathcal{H}$-homomorphism $\theta$ from $S^{\lambda_i}$ onto $M_i/M_{i+1}$ such that $\theta(z_{\lambda_i}) = m_{\lambda t_{\lambda_i}} + M_{i+1}$. Since both $S^{\lambda_i}$ and $M_i/M_{i+1}$ have rank equal to the number of standard $\lambda_i$–tableaux, $\theta$ is an isomorphism. The Corollary now follows.
5 The double annihilator of \( m_\mu \)

The purpose of this section is to compute the double annihilator of the element \( m_\mu \) of \( \mathcal{H} \) for any multicomposition \( \mu \) of \( n \). This will enable us to calculate a basis for \( \text{Hom}_{\mathcal{H}}(M^\nu, M^\mu) \) in the next section.

(5.1) Definition Suppose \( S \) is a subset of \( \mathcal{H} \) and define \( r(S) = \{ h \in \mathcal{H} \mid Sh = 0 \} \) and \( l(S) = \{ h \in \mathcal{H} \mid hs = 0 \} \). The double annihilator of \( S \) is \( lr(S) = l(r(S)) \).

It is elementary that \( lr(S) \) contains the left ideal of \( \mathcal{H} \) generated by \( S \). If \( \mathcal{H} \) is a quasi-Frobenius algebra, then \( lr(S) \) is equal to the left ideal of \( \mathcal{H} \) generated by \( S \) [3, (61.2)]. Although \( \mathcal{H} \) is quasi-Frobenius for \( r \leq 2 \) (see, for example, [4, section 2]), we do not know whether it is quasi-Frobenius for \( r > 2 \). If \( \mathcal{H} \) is quasi-Frobenius for all \( r \), then the main result of this section, namely Theorem 5.16, is immediate.

A connection between double annihilators and homomorphisms is provided by the following easy lemma.

(5.2) Lemma Suppose that \( m \in \mathcal{H} \) and that \( lr(m) = \mathcal{H}m \). Let \( I \) be a right ideal of \( \mathcal{H} \).

(i) For all \( \varphi \in \text{Hom}_{\mathcal{H}}(m, \mathcal{H}, I) \) there exists \( h_\varphi \in \mathcal{H} \) such that \( \varphi(m) = h_\varphi m \).

(ii) \( \text{Hom}_{\mathcal{H}}(m, \mathcal{H}, I) \cong \mathcal{H}m \cap I \).

Proof: For all \( x \in r(m) \) we have \( \varphi(m)x = \varphi(mx) = 0 \). Therefore, \( \varphi(m) \in lr(m) = \mathcal{H}m \); so \( \varphi(m) = h_\varphi m \) for some \( h_\varphi \in \mathcal{H} \). The proof of part (ii) is now straightforward. \( \square \)

The main result of this section is that \( lr(m_\mu) = \mathcal{H}m_\mu \). Because the Iwahori-Hecke algebra \( \mathcal{H}(\mathfrak{S}_n) \) is quasi-Frobenius, our first step is easy.

(5.3) Lemma Suppose that \( \mu \) is a multicomposition of \( n \). Then \( lr(x_\mu) = \mathcal{H}x_\mu \).

Proof: Since \( \mathcal{H}(\mathfrak{S}_n) \) is a quasi-Frobenius algebra, we have \( lr(x_\mu) \cap \mathcal{H}(\mathfrak{S}_n) = \mathcal{H}(\mathfrak{S}_n)x_\mu \).

Now suppose that \( z \in lr(x_\mu) \). By Theorem 2.2, we may write

\[
z = \sum_{c \in \mathcal{C}} L_1^{c_1} \ldots L_n^{c_n} h_c
\]

where \( \mathcal{C} = \{ c = (c_1, \ldots, c_r) \mid 0 \leq c_j < r \text{ for } 1 \leq j \leq n \} \) and \( h_c \in \mathcal{H}(\mathfrak{S}_n) \). Then, for all \( y \in r(x_\mu) \cap \mathcal{H}(\mathfrak{S}_n) \), we have

\[
0 = zy = \sum_{c \in \mathcal{C}} L_1^{c_1} \ldots L_n^{c_n} h_c y.
\]

Therefore, by Theorem 2.2, we have \( h_c y = 0 \) for all \( c \in \mathcal{C} \). Thus, \( h_c \in lr(x_\mu) \cap \mathcal{H}(\mathfrak{S}_n) = \mathcal{H}(\mathfrak{S}_n)x_\mu \), and so \( z \in \mathcal{H}x_\mu \). \( \square \)

For the remainder of this section we fix a multicomposition \( \mu \) and define \( a = (a_1, \ldots, a_r) \) by \( a_k = \sum_{i=1}^{k-1} |\mu^{(i)}| \); so \( 0 = a_1 \leq a_2 \leq \ldots \leq a_r \leq n \). Our main task in this section is to prove that \( lr(u_a^\mu) = \mathcal{H}u_a^\mu \). To achieve this goal, we use the next result.
(5.4) **Lemma** Assume that $I$ is a left ideal of $\mathcal{H}$ and suppose that $S$ is a subset of $r(I)$ such that $l(S) \subseteq I$. Then $lr(I) = I$.

**Proof:** By assumption $S \subseteq r(I)$ and $l(S) \subseteq I$. Also, $I \subseteq lr(I)$. Therefore, $lr(I) \subseteq l(S) \subseteq I \subseteq lr(I)$; so $lr(I) = I$ as required. □

Our definition of $v^+_a$ expresses $v^+_a$ as a double product

$$v^+_a = \prod_{k=1}^r \prod_{m=1}^{a_k} (L_m - Q_k).$$

We now introduce notation which allows us to reverse the order of the multiplication here.

(5.5) **Definition** Suppose that $1 \leq i \leq n$.

(i) Let $\gamma_i = k$ if $k$ is maximal such that $a_k < i$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

(ii) Let $v_i = \prod_{k=\gamma_i+1}^r (L_i - Q_k)$.

Then $v^+_a = v_1v_2\ldots v_n$ and all of these factors commute.

(5.6) **Definition** Suppose that $1 \leq i \leq n$. Let $y_i = T_{i-1}\ldots T_2T_1 \prod_{k=1}^{\gamma_i} (L_1 - Q_k)$.

(5.7) **Example** Suppose that $r = 4$, $n = 5$ and $a = (0, 2, 4, 5)$. Then

$$v^+_a = (L_1 - Q_2)(L_2 - Q_2) \times (L_1 - Q_3)(L_2 - Q_3)(L_3 - Q_3)(L_4 - Q_3) \times (L_1 - Q_4)(L_2 - Q_4)(L_3 - Q_4)(L_4 - Q_4)(L_5 - Q_4).$$

Also $\gamma = (1, 1, 2, 2, 3)$ and, for $1 \leq i \leq n$, the product of the factors in the $i$th column of the above array is $y_i$. We have

$$y_1 = (L_1 - Q_1),$$

$$y_2 = T_1(L_1 - Q_1),$$

$$y_3 = T_2T_1(L_1 - Q_1)(L_1 - Q_2),$$

$$y_4 = T_3T_2T_1(L_1 - Q_1)(L_1 - Q_2),$$

$$y_5 = T_4T_3T_2T_1(L_1 - Q_1)(L_1 - Q_2)(L_1 - Q_3).$$

(5.8) **Lemma** Suppose that $1 \leq i \leq n$. Then $v_1v_2\ldots v_iy_i = 0$. 
PROOF: Assume that \( k \) satisfies \( \gamma_i + 1 \leq k \leq r \). Then \( v_1 v_2 \ldots v_i \) has a factor \((L_1 - Q_k)(L_2 - Q_k)\ldots\ldots(L_i - Q_k)\). By (2.1)(iv), this factor commutes with \( T_{i-1} \ldots T_2 T_1 \). Hence, \( v_1 v_2 \ldots v_i T_{i-1} \ldots T_2 T_1 \) has a right factor \( \prod_{k=n+1}^{r}(L_1 - Q_k) = \prod_{k=1}^{r}(T_0 - Q_k) = 0; \)
so it follows that \( v_1 v_2 \ldots v_i y_i = 0 \) as claimed. \( \square \)

Since \( u_a^i = v_1 v_2 \ldots v_n \) we have the following Corollary.

(5.9) Corollary Suppose that \( 1 \leq i \leq n \). Then \( u_a^i y_i = 0 \).

It turns out that \( y_1, y_2, \ldots, y_n \) generate \( r(u_a^i) \). More importantly, together with Lemma 5.4 the elements \( y_i \) will allow us to prove that \( \text{I}r(u_a^i) = \mathcal{H} u_a^i \). For our proof, we require the technical Lemma 5.13 below; for this result we need some preparation.

(5.10) Lemma Suppose that \( 1 \leq i < n \) and that \( a, b \in \{0, 1, \ldots, r - 1\} \).

(i) If \( a \leq b \) then \( L_i^b L_{i+1}^a T_i = T_i L_i^b L_{i+1}^a + (q - 1) \sum_{c=1}^{a-b} L_i^{b-c} L_{i+1}^{a+c} \).

(ii) If \( a > b \) then \( L_i^b L_{i+1}^a T_i = qT_i^{-1} L_i^b L_{i+1}^a - (q - 1) \sum_{c=1}^{a-b} L_i^{b+c} L_{i+1}^{a-c} \).

PROOF: (i) Assume that \( a \leq b \). If \( b = a \) then \( T_i \) commutes with \( L_i^b L_{i+1}^a \) by (2.1)(iii), so the result is correct in this case. Now suppose that \( b > a \). Since \( L_{i+1} T_i = T_i L_i + (q - 1)L_{i+1} \) we have

\[
L_i^b L_{i+1}^a T_i = L_i^b L_{i+1}^{a-1}(T_i L_i + (q - 1)L_{i+1}),
\]

which gives the required result by induction on \( b - a \).

(ii) Either argue similarly, or apply the antiautomorphism \( * \) to the result of part (i) and rearrange, interchanging \( a \) and \( b \). \( \square \)

We next generalize Lemma 5.10 as follows.

(5.11) Lemma Suppose that \( 1 \leq i \leq n \) and \( b_j \in \{0, 1, \ldots, r - 1\} \) for \( j = 1, 2, \ldots, n \). Then there exists an integer \( b \) and \( \epsilon_1, \ldots, \epsilon_{i-1} \in \{\pm 1\} \) such that

\[
L_i^{b_1} L_2^{b_2} \ldots L_i^{b_i} T_{i-1} \ldots T_2 T_1 = x_1 + x_2,
\]

where \( x_1 = q^{b_i T_i^{\epsilon_{i-1}} \ldots T_2^{\epsilon_2} T_1^{\epsilon_1} L_i^{b_i} L_2^{b_2} \ldots L_i^{b_1}} \) and \( x_2 \) is a linear combination of terms of the form \( T_w L_1^{c_1} L_2^{c_2} \ldots L_i^{c_i} \) where

(i) \( w \in \mathfrak{S}_n \); and
(ii) $c_1, \ldots, c_i \in \{0, 1, \ldots, r - 1\}$ with $c_1 + c_2 + \cdots + c_i = b_1 + b_2 + \cdots + b_i$; and

(iii) either $\prod_{j=1}^{i}(c_j + 1) < \prod_{j=1}^{i}(b_j + 1)$, or $c_1, c_2, \ldots, c_i$ is a permutation of $b_1, b_2, \ldots, b_i$ and $c_1 < b_i$.

PROOF: The result is true when $i = 1$. (In this case, $x_1 = L^b_i$ and $x_2 = 0$.) Assume, inductively, that $i < n$ and that the result as stated is true. By Lemma 5.10, there exists $b' \in \{0, 1\}$ and $\epsilon_i = \pm 1$ such that

$$L_i \prod_{j=1}^{i}(c_j + 1)T_i = q^{b'}T_i^{\epsilon_i}L_i^{b_i+1}L_i^{b_i} \pm (q - 1) \sum L_i^{d_i}T_i^{d_{i-1}}$$

where $d_i, d_{i+1} \in \{0, 1, \ldots, r - 1\}$ and $d_i + d_{i+1} = b_i + b_{i+1}$ and either $(d_i + 1)(d_{i+1} + 1) < (b_i + 1)(b_{i+1} + 1)$ or $d_i = b_i < b_{i+1}$. Using this result, and (2.1)(ii), we obtain

$$L_1^{b_1}L_2^{b_2} \cdots L_i^{b_i}T_{i-1} \cdots T_2T_1 = q^{b'}T_i^{\epsilon_i}L_1^{b_1} \cdots L_{i-1}^{b_{i-1}}L_i^{b_i} \cdots T_{i-1}T_i^{d_{i+1}} \pm (q - 1) \sum L_1^{b_1} \cdots L_{i-1}^{b_{i-1}}L_i^{d_i}T_{i-1} \cdots T_2T_1T_i^{d_{i+1}},$$

where $b'$, $d_i$ and $d_{i+1}$ are as above. The result now follows from our inductive hypothesis.

For the statement and proof of our next Lemma, we need some more notation.

(5.12) Notation Suppose that $0 \leq i \leq n$. Let $V_i$ be the $R$-module with basis

$$\{ T_wL_1^{b_1}L_2^{b_2} \cdots L_n^{b_n}v_1v_2 \cdots v_i \mid \ w \in \mathfrak{S}_n, \ 0 \leq b_j < \gamma_j, \text{ for } 1 \leq j \leq i, \text{ and } 0 \leq h_j < r, \text{ for } i + 1 \leq j \leq n \}.$$  

Result (2.4) shows the elements in these sets are indeed linearly independent. Note also that $V_0 = \mathcal{H}$ and $V_n = \mathcal{H}u_+^r$.

We shall use (2.4) extensively in the proof of Lemma 5.13.

(5.13) Lemma Suppose that $1 \leq i \leq n$ and that $z \in V_{i-1}$ and $zy_i = 0$. Then $z \in V_i$.

PROOF: Since $z \in V_{i-1}$ we may write $z$ as a linear combination of linearly independent terms

$$T_wL_1^{b_1} \cdots L_{i-1}^{b_{i-1}}f(L_i)L_i^{b_i+1} \cdots L_n^{b_n}v_1v_2 \cdots v_{i-1}$$

where $w \in \mathfrak{S}_n$ and $0 \leq b_j < \gamma_j$ for $1 \leq j \leq i - 1$ and $0 \leq b_j < r$ for $i + 1 \leq j \leq n$, and $f(X)$ is a polynomial in $R[X]$ of degree less than $r$. Write

$$f(X) = g(X) \prod_{k=\gamma_{i+1}}^{r} (X - Q_k) + h(X),$$
where \( g(X) \) and \( h(X) \) are polynomials in \( R[X] \) such that \( \deg g(X) < \gamma_1 \) and \( \deg h(X) < r - \gamma_1 \). We may apply the "Euclidean algorithm" in this way, even though \( R[X] \) need not be a Euclidean domain, because the divisor is a monic polynomial.

Note that \( \prod_{k=1}^{n+1} (L_i - Q_k) = v_i \). As a consequence, we have written \( z \) as \( z_1 + z_2 \) where \( z_1 \in V_i \) and \( z_2 \) is a linear combination of terms
\[
T_{u} L_{1}^{b_1} \cdots L_{i-1}^{b_{i-1}} h(L_i) L_{i+1}^{b_{i+1}} \cdots L_{n}^{b_{n}} v_1 v_2 \cdots v_{i-1}.
\]
We shall prove that \( z_2 = 0 \).

Note that \( z_1 y_i = 0 \), by Lemma 5.8, since \( z_1 \in V_i \). Therefore, because \( y_i = 0 \) by assumption, it follows that \( z_2 y_i = 0 \).

For each \( n \)-tuple \( c = (c_1, \ldots, c_n) \) with \( 0 \leq c_i < r \), for all \( i \) with \( 1 \leq i \leq n \), let \( U_c = H^c(\mathfrak{g}) L_{1}^{a_1} \cdots L_{n}^{a_n} \). Then we may write
\[
z_2 = \sum_{c \in \mathcal{C}} z_c \quad \text{where} \quad z_c \in U_c
\]
and \( \mathcal{C} = \{ c = (c_1, \ldots, c_n) \mid 0 \leq c_j < r \text{ for } 1 \leq j \leq n \text{ and } c_i < r - \gamma_i \} \).

Assume that \( z_2 \neq 0 \). Among the \( c \) such that \( z_c \neq 0 \) choose \( d = (d_1, \ldots, d_n) \) such that
\[(5.14) \quad \begin{align*}
& (i) \quad d_1 + d_2 + \cdots + d_n \text{ is maximal; and,} \\
& (ii) \quad (d_1 + 1)(d_2 + 1) \cdots (d_n + 1) \text{ is maximal subject to (i); and,} \\
& (iii) \quad d_i \text{ is maximal subject to (i) and (ii)}. 
\end{align*}
\]
Then \( z_d = h_d L_{1}^{d_1} \cdots L_{n}^{d_n} \) for some non-zero element \( h_d \) of \( H^c(\mathfrak{g}) \).

Now, \( T_{i-1} \cdots T_2 T_1 \) commutes with each of the elements \( L_{i+1}, L_{i+2}, \ldots, L_n \). Therefore, by Lemma 5.11 there exists an invertible element \( h \in H^c(\mathfrak{g}) \) (namely, \( h = q^{b_1 T_{i-1}} \cdots T_2 T_1 \)) such that
\[
z_2 T_{i-1} \cdots T_2 T_1 = h_d h L_{1}^{d_1} L_{2}^{d_2} L_{3}^{d_3} \cdots L_{i}^{d_i} L_{i+1}^{d_{i+1}} \cdots L_{n}^{d_{n}} + \text{ terms in the sets } U_e,
\]
where \( e = (e_1, \ldots, e_n) \) and \( e_1 + \cdots + e_n \leq d_1 + \cdots + d_n \), and either
\[
(\text{i}) \quad e_1 + \cdots + e_n < d_1 + \cdots + d_n \quad \text{(by Lemma 5.11(ii) and (5.14)(i)); or,}
\]
\[
(\text{ii}) \quad (e_1 + 1) \cdots (e_n + 1) < (d_1 + 1) \cdots (d_n + 1) \quad \text{(using Lemma 5.11(iii) and (5.14)(ii)); or,}
\]
\[
(\text{iii}) \quad e_1, \ldots, e_n \text{ is a permutation of } d_1, \ldots, d_n \text{ and } e_1 < d_1 \quad \text{(using Lemma 5.11(iii) and}
\]
\[
(\text{iv}) \quad e_1, \ldots, e_n \text{ is a permutation of } d_1, \ldots, d_n \text{ but } e \neq (d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \quad \text{(consider the term } z_1 \text{ in Lemma 5.11).}
\]
In particular, no \( e \) on the right hand side is equal to \((d_1, d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)\).

Next we postmultiply our expression for \( z_2 T_{i-1} \cdots T_2 T_1 \) by \( \prod_{m=1}^{n} (L_1 - Q_m) \) to obtain \( z_2 y_i \). Note that if \( v_e \in U_e \) and \( k \) is any positive integer then \( v_e L_1^k \) is a linear combination of terms \( v_e \in U_f \) where \( f = (f_1, e_2, \ldots, e_n) \) and \( f_1 < r \). Thus, if \( e_1 + \cdots + e_n < d_1 + \cdots + d_n \) and \( e_1 + k \geq r \) then \( v_e L_1^k \) is a linear combination of terms \( v_f \) with \( f_1 + e_2 + \cdots + e_n < d_1 + \cdots + d_n + k \). Therefore,
\[
z_2 y_i = h_d h L_{1}^{d_1+\gamma_1} L_{2}^{d_2} L_{3}^{d_3} \cdots L_{i-1}^{d_{i-1}} L_{i+1}^{d_{i+1}} \cdots L_{n}^{d_{n}} + \text{ terms in the sets } U_f.
\]
where \( f \neq (d_i + \gamma_i, d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \).

Note that \( d_i + \gamma_i < r \) since \( d \in \mathcal{C} \). Therefore, since \( h_d \neq 0 \) and \( h_i \) is invertible, \( z_2 y_i \neq 0 \) giving a contradiction. Hence, our assumption that \( z_2 \neq 0 \) is false. Thus, \( z = z_1 \in V_i \) as required.

\( \square \)

(5.15) **Corollary** We have \( \text{lr}(u^+_a) = \mathcal{H} u^+_a \).

**Proof:** Let \( I = \mathcal{H} u^+_a \) and \( S = \{y_1, y_2, \ldots, y_n\} \). Then Corollary 5.9 shows that \( S \subseteq r(I) \).

Suppose that \( z \in \mathcal{H} \) and \( z y_i = 0 \) for all \( i \) with \( 1 \leq i \leq n \). Since \( V_0 = \mathcal{H} \) and \( V_n = \mathcal{H} u^+_a = I \), it follows from Lemma 5.13 that \( z \in I \). Thus, \( I(S) \subseteq I \). Lemma 5.4 now implies that \( \text{lr}(I) = I \); that is, \( \text{lr}(u^+_a) = \mathcal{H} u^+_a \).

Finally, we can prove the main result of this section.

(5.16) **Theorem** Suppose that \( \mu \) is a multicomposition of \( n \). Then \( \text{lr}(m_\mu) = \mathcal{H} m_\mu \).

**Proof:** Write \( m_\mu = u^+_a x_\mu \) as in Definition 3.5. By applying the definitions, Corollary 5.15 and Lemma 5.3, we obtain

\[
\mathcal{H} m_\mu \subseteq \text{lr}(m_\mu) \subseteq \text{lr}(u^+_a) \cap \text{lr}(x_\mu) = \mathcal{H} u^+_a \cap \mathcal{H} x_\mu.
\]

However, \( (u^+_a \mathcal{H} \cap x_\mu \mathcal{H})^* = (m_\mu \mathcal{H})^* \) by Corollary 4.13. Hence, we have equality throughout and \( \text{lr}(m_\mu) = \mathcal{H} m_\mu \) as claimed.

\( \square \)

(5.17) **Corollary** Suppose that \( \mu \) and \( \nu \) are multicompositions of \( n \). Then

(i) For every element \( \varphi \) in \( \text{Hom}_\mathcal{R}(M^\nu, M^\mu) \) there exists \( h_\varphi \in \mathcal{H} \) such that \( \varphi(m_\nu) = h_\varphi m_\nu \);

in particular, \( \varphi(m_\nu) \in M^{\nu*} \cap M^\mu \).

(ii) \( \text{Hom}_\mathcal{R}(M^\nu, M^\mu) \cong M^{\nu*} \cap M^\mu \).

**Proof:** By Theorem 5.16 we may apply Lemma 5.2(i) with \( m = m_\nu \) to obtain that \( \varphi(m_\nu) = h_\varphi m_\nu \) for some \( h_\varphi \in \mathcal{H} \). Thus, \( \varphi(m_\nu) \in M^{\nu*} \). It is clear that \( \varphi(m_\nu) \in M^\mu \).

Part (ii) follows from Lemma 5.2(ii), where an explicit isomorphism is given by \( \varphi \mapsto \varphi(m_\nu) \).

\( \square \)

6 **The cyclotomic \( q \)-Schur algebra**

In order to make our results as general as possible, let \( \Lambda_r \) be a poset ideal in the set of all multicompositions of \( n \). Thus, if \( \mu \in \Lambda_r \) and \( \nu > \mu \) then \( \nu \in \Lambda_r \). We also let \( \Lambda_r^+ \) be the set of multipartitions in \( \Lambda_r \).
(6.1) Definition The cyclotomic $q$–Schur algebra is the endomorphism algebra

$$\mathcal{S} = \text{End}_{\mathcal{H}} \left( \bigoplus_{\mu \in \Lambda^c} M^\mu \right).$$

Thus, $\mathcal{S} \cong \bigoplus_{\mu, \nu \in \Lambda^c} \text{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$.

Suppose that $\varphi \in \text{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$. Then $\varphi(m_\nu h) = \varphi(m_\nu)h$ for all $h \in \mathcal{H}$; thus $\varphi$ is completely determined by $\varphi(m_\nu)$. Moreover, $\varphi(m_\nu) \in M^{\nu^*} \cap M^\mu$ by Corollary 5.17. These remarks motivate us to construct a basis of $M^{\nu^*} \cap M^\mu$.

(6.2) Definition Suppose that $\mu$ and $\nu$ are multicompositions of $n$ and that $\lambda$ is a multipartition of $n$. Assume that $S \in \mathcal{T}_0(\lambda, \mu)$ and that $T \in \mathcal{T}_0(\lambda, \nu)$. Let

$$m_{ST} = \sum_{s, t} m_{st},$$

where the sum is over all $s, t \in \text{Std}(\lambda)$ with $\mu(s) = S$ and $\nu(t) = T$.

Note that $m_{ST} = m_{TS}$.

(6.3) Proposition Suppose that $\mu$ and $\nu$ are multicompositions of $n$. Then

$$\{ m_{ST} \mid S \in \mathcal{T}_0(\lambda, \mu) \text{ and } T \in \mathcal{T}_0(\lambda, \nu) \text{ for some multipartition } \lambda \text{ of } n \}$$

is a basis of $M^{\nu^*} \cap M^\mu$.

Proof: Since

$$m_{ST} = \sum_{s \in \text{Std}(\lambda)} (m_{T^s})^* = \sum_{t \in \text{Std}(\lambda)} m_{S^t},$$

Lemma 4.10 shows that $m_{ST} \in M^{\nu^*} \cap M^\mu$. Moreover, the elements $m_{ST}$ are linearly independent since they involve distinct elements $m_{st}$ of the standard basis of $\mathcal{H}$.

Now suppose that $h \in M^{\nu^*} \cap M^\mu$. Since $h \in \mathcal{H}$, we may express $h$ in terms of the standard basis $\mathcal{H}$; say

$$h = \sum_{m_{st} \in \mathcal{H}} r_{st} m_{st},$$

where $r_{st} \in R$. Since $h \in M^\mu$, we have $r_{st} = r_{s't'}$ if $\mu(s) = \mu(s')$, by Theorem 4.14. Similarly, since $h \in M^{\nu^*}$, we have $r_{st} = r_{s't'}$ if $\nu(t) = \nu(t')$. Thus, if $\mu(s) = \mu(s')$ and $\nu(t) = \nu(t')$ then $r_{st} = r_{s't'}$. Furthermore, $r_{st} = 0$ unless both $\mu(s)$ and $\nu(t)$ are semistandard. Therefore, $h$ is a linear combination of elements $m_{ST}$. This completes the proof of the Proposition. \qed

Proposition 6.3 shows that, in our next Definition, $\varphi_{ST}$ is a well defined $\mathcal{H}$–homomorphism from $M^\nu$ into $M^\mu$. 


(6.4) Definition Suppose that $\mu, \nu \in \Lambda_r$ and $\lambda \in \Lambda^+$. Assume that $S \in T_0(\lambda, \mu)$ and $T \in T_0(\lambda, \nu)$. Define $\varphi_{ST} \in \text{Hom}_F(M^\nu, M^\mu)$ by
$$\varphi_{ST}(m_\nu h) = m_{ST} h$$
for all $h \in H$. Extend $\varphi_{ST}$ to an element of the cyclotomic $q$-Schur algebra $\mathcal{S}$ by defining $\varphi_{ST}$ to be zero on $M^\kappa$ when $\nu \neq \kappa \in \Lambda_r$.

(6.5) Definition Suppose that $\lambda \in \Lambda^+$. Let $\overline{\mathcal{S}}_{\lambda}$ be the $R$-submodule of $\mathcal{S}$ spanned by
$$\left\{ \varphi_{UV} \mid U \in T_0(\alpha, \mu), V \in T_0(\alpha, \nu) \text{ for some } \mu, \nu \in \Lambda_r \text{ and } \alpha \in \Lambda^+_r \right\}.$$

This brings us to one of the main results of our paper. The work in Sections 4 and 5 was aimed at proving part (i) of the Theorem.

(6.6) Theorem (The Semistandard Basis Theorem)
(i) The cyclotomic $q$-Schur algebra $\mathcal{S}$ is free as an $R$-module with basis
$$\left\{ \varphi_{ST} \mid S \in T_0(\lambda, \mu), T \in T_0(\lambda, \nu) \text{ for some } \mu, \nu \in \Lambda_r \text{ and } \lambda \in \Lambda^+_r \right\}.$$

(ii) Suppose that $\mu, \nu \in \Lambda_r$ and $\lambda \in \Lambda^+_r$ and let $\varphi \in \mathcal{S}$. Then, for every $\kappa \in \Lambda_r$ and every $T' \in T_0(\lambda, \kappa)$ there exists $r_{T'} \in R$ such that for all $S \in T_0(\lambda, \mu)$, we have
$$\varphi_{ST} \varphi \equiv \sum_{\kappa \in \Lambda_r} \sum_{T \in T_0(\lambda, \kappa)} r_{T'} \varphi_{ST'} \mod \overline{\mathcal{S}}_{\lambda}.$$

PROOF: (i) By Corollary 5.17(ii), $\text{Hom}_F(M^\nu, M^\mu) \cong M^{\nu \ast} \cap M^\mu$, where an isomorphism is given by $\varphi \mapsto \varphi(m_\nu)$. Hence, by Proposition 6.3 and Definition 6.4, the set $\{ \varphi_{ST} \}$ is a basis of $\mathcal{S}$.

(ii) It is sufficient to consider the case where $\varphi \in \text{Hom}_F(M^\kappa, M^\nu)$ for some $\kappa \in \Lambda_r$. Suppose that $\varphi(m_\kappa) = m_\nu h$ where $h \in H$. Then, for all $S \in T_0(\lambda, \mu)$ and $T \in T_0(\lambda, \nu)$, we have
$$\varphi_{ST} \varphi(m_\kappa) = m_{ST} h \in M^{\kappa \ast} \cap M^\mu.$$ By Proposition 6.3, $m_{ST} h = \sum r_{UV} m_{UV}$, where $r_{UV} \in R$ and the sum is over $U \in T_0(\alpha, \mu)$ and $V \in T_0(\alpha, \kappa)$ for some $\alpha \in \Lambda^+_r$. By applying Proposition 3.25 we deduce that
$$m_{ST} h = \sum_{T \in T_0(\lambda, \kappa)} r_{T'} m_{ST'} + \sum_{U', V'} r_{U'V'} m_{U'V'}$$
where $r_{T'}, r_{U'V'} \in R$ and the second sum is over $U' \in T_0(\alpha, \mu)$ and $V' \in T_0(\alpha, \nu)$ for some $\alpha \in \Lambda^+_r$ with $\alpha \triangleright \lambda$. Therefore,
$$\varphi_{ST} \varphi \equiv \sum_{T \in T_0(\lambda, \kappa)} r_{T'} \varphi_{ST'} \mod \overline{\mathcal{S}}_{\lambda}.$$ This completes the proof. □
(6.7) **Definition** We call the basis $\{\varphi_{ST}\}$ the semistandard basis of $\mathcal{S}$.

(6.8) **Remark** If $d \geq n$ then $\Lambda^*_+$ consists of all multipartitions of $n$. It is straightforward to prove that if $d > n$, then the algebra $\mathcal{S}$ is Morita equivalent to the algebra we obtain by taking $d = n$.

In order to show that the semistandard basis of $\mathcal{S}$ is a cellular basis we need an appropriate antiautomorphism $\ast$ for $\mathcal{S}$.

(6.9) **Proposition** Let $\ast : \mathcal{S} \rightarrow \mathcal{S}$ be the unique $R$–linear map such that $\varphi_{ST}^\ast = \varphi_{TS}$ for all elements $\varphi_{ST}$ in the semistandard basis. Then $\ast$ is an antiautomorphism of $\mathcal{S}$.

**Proof:** Assume that $S \in \mathcal{T}_0(\lambda, \mu)$ and that $T \in \mathcal{T}_0(\lambda, \nu)$ for some $\mu, \nu \in \Lambda_r$ and $\lambda \in \Lambda^*_+$. Then

$$\varphi_{ST}(m_\nu) = m_{ST} = (m_{TS})^\ast = (\varphi_{TS}(m_\mu))^\ast.$$ 

The proposition now follows from the next general lemma.

(6.10) **Lemma** Suppose that $\mathcal{H}$ is an algebra with an antiautomorphism $\ast$. Let $\{m_\mu \mid \mu \in \Lambda\}$ be a set of elements of $\mathcal{H}$ such that $m_\mu^\ast = m_\mu$ for all $\mu \in \Lambda$ and let

$$S = \text{End}_\mathcal{H} \left( \bigoplus_{\mu \in \Lambda} m_\mu \mathcal{H} \right).$$

Assume that, for all $\mu, \nu \in \Lambda$, every $\mathcal{H}$–homomorphism from $m_\nu \mathcal{H}$ to $m_\mu \mathcal{H}$ is given by left multiplication by an element of $\mathcal{H}$. Then the following hold.

(i) For each $\varphi \in \text{Hom}_\mathcal{H}(m_\nu \mathcal{H}, m_\mu \mathcal{H})$ there is a unique $\varphi^\ast \in \text{Hom}_\mathcal{H}(m_\mu \mathcal{H}, m_\nu \mathcal{H})$ such that $\varphi^\ast(m_\mu) = (\varphi(m_\nu))^\ast$.

(ii) The map $\ast$ is an antiautomorphism of $\mathcal{S}$.

**Proof:** (i) Suppose that $\varphi \in \text{Hom}_\mathcal{H}(m_\nu \mathcal{H}, m_\mu \mathcal{H})$. Then $\varphi(m_\nu) = x_1 m_\nu = m_\mu y$ for some $x_1, y \in \mathcal{H}$. Define $\varphi^\ast \in \text{Hom}_\mathcal{H}(m_\mu \mathcal{H}, m_\nu \mathcal{H})$ by $\varphi^\ast(m_\mu h) = y^\ast m_\mu h$ for all $h \in \mathcal{H}$. Then $\varphi^\ast$ is a well–defined $\mathcal{H}$–homomorphism; also, it maps into $m_\nu \mathcal{H}$, since $y^\ast m_\mu = m_\nu x_1^\ast$. Since $\varphi^\ast(m_\mu) = (\varphi(m_\nu))^\ast$, the proof of (i) is complete.

(ii) Suppose that $\psi \in \text{Hom}_\mathcal{H}(m_\mu \mathcal{H}, m_\lambda \mathcal{H})$ for some $\lambda \in \Lambda$. Then $\psi(m_\mu) = x_2 m_\mu$ for some $x_2 \in \mathcal{H}$. We have,

$$\psi(\varphi)^\ast(m_\lambda) = (\psi \varphi(m_\nu))^\ast = (x_2 x_1 m_\nu)^\ast = m_\nu x_1^\ast x_2^\ast$$

$$= \varphi^\ast(m_\mu) x_2^\ast = \varphi^\ast(m_\mu x_2^\ast) = \varphi^\ast \psi^\ast(m_\lambda).$$

Therefore, $(\psi \varphi)^\ast = \varphi^\ast \psi^\ast$, and it follows that $\ast$ is an antiautomorphism of $\mathcal{S}$. \hfill \Box

(6.11) **Corollary** The $R$–module $\mathcal{F}$ is a two–sided ideal of $\mathcal{S}$. 

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This text is a continuation of the discussion on cyclotomic $q$–Schur algebras, where the definitions, remarks, and lemmas are presented to establish the properties of these algebras. The focus is on the semistandard basis and the antiautomorphism, which are essential for understanding the structure of these algebras.
PROOF: Theorem 6.6(ii) shows that \( \mathcal{F} \) is a right ideal. By applying the antiautomorphism \(*\), we deduce that it is also a left ideal. \( \square \)

(6.12) Theorem The semistandard basis of \( \mathcal{S} \) is a cellular basis.

PROOF: This follows at once from Theorem 6.6, Corollary 6.11 and Proposition 6.9. \( \square \)

We now apply the theory of cellular algebras to the representation theory of \( \mathcal{S} \), just as we treated \( \mathcal{H} \) in Section 3.

Suppose that \( \lambda \) is a multipartition of \( n \). Let \( T^\lambda = \lambda(t^\lambda) \) (see Definition 4.2). Then \( T^\lambda \) is the unique semistandard \( \lambda \)–tableau of type \( \lambda \) (cf. Example 4.3). Define \( \varphi_\lambda = \varphi_{T^\lambda \lambda} \); then \( \varphi_\lambda \) is the identity map on \( M^\lambda \).

(6.13) Definition Suppose that \( \lambda \in \Lambda_r^+ \), The Weyl module \( W^\lambda \) is the submodule of \( \mathcal{S}/\mathcal{F}_\lambda \) given by \( W^\lambda = \mathcal{S}(\varphi_\lambda + \mathcal{F}_\lambda) \).

If \( S \) is a semistandard \( \lambda \)–tableau let \( \varphi_S = \varphi_{ST^\lambda}(\varphi_\lambda + \mathcal{F}_\lambda) = \varphi_{ST^\lambda} + \mathcal{F}_\lambda \). Then, from Theorem 6.12, we obtain the following result.

(6.14) Corollary The Weyl module \( W^\lambda \) is a free \( R \)–module with basis

\[ \{ \varphi_S \mid S \in T_0(\lambda, \mu) \text{ for some } \mu \in \Lambda_r \} \].

Define a bilinear form \( \langle , \rangle \) on \( W^\lambda \) by requiring that

\[ \varphi_{T^\lambda S}\varphi_{T^\lambda} = \langle \varphi_S, \varphi_T \rangle \varphi_\lambda \mod \mathcal{F}_\lambda \]

for all semistandard \( \lambda \)–tableaux \( S \) and \( T \). This bilinear form is well–defined and symmetric, and also satisfies \( \langle su, v \rangle = \langle u, s^*v \rangle \) for all \( u, v \in W^\lambda \) and all \( s \in \mathcal{S} \). Consequently, \( \text{rad } W^\lambda = \{ u \in W^\lambda \mid \langle u, v \rangle = 0 \text{ for all } v \in W^\lambda \} \) is a submodule of \( W^\lambda \).

(6.15) Definition Suppose that \( \lambda \in \Lambda_r^+ \). Let \( F^\lambda = W^\lambda / \text{rad } W^\lambda \).

(6.16) Theorem Suppose that \( R \) is a field. Then

\[ \{ F^\lambda \mid \lambda \in \Lambda_r^+ \} \]

is a complete set of non–isomorphic irreducible \( \mathcal{S} \)–modules. Moreover, each \( F^\lambda \) is absolutely irreducible.
PROOF: Let $\lambda \in \Lambda^+_+$. From the definition of the bilinear form on $W^\lambda$, we have

$$\varphi_{\tau^1 \tau^1} \varphi_{\tau^1 \tau^1} \equiv \langle \varphi_{\tau^1}, \varphi_{\tau^1} \rangle \varphi_{\lambda} \mod \mathcal{I}.$$ 

However, $\varphi_{\tau^1 \tau^1} \varphi_{\tau^1 \tau^1} = \varphi_{\lambda}$ is the identity on $M^\lambda$; so $\langle \varphi_{\tau^1}, \varphi_{\tau^1} \rangle = 1$. Consequently, $F^\lambda$ is non-zero. The Theorem now follows from [12, (3.4)].

If $\lambda, \mu \in \Lambda^+_+$, let $d_{\lambda \mu}$ denote the composition multiplicity of $F^\mu$ as a composition factor of $W^\lambda$. Then

$$(d_{\lambda \mu})_{\lambda, \mu \in \Lambda^+_+}$$

is the decomposition matrix of $\mathcal{I}$. The theory of cellular algebras [12, (3.6)] yields the following.

(6.17) Corollary The decomposition matrix of $\mathcal{I}$ is unitriangular. That is, for $\lambda, \mu \in \Lambda^+_+$, we have $d_{\mu \mu} = 1$ and $d_{\lambda \mu} \neq 0$ only if $\lambda \geq \mu$.

Finally, Theorem 6.16 combined with [12, (3.10)], gives us our last result.

(6.18) Corollary The cyclotomic $q$–Schur algebra is quasi-hereditary.

References


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